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Research Article

Inequality of Ostrowski Type for Mappings with Bounded Fourth Order Partial Derivatives

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A general Ostrowski's type inequality for double integrals is given. We utilize function whose partial derivative of order four exists and is bounded.

1. Introduction

In 1938, Ostrowski [1] introduced the following integral inequality.

Theorem 1. *Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be continuous mapping on $[a,b]$ and differentiable on (a,b), whose derivative $f' : (a,b) \longrightarrow \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in[a,b]}|f'(t)| < \infty$, then for all $x \in [a,b]$*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty . \tag{1}$$

The constant $1/4$ is sharp in the sense that it cannot be replaced by a smaller one.

In 1975, Milovanović [2] proposed the following generalization of (1) for a function f of several variables whose first order partial derivatives are bounded.

Theorem 2. *Let $f\colon \mathbb{R}^m \longrightarrow \mathbb{R}$ be a differentiable function defined on $\overline{D}$ and let $|\partial f/\partial x_i| \leq M_i$ ($M_i > 0; i = 1,\ldots,m$) in D. Then, for every $X = (x_1,\ldots,x_m) \in \overline{D}$,*

$$\left| f(x_1,\ldots,x_m) - \frac{1}{\prod_{i=1}^m (b_i - a_i)}\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} f(y_1,\ldots,y_m)\,dy_1\cdots dy_m \right| \leq \sum_{i=1}^m \left[\frac{1}{4} + \frac{(x_i-(a_i+b_i)/2)^2}{(b_i-a_i)^2}\right](b_i-a_i)M_i. \tag{2}$$

In 1998, Barnett and Dragomir [3] proved the following Ostrowski type inequality for mappings of two variables with bounded second order partial derivatives.

Theorem 3. *Let $f\colon [a,b]\times[c,d] \longrightarrow \mathbb{R}$ continuous on $[a,b]\times[c,d]$, $f''_{x,y} = \partial^2 f/\partial x\partial y$ exists on $(a,b)\times(c,d)$ and is bounded, i.e.,*

$$\left\|f''_{s,t}\right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)} \left|\frac{\partial^2 f(x,y)}{\partial x \partial y}\right| < \infty, \tag{3}$$

Then we have the inequality

$$\begin{aligned}&\left|\int_a^b\int_c^d f(s,t)\,ds dt - \left[(b-a)\int_c^d f(x,t)\,dt + (d-c)\int_a^b f(s,y)\,ds - (d-c)(b-a)f(x,y)\right]\right| \\ &\leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2\right]\left\|f''_{s,t}\right\|_\infty ,\end{aligned} \tag{4}$$

for all $(x,y) \in [a,b]\times[c,d]$.

In [4], Xue et al. derive the following inequality of Ostrowski type.

Theorem 4. *Let $f: [a,b] \times [c,d] \longrightarrow \mathbb{R}$ be an absolutely continuous function such that the partial derivatives of order two exist and suppose that there exist constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq \partial^2 f(t,s)/\partial t\partial s \leq \Gamma$ for all $(t,s) \in [a,b] \times [c,d]$. Then we have*

$$\begin{aligned}
&\Bigg|(1-\lambda)^2 f(x,y) \\
&+\frac{\lambda}{2}(1-\lambda)\left[f(a,y)+f(b,y)+f(x,c)+f(x,d)\right] \\
&+\left(\frac{\lambda}{2}\right)^2\left[f(a,c)+f(b,c)+f(a,d)+f(b,d)\right] \\
&-\frac{1}{b-a}\left[(1-\lambda)\int_a^b f(t,y)\,dt\right. \\
&\left.+\frac{\lambda}{2}\int_a^b\left[f(t,c)+f(t,d)\right]dt\right] \\
&-\frac{1}{d-c}\left[(1-\lambda)\int_c^d f(x,s)\,ds\right. \\
&\left.+\frac{\lambda}{2}\int_c^d\left[f(a,s)+f(b,s)\right]ds\right] \\
&-\frac{\Gamma+\gamma}{2}(1-\lambda)^2\left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right) \\
&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,ds\,dt\Bigg| \leq \frac{\Gamma-\gamma}{2} \\
&\cdot\frac{1}{(b-a)(d-c)}\left[\left(\lambda^2+(1-\lambda)^2\right)\frac{(b-a)^2}{4}\right. \\
&\left.+\left(x-\frac{a+b}{2}\right)^2\right]\times\left[\left(\lambda^2+(1-\lambda)^2\right)\frac{(d-c)^2}{4}\right. \\
&\left.+\left(y-\frac{c+d}{2}\right)^2\right],
\end{aligned} \tag{5}$$

