

## Research Article

# Constructive Existence of (1,1)-Solutions to Two-Point Value Problems for Fuzzy Linear Multiterm Fractional Differential Equations

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In this paper, we consider the following two-point boundary value problems of fuzzy linear fractional differential equations:  $({}^c D_{1,1}^\alpha y)(t) \oplus b(t) \otimes ({}^c D_{1,1}^\beta y)(t) \oplus c(t) \otimes y(t) = f(t)$ ,  $t \in (0, 1)$ ,  $y(0) = y_0$  and  $y(1) = y_1$ , where  $b, c \in C(I)$ ,  $b(t), c(t) \geq 0$ ,  $y, f \in C(I, \mathbf{R}_F)$ ,  $I = [0, 1]$ ,  $y_0, y_1 \in \mathbf{R}_F$  and  $1 < \beta < \alpha \leq 2$ . Our existence result is based on Banach fixed point theorem and the approximate solution of our problem is obtained by applying the Haar wavelet operational matrix.

## 1. Introduction

A lot of researchers have studied fuzzy differential equations especially fuzzy boundary value problems (FBVPs) because they are effective tools for modeling processes. FBVPs arise in many applications such as modeling of fuzzy optimal control problem [1] and HIV infection [2]. Many theoretical researches have been carried out on fractional differential equations over the last years [3–7].

Currently, the approximative methods for solving fuzzy fractional differential equation include the operational matrix method based on orthogonal functions [8–10], linearization formula [11], Homotopy Analysis Method [12–14], and Variation of constant formula [15].

O'Regan [16] and Lakshmikantham [17] proved that two-point boundary value problems of fuzzy differential equations are equivalent to fuzzy integral equations.

Lakshmikantham et al. [18] considered Riemann–Liouville differentiability concept based on the Hukuhara differentiability to solve fuzzy fractional differential equations. Prakash [19] considered initial value problems for differential equations of fractional order with uncertainty. Mazandarania [20] investigated the solution to fuzzy fractional initial value problem (FFIVP) under Caputo-type fuzzy fractional derivatives by a modified fractional Euler method. As we

can see, fuzzy initial value problems were studied by many researchers, but few fuzzy boundary value problems were considered in special cases. Nieto [21] considered second order fuzzy differential by the sense of (1, 1), (1, 2), (2, 1), (2, 2)–derivatives. Also Nieto [22] investigated the existence and uniqueness of solutions for a first-order linear fuzzy differential equation with impulsive boundary value condition. Ngo et al. [23] proved the existence and uniqueness results of the solution to initial value problem of Caputo–Katugampola (CK) fractional differential equations in fuzzy setting and [24] present that a fractional fuzzy differential equation and a fractional fuzzy integral equation are not equivalent in general. Wang [25] considered the existence and uniqueness of solution for a class of FFDEs:

$$\begin{aligned} {}^C gHD_*^q u(t) &= f(t, u(t)), \quad 0 < t < T, \\ u(0) &= \lambda u(T), \\ u'_{gH}(0) &= u_0 \in E_C^1, \end{aligned} \tag{1}$$

where  ${}^C gHD_*^q$  is the fuzzy  $gH$ –fractional Caputo derivative,  $u'_{gH}$  is generalized Hukuhara derivative of  $u$ ,  $E_C^1$  is the space of fuzzy number,  $f : [0, T] \times E_C^1 \rightarrow E_C^1 \setminus R$  is a continuous

fuzzy valued function,  $q \in (1, 2]$  is a real number, and  $\lambda \in (0, 1) \cup (1, +\infty)$ .

Gasilov [26] presented a new approach to a nonhomogeneous fuzzy boundary value problem.

But researchers who studied fuzzy differential equations by using  $r$ -cut did not consider if the solutions of  $r$ -cut equations constitute intervals. So they had to recheck whether the solutions of  $r$ -cut equations constitute intervals or not after solving a problem. For instance, in [27] they only consider the existence of solutions of  $r$ -cut equation. And the existence of fuzzy solutions was considered in specific example. In that specific example, they noted that the fuzzy solutions do not exist even if the solutions of  $r$ -cut equation exist.

These facts lead to the following: the existence of solutions of fuzzy problem is not equivalent to the existence of solutions of corresponding  $r$ -cut equation. So, it is necessary to obtain a new  $r$ -cut problem that guarantees the existence of fuzzy solutions.