for all $(x,y) \in [a+\lambda((b-a)/2), b-\lambda((b-a)/2)] \times [c+\lambda((d-c)/2), d-\lambda((d-c)/2)]$ and $\lambda \in [0,1]$.

More recently, Sarikaya et al. [5] establish weighted Ostrowski type inequalities considering function whose second order partial derivatives are bounded as follows.

Theorem 5. *Let $f: [a,b] \times [c,d] \longrightarrow \mathbb{R}$ be an absolutely continuous function such that the partial derivatives of order two exist and are bounded, i.e.,*

$$\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right| < \infty, \tag{6}$$

for all $(t,s) \in [a,b] \times [c,d]$. Then we have

$$\begin{aligned}
&\Bigg|m_\circ(a,b)\,m_\circ(c,d)\,(x-\mu(a,b))\,(y-\mu(c,d))\,f(x,y) \\
&-m_\circ(c,d)\,(y-\mu(c,d))\left[\int_a^x\left(\int_a^t\omega(u)\,du\right)f(t,y)\,dt\right. \\
&\left.+\int_x^b\left(\int_b^t\omega(u)\,du\right)f(t,y)\,dt\right]-m_\circ(a,b)\,(x \\
&-\mu(a,b))\left[\int_c^y\left(\int_c^s\omega(v)\,dv\right)f(x,s)\,ds\right. \\
&\left.+\int_y^d\left(\int_d^s\omega(v)\,dv\right)f(x,s)\,ds\right]-m_\circ(a,b)\,m_\circ(c,d) \\
&\cdot\int_a^b\int_c^d f(t,s)\,ds\,dt\Bigg| \leq \frac{m_\circ(a,b)\,m_\circ(c,d)}{4}\left[(x\right. \\
&\left.-\mu(a,b))^2+\sigma^2(a,b)\right]\left[(y-\mu(c,d))^2+\sigma^2(c,d)\right] \\
&\cdot\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \leq \frac{m_\circ(a,b)\,m_\circ(c,d)}{4}\left(\left|x-\frac{a+b}{2}\right|\right. \\
&\left.+\frac{b-a}{2}\right)^2\left(\left|y-\frac{c+d}{2}\right|+\frac{d-c}{2}\right)^2\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
m_i(a,b) &= \int_a^b t^i\omega(t)\,dt, \quad i=0,1,\ldots, \\
\mu(a,b) &= \frac{m_1(a,b)}{m_\circ(a,b)}, \\
\sigma^2(a,b) &= \frac{m_2(a,b)}{m_\circ(a,b)}-\mu^2(a,b).
\end{aligned} \tag{8}$$

For other related work, we refer the reader to [6–15].

In this paper, motivated by the ideas in both [4, 5], we shall derive a new inequality of Ostrowski's type similar to the inequalities (5) and (7), involving functions of two independent variables.

2. Main Results

In order to introduce our main results, we commence with the following lemma.

Lemma 6. *Let $f: [a,b] \times [c,d] \longrightarrow \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 4 exists for all $(x,y) \in [a+h((b-a)/2), b-h((b-a)/2)] \times [c+h((d-c)/2), d-h((d-c)/2)]$ and $h \in [0,1]$. Then for any two mappings $K_1(t;x) : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ and $K_2(s;y) : [c,d] \times [c,d] \longrightarrow \mathbb{R}$, where*

$$K_1(t;x) := \begin{cases} \dfrac{1}{2}\left[t-\left(a+h\dfrac{b-a}{2}\right)\right]^2, & t \in [a,x] \\ \dfrac{1}{2}\left[t-\left(b-h\dfrac{b-a}{2}\right)\right]^2, & t \in (x,b] \end{cases} \tag{9}$$

and

$$K_2(s;y) := \begin{cases} \frac{1}{2}\left[s-\left(c+h\frac{d-c}{2}\right)\right]^2, & t\in[c,y] \\ \frac{1}{2}\left[s-\left(d-h\frac{d-c}{2}\right)\right]^2, & t\in(y,d], \end{cases} \quad (10)$$

the identity

$$\begin{aligned}
E(f;h) &= \int_a^b\int_c^d K_1(t;x)K_2(s;y)\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt = (1 \\
&- h)^2(b-a)(d-c)\left[f(x,y)+\left(\frac{a+b}{2}-x\right)\right. \\
&\cdot\left(\frac{c+d}{2}-y\right)f_{ts}(x,y)+\left(\frac{a+b}{2}-x\right)f_t(x,y) \\
&\left.+\left(\frac{c+d}{2}-y\right)f_s(x,y)\right]+h^2(1-h) \\
&\cdot\frac{(b-a)(d-c)}{8}\left[(d-c)\left(\frac{a+b}{2}-x\right)\right. \\
&\cdot(f_{ts}(x,c)-f_{ts}(x,d))+(b-a)\left(\frac{c+d}{2}-y\right) \\
&\left.\cdot(f_{ts}(a,y)-f_{ts}(b,y))\right]+h^4 \\
&\cdot\frac{(b-a)^2(d-c)^2}{64}\left[f_{ts}(a,c)-f_{ts}(a,d)-f_{ts}(b,c)\right. \\
&\left.+f_{ts}(b,d)\right]+h^2\frac{(b-a)^2}{8}\left\{(1-h)(d-c)\right. \\
&\cdot\left[f_t(a,y)-f_t(b,y)\right] \\
&+h\frac{d-c}{2}\left[f_t(a,c)+f_t(b,c)+f_t(a,d)-f_t(b,d)\right] \\
&\left.-\int_c^d\left[f_t(a,s)-f_t(b,s)\right]ds\right\}+\frac{h^2(d-c)^2}{8}\left\{(1-h)\right. \\
&\cdot(b-a)\left[f_s(x,c)-f_s(x,d)\right]+h \\
&\cdot\frac{b-a}{2}\left[f_s(a,c)+f_s(b,c)+f_s(a,d)-f_s(b,d)\right] \\
&\left.-\int_a^b\left[f_s(t,c)-f_s(t,d)\right]dt\right\}+(1-h)(d-c)\left(\frac{c+d}{2}\right. \\
&\left.-y\right)\left[h\frac{b-a}{2}\left[f_s(a,y)+f_s(b,y)\right]-\int_a^b f_s(t,y)\,dt\right] \\
&+(1-h)(b-a)\left(\frac{a+b}{2}-x\right) \\
&\cdot\left[h\frac{d-c}{2}\left[f_t(x,c)+f_t(x,d)\right]\right. \\
&\left.-\int_c^d f_t(x,s)\,ds\right]+h(1-h)\frac{(b-a)(d-c)}{2}\left[f(a,y)\right. \\
&+f(b,y)+f(x,c)+f(x,d)\big]+h^2 \\
&\cdot\frac{(b-a)(d-c)}{4}\left[f(a,c)\right. \\
&+f(b,c)+f(a,d)+f(b,d)\big]-(1-h)\left[(d-c)\right. \\
&\left.\cdot\int_a^b f(t,y)\,dt+(b-a)\int_c^d f(x,s)\,ds\right]-\frac{h}{2}\left[(d-c)\right. \\
&\cdot\int_a^b\left[f(t,c)+f(t,d)\right]dt+(b-a) \\
&\left.\cdot\int_c^d\left[f(a,s)+f(b,s)\right]ds\right]+\int_a^b\int_c^d f(t,s)\,dsdt.
\end{aligned} \quad (11)$$

holds.