Motivated by the results mentioned above, we consider the following fuzzy boundary value problem:

$$\begin{aligned} &({}^c D_{1,1}^\alpha y)(t) \oplus b(t) \otimes ({}^c D_{1,1}^\beta y)(t) \oplus c(t) \otimes y(t) \\ &= f(t), \quad t \in (0, 1), \\ &y(0) = y_0, \\ &y(1) = y_1, \end{aligned} \tag{2}$$

where  $b, c \in C(I)$ ,  $b(t), c(t) \geq 0$ ,  $y, f \in C(I, \mathbf{R}_F)$ ,  $I = [0, 1]$ ,  $y_0, y_1 \in \mathbf{R}_F$ ,  $1 < \beta < \alpha \leq 2$ ,  ${}^c D_{1,1}^\alpha$  is fuzzy fractional Caputo derivative, and  $\mathbf{R}_F$  is the space of fuzzy number.

We obtain existence result by using Banach fixed point theorem and obtain its approximate solution by applying the Haar wavelet operational matrix. Also we present a new  $r$ -cut problem which involves inequalities to obtain the conditions of existence of fuzzy solutions and prove that these inequalities guarantee that the solutions of  $r$ -cut equations constitute fuzzy solutions. Our paper is organized as follows: In Section 2, we recall some definitions and basic results and prove some lemmas that will be useful to our main results. Section 3 investigated the constructive existence of solutions to our problem. In Section 4, a method to find out the solutions is given. Section 5 presented two examples to illustrate our results. In Section 6, we summarize our main results.

## 2. Preliminaries and Basic Results

*Definition 1* (see [28]). We denote the set of all fuzzy numbers on  $\mathbf{R}$  by  $\mathbf{R}_F$ . A fuzzy number is a mapping  $\mu : \mathbf{R} \rightarrow [0, 1]$  with the following properties:

- (i)  $\mu$  is normal, i.e.,  $\exists x_0 \in \mathbf{R}; \mu(x_0) = 1$
- (ii)  $\mu$  is a convex fuzzy subset, i.e.,

$$\mu(\lambda x + (1 - \lambda)y) \geq \min\{\mu(x), \mu(y)\}, \tag{4}$$

$$\forall x, y \in \mathbf{R}, \forall \lambda \in [0, 1]$$

(iii)  $\mu$  is upper semicontinuous on  $\mathbf{R}$

(iv) The set  $\overline{\text{supp}(\mu)}$  is compact in  $\mathbf{R}$  (where  $\text{supp}(\mu) := \{x \in \mathbf{R} \mid \mu(x) > 0\}$ )

Then  $\mathbf{R}_F$  is called the space of fuzzy numbers.

*Definition 2* (see [27]). Let  $u, v \in \mathbf{R}_F$ ,  $[u]^r := [u_-^{(r)}, u_+^{(r)}]$ ,  $[v]^r := [v_-^{(r)}, v_+^{(r)}]$ . The distance structure  $D : \mathbf{R}_F \times \mathbf{R}_F \rightarrow \mathbf{R}_+$  is defined as follows:

$$D(u, v) := \sup_{r \in [0, 1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\}. \tag{5}$$

Let  $x, y \in \mathbf{R}_F$ . If there exists  $z \in \mathbf{R}_F$  such that  $x = y \oplus z$ , then  $z$  is called the H-difference (Hukuhara difference) of  $x, y$  and it is denoted as  $x \ominus_H y$ .

*Definition 3* (see [29]). Let  $f : I \rightarrow \mathbf{R}_F$  and fix  $x_0 \in (a, b)$ .

(i) We say that  $f$  is (1)-differentiable at  $x_0$ , if there exists an element  $f'_G(x_0) \in \mathbf{R}_F$  such that, for all  $h > 0$  sufficiently near to 0,  $f(x_0 + h) \ominus_H f(x_0)$ ,  $f(x_0) \ominus_H f(x_0 - h)$  and the limits

$$\begin{aligned} &\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) \ominus_H f(x_0)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{f(x_0) \ominus_H f(x_0 - h)}{h} = f'_G(x_0) \end{aligned} \tag{6}$$

exist.

(ii) We say that  $f$  is (2)-differentiable at  $x_0$ , if there exists an element  $f'_G(x_0) \in \mathbf{R}_F$  such that, for all  $h > 0$  sufficiently near to 0,  $f(x_0) \ominus_H f(x_0 + h)$ ,  $f(x_0 - h) \ominus_H f(x_0)$  and the limits

$$\begin{aligned} &\lim_{h \rightarrow 0^+} \frac{f(x_0) \ominus_H f(x_0 + h)}{-h} \\ &= \lim_{h \rightarrow 0^+} \frac{f(x_0 - h) \ominus_H f(x_0)}{-h} = f'_G(x_0) \end{aligned} \tag{7}$$

exist.