Proof. By definitions of $K_1(t;x)$ and $K_2(s;y)$ in both (9) and (10), we have

$$\begin{aligned}
E(f;h) &= \int_a^b\int_c^d K_1(t;x)K_2(s;y)\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt = \frac{1}{4} \\
&\cdot\int_a^x\int_c^y\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2 \\
&\cdot\left[s-\left(c+h\frac{d-c}{2}\right)\right]^2\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt \\
&+\frac{1}{4}\int_a^x\int_y^d\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2 \\
&\cdot\left[s-\left(d-h\frac{d-c}{2}\right)\right]^2\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt \\
&+\frac{1}{4}\int_x^b\int_c^y\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2 \\
&\cdot\left[s-\left(c+h\frac{d-c}{2}\right)\right]^2\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt \\
&+\frac{1}{4}\int_x^b\int_y^d\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2 \\
&\cdot\left[s-\left(d-h\frac{d-c}{2}\right)\right]^2\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt \\
&= I_1+I_2+I_3+I_4.
\end{aligned} \quad (12)$$

For I_1, integration by parts yields

$$\begin{aligned}
I_1 &= \frac{1}{4}\int_a^x\int_c^y\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2 \\
&\cdot\left[s-\left(c+h\frac{d-c}{2}\right)\right]^2\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt \\
&= \frac{1}{4}\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2\left[s-\left(c+h\frac{d-c}{2}\right)\right]^2 \\
&\cdot f_{ts}(x,y)-\frac{h^2}{16}\left[(d-c)^2\left[x-\left(a+h\frac{b-a}{2}\right)\right]^2\right.
\end{aligned}$$

$$
\begin{aligned}
&\cdot f_{ts}(x,c)+(b-a)^2\left[y-\left(c+h\frac{d-c}{2}\right)\right]^2 f_{ts}(a,y)\Big]\\
&+\frac{h^4(b-a)^2(d-c)^2}{64}f_{ts}(a,c)+\frac{h^2(d-c)^2}{8}\int_x^b\Big[t\\
&-\left(b-h\frac{b-a}{2}\right)\Big]f_{ts}(t,c)\,dt-\frac{h^2(b-a)^2}{8}\int_c^y\Big[s\\
&-\left(c+h\frac{d-c}{2}\right)\Big]f_{ts}(b,s)\,ds+\frac{1}{2}\int_c^y\Big[s\\
&-\left(c+h\frac{d-c}{2}\right)\Big]\left[x-\left(b-h\frac{b-a}{2}\right)\right]^2 f_{ts}(x,s)\,ds\\
&-\frac{1}{2}\int_x^b\left[t-\left(b-h\frac{b-a}{2}\right)\right]\left[y-\left(c+h\frac{d-c}{2}\right)\right]^2\\
&\cdot f_{ts}(t,y)\,dt+\int_a^x\int_c^y\left[t-\left(a+h\frac{b-a}{2}\right)\right]\\
&\cdot\left[s-\left(c+h\frac{d-c}{2}\right)\right]f_{ts}(t,s)\,dsdt.
\end{aligned}
\tag{13}
$$

Similarly, I_2, I_3, and I_4 can be obtained.

Thus, by adding I_1, I_2, I_3, and I_4, we easily deduce

$$
\begin{aligned}
E(f;h)&=\int_a^b\int_c^d K_1(t;x)K_2(s;y)\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt=\frac{1}{4}[(h\\
&-1)(b-a)(a+b-2x)][(h-1)(d-c)(c+d-2y)]\\
&\cdot f_{ts}(x,y)-\frac{h^2}{16}(d-c)^2(h-1)(b-a)(a+b-2x)\\
&\cdot[f_{ts}(x,c)-f_{ts}(x,d)]-\frac{h^2}{16}(b-a)^2(h-1)(d-c)(c\\
&+d-2y)[f_{ts}(a,y)-f_{ts}(b,y)]\\
&+\frac{h^4(b-a)^2(d-c)^2}{64}[f_{ts}(a,c)-f_{ts}(a,d)-f_{ts}(b,c)\\
&+f(b,d)]+\frac{h^2(b-a)^2}{8}\left[\int_c^y\left[s-\left(c+h\frac{d-c}{2}\right)\right]\right.\\
&\cdot[f_{ts}(a,s)-f_{ts}(b,s)]\,ds+\int_y^d\left[s-\left(d-h\frac{d-c}{2}\right)\right]\\
&\left.\cdot[f_{ts}(a,s)-f_{ts}(b,s)]\,ds\right]\\
&+\int_a^x\int_c^y\left[t-\left(a+h\frac{b-a}{2}\right)\right]\left[s-\left(c+h\frac{d-c}{2}\right)\right]\\
&\cdot f_{ts}(t,s)\,dsdt+\int_a^x\int_y^d\left[t-\left(a+h\frac{b-a}{2}\right)\right]\\
&\cdot\left[s-\left(d-h\frac{d-c}{2}\right)\right]f_{ts}(t,s)\,dsdt\\
&+\int_x^b\int_c^y\left[t-\left(b-h\frac{b-a}{2}\right)\right]\left[s-\left(c+h\frac{d-c}{2}\right)\right]\\
&\cdot f_{ts}(t,s)\,dsdt+\int_x^b\int_y^d\left[t-\left(b-h\frac{b-a}{2}\right)\right]\\
&\cdot\left[s-\left(d-h\frac{d-c}{2}\right)\right]f_{ts}(t,s)\,dsdt.
\end{aligned}
\tag{14}
$$

By further algebraic manipulations and assuming result by [4], the proof of Lemma 6 is completed. □

Theorem 7. *Let $f\colon [a,b]\times[c,d]\longrightarrow\mathbb{R}$ such that $f\in C^4([a,b]\times[c,d])$ be an absolutely continuous function such that the partial derivative of order 4 exists and is bounded; i.e.,*

$$
\left\|\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}\right\|_\infty=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}\right|<\infty,\tag{15}
$$

for all $(t,s)\in[a,b]\times[c,d]$. Then for all $(x,y)\in[a+h((b-a)/2),b-h((b-a)/2)]\times[c+h((d-c)/2),d-h((d-c)/2)]$ and $h\in[0,1]$, we have

$$
\begin{aligned}
|E(f;h)|\le&\left[\frac{h^3(b-a)^3}{24}\right.\\
&+\frac{(1-h)(b-a)}{2}\left(\frac{(1-h)^2(b-a)^2}{12}\right.\\
&\left.\left.+\left(x-\frac{a+b}{2}\right)^2\right)\right]\times\left[\frac{h^3(d-c)^3}{24}\right.\\
&+\frac{(1-h)(d-c)}{2}\left(\frac{(1-h)^2(d-c)^2}{12}\right.\\
&\left.\left.+\left(y-\frac{c+d}{2}\right)^2\right)\right]\left\|\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}\right\|_\infty,
\end{aligned}
\tag{16}
$$

where the functional $E(f;h)$ is given by (11).

Proof. By considering (11), we have

$$
\begin{aligned}
|E(f;h)|&=\left|\int_a^b\int_c^d K_1(t;x)K_2(s;y)\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}dsdt\right|\\
&\le\int_a^b\int_c^d|K_1(t;x)||K_2(s;y)|\left|\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}\right|dsdt\\
&\le\left\|\frac{\partial^4 f(t,s)}{\partial t^2\partial s^2}\right\|_\infty\int_a^b K_1(t;x)\,dt\cdot\int_c^d K_2(s;y)\,ds.
\end{aligned}
\tag{17}
$$

But,

$$
\begin{aligned}
&\int_a^b K_1(t;x)\,dt=\frac{h^3}{24}(b-a)^3\\
&\quad+\frac{(1-h)(b-a)}{2}\left(\frac{(1-h)^2(b-a)^2}{12}+\left[x-\frac{a+b}{2}\right]^2\right),
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
&\int_c^d K_2(s;y)\,ds=\frac{h^3}{24}(d-c)^3\\
&\quad+\frac{(1-h)(d-c)}{2}\left(\frac{(1-h)^2(d-c)^2}{12}+\left[y-\frac{c+d}{2}\right]^2\right).
\end{aligned}
\tag{19}
$$

Now, substituting (18), (19) into (17) gives (16) and, hence, completes the proof. □

Corollary 8. *Under the assumption of Theorem 7 with $h = 0$, we have*

$$\begin{aligned}&\Bigg| f(x,y) + \left(\frac{a+b}{2} - x\right)\left(\frac{c+d}{2} - y\right) f_{ts}(x,y) \\ &+ \left(\frac{a+b}{2} - x\right) f_t(x,y) + \left(\frac{c+d}{2} - y\right) f_s(x,y) \\ &- \left[\frac{1}{(b-a)}\left(\frac{c+d}{2} - y\right)\int_a^b f_s(t,y)\,dt \right. \\ &\left. + \frac{1}{(d-c)}\left(\frac{a+b}{2} - x\right)\int_c^d f_t(x,s)\,ds\right] \\ &- \left[\frac{1}{(b-a)}\int_a^b f(t,y)\,dt + \frac{1}{(d-c)}\int_c^d f(x,s)\,ds\right] \\ &+ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,ds\,dt\Bigg| \le \frac{1}{4}\left(\frac{(b-a)^2}{12} \right. \\ &\left. + \left(x - \frac{a+b}{2}\right)^2\right)\left(\frac{(d-c)^2}{12} + \left(y - \frac{c+d}{2}\right)^2\right) \\ &\cdot \left\| \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} \right\|_\infty .\end{aligned} \tag{20}$$

Corollary 9. *Under the assumption of Theorem 7 with $h = 0$, $x = (a+b)/2$, and $y = (c+d)/2$ we have*

$$\begin{aligned}&\Bigg| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \left[\frac{1}{(b-a)}\int_a^b f\left(t, \frac{c+d}{2}\right) dt \right. \\ &\left. + \frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2}, s\right) ds\right] + \frac{1}{(b-a)(d-c)} \\ &\cdot \int_a^b\int_c^d f(t,s)\,ds\,dt\Bigg| \\ &\le \frac{(b-a)^2(d-c)^2}{576}\left\| \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} \right\|_\infty .\end{aligned} \tag{21}$$

Corollary 10. *Under the assumption of Theorem 7 with $h = 0$, $x = (a+b)/4$, and $y = (c+d)/4$ we have*

$$\begin{aligned}&\Bigg| f\left(\frac{a+b}{4}, \frac{c+d}{4}\right) + \frac{(a+b)(c+d)}{16} f_{ts}\left(\frac{a+b}{4}, \frac{c+d}{4}\right) \\ &+ \left(\frac{a+b}{4}\right) f_t\left(\frac{a+b}{4}, \frac{c+d}{4}\right) + \left(\frac{c+d}{4}\right) \\ &\cdot f_s\left(\frac{a+b}{4}, \frac{c+d}{4}\right) - \frac{1}{4}\left[\frac{(d-c)}{(b-a)}\int_a^b f_s\left(t, \frac{c+d}{4}\right) dt \right. \\ &\left. + \frac{(b-a)}{(d-c)}\int_c^d f_t\left(\frac{a+b}{4}, s\right) ds\right] \\ &- \left[\frac{1}{(b-a)}\int_a^b f\left(t, \frac{c+d}{4}\right) dt \right. \\ &\left. + \frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{4}, s\right) ds\right] \\ &+ \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,ds\,dt\Bigg| \le \frac{1}{4}\left[\frac{(b-a)^2}{12} \right. \\ &\left. + \frac{(a+b)^2}{16}\right]\left[\frac{(d-c)^2}{12} + \frac{(c+d)^2}{16}\right]\left\| \frac{\partial^4 f(t,s)}{\partial t^2 \partial s^2} \right\|_\infty .\end{aligned} \tag{22}$$

Remark 11. In Corollaries 8, 9, and 10 we assume that the involved integrals can more easily be computed than the original double integral.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] A. Ostrowski, “Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert,” *Commentarii Mathematici Helvetici*, vol. 10, no. 1, pp. 226-227, 1937.

[2] G. V. Milovanovic, *On Some Integral Inequalities*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Math. Fiz., (498/541), pp, 119-124, 1975.

[3] N. S. Barnett and S. S. Dragomir, “An Ostrowski type inequality for double integrals and applications for cubature formulae,” *Soochow Journal of Mathematics*, vol. 27, no. 1, pp. 1–10, 2001.

[4] Q. Xue, J. Zhu, and W. Liu, “A new generalization of Ostrowski-type inequality involving functions of two independent variables,” *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2219–2224, 2010.

[5] M. Z. Sarikaya, H. Yaldiz, and S. Erden, “On the weighted Ostrowski type inequalities for double integrals,” *Kragujevac Journal of Mathematics*, vol. 38, no. 2, pp. 303–314, 2014.

[6] W. G. Alshanti and A. Qayyum, “A note on new Ostrowski type inequalities using a generalized kernel,” *Bulletin of Mathematical Analysis and Applications*, vol. 9, no. 1, pp. 74–91, 2017.

[7] W. G. Alshanti, A. Qayyum, and M. A. Majid, “Ostrowski type inequalities by using generalized quadratic kernel,” *Journal of Inequalities and Special Functions*, vol. 8, no. 4, pp. 111–135, 2017.

[8] W. G. Alshanti, “A Perturbed Version of General Weighted Ostrowski Type Inequality and Applications,” *International Journal of Analysis and Applications*, vol. 16, no. 4, pp. 503–517, 2018.

[9] S. Erden, H. Budak, and M. Z. Sarikaya, “An Ostrowski type inequality for twice differentiable mappings and applications,” *Mathematical Modelling and Analysis*, vol. 21, no. 4, pp. 522–532, 2016.

[10] S. S. Dragomir, P. Cerone, N. S. Barnett, and J. Roumeliotis, “An inequality of the Ostrowski type for double integrals and

applications for cubature formulae," *RGMIA Research Report Collection*, vol. 2, no. 6, pp. 1–13, 1999.

[11] B. G. Pachpatte, "On an inequality of Ostrowski type in three independent variables," *Journal of Mathematical Analysis and Applications*, vol. 249, no. 2, pp. 583–591, 2000.

[12] B. G. Pachpatte, "On multivariate Ostrowski type inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 4, pp. 1–12, 2002.

[13] M. Z. Sarikaya, "On the Ostrowski type integral inequality for double integrals," *Demonstratio Mathematica*, vol. 45, no. 3, pp. 533–540, 2012.

[14] M. Z. Sarikaya and H. Ogunmez, "On the weighted Ostrowski-type integral inequality for double integrals," *Arabian Journal for Science and Engineering*, vol. 36, no. 6, pp. 1153–1160, 2011.

[15] G. Farid, "New Ostrowski-type inequalities and their applications in two coordinates," *Acta Mathematica Universitatis Comenianae*, vol. 85, no. 1, pp. 107–112, 2016.