If  $f$  is  $(n)$ -differentiable at  $x_0$ , we denote its first derivatives by  $D_n^{(1)} f(x_0)$  for  $n = 1, 2$ .

**Lemma 4** (see [21]). Let  $f : (a, b) \rightarrow \mathbf{R}_F$  be fuzzy valued function, where  $[f(t)]^r := [f_1(t, r), f_2(t, r)]$  for each  $r \in [0, 1]$ .

- (i) If  $f$  is (1)-differentiable, then  $f_1(t, r)$  and  $f_2(t, r)$  are differentiable functions and  $[D_1^{(1)} f(t)]^r = [f'_1(t, r), f'_2(t, r)]$ .
- (ii) If  $f$  is (2)-differentiable, then  $f_1(t, r)$  and  $f_2(t, r)$  are differentiable functions and  $[D_2^{(1)} f(t)]^r = [f'_2(t, r), f'_1(t, r)]$ .

*Definition 5* (see [21]). Let  $f : (a, b) \rightarrow \mathbf{R}_F$  and  $n, m = 1, 2$ . We say that  $f$  is  $(n, m)$ -differentiable at  $x_0 \in (a, b)$ , if  $D_n^{(1)} f(x_0)$  exists, and is  $(m)$ -differentiable at  $x_0$ . The second derivative of  $f$  is denoted by  $D_{n,m}^{(2)} f(x_0)$  for  $n, m = 1, 2$ .

**Lemma 6** (see [30]). Let  $f : (a, b) \rightarrow \mathbf{R}_F$  be fuzzy valued function and denote its  $r$ -level sets by  $[f(t)]^r := [f_1(t, r), f_2(t, r)]$  for each  $r \in [0, 1]$ .

(i) If  $D_1^{(1)} f(t)$  is (1)-differentiable, then  $f_1'(t, r), f_2'(t, r)$  are differentiable and

$$\left[ D_{1,1}^{(2)} f(t) \right]^r = \left[ f_1''(t, r), f_2''(t, r) \right]. \tag{8}$$

(ii) If  $D_1^{(1)} f(t)$  is (2)-differentiable, then  $f_1'(t, r), f_2'(t, r)$  are differentiable and

$$\left[ D_{1,2}^{(2)} f(t) \right]^r = \left[ f_2''(t, r), f_1''(t, r) \right]. \tag{9}$$

**Definition 7.** The Riemann-Liouville fractional integral operator of order  $\alpha > 0$  of a real function  $\varphi : I \rightarrow \mathbf{R}$  is defined as

$$I_{a+}^\alpha \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \varphi(s) ds, \tag{10}$$

where  $\Gamma(\cdot)$  is the Euler gamma function.

**Definition 8.** Let  $\varphi : I \rightarrow \mathbf{R}$ , the Caputo fractional derivative of order  $\alpha > 0, m-1 < \alpha < m, m \in \mathbf{N} = \{1, 2, \dots\}$ , be defined as

$$\left( {}^c D_{a+}^\alpha \varphi \right)(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} \varphi^{(m)}(s) ds. \tag{11}$$

**Definition 9.** The fuzzy fractional Caputo differentiability of fuzzy valued function is defined as follows:

$$\left( {}^c D_{1,1}^\alpha f \right)(t) = \frac{1}{\Gamma(2-\alpha)} \int_a^t (t-s)^{1-\alpha} \left( D_{1,1}^{(2)} f \right)(s) ds, \tag{12}$$

where  $1 < \alpha \leq 2, t > 0, f : I \rightarrow \mathbf{R}_F$ . Then we say that  $f$  is  ${}^c[(1, 1) - \alpha]$ -differentiable.

**Lemma 10** (see [23, 24, 31]). Let  $f : I \rightarrow \mathbf{R}_F, \alpha \in (1, 2), [f(t)]^r := [f_1(t, r), f_2(t, r)]$ . If  $f$  is  ${}^c[(1, 1) - \alpha]$ -differentiable, it holds that

$$\left[ \left( {}^c D_{1,1}^\alpha f \right)(t) \right]^r = \left[ {}^c D_{0+}^\alpha f_1(t, r), {}^c D_{0+}^\alpha f_2(t, r) \right]. \tag{13}$$

**Lemma 11** (see [27]). Let  $\{U_r \mid r \in (0, 1]\}$  be a family of real intervals such that the following three conditions are satisfied:

- (i)  $U_r$  is a nonempty compact interval for all  $r \in [0, 1]$
- (ii) if  $0 < r_1 < r_2 \leq 1$  then  $U_{r_2} \subseteq U_{r_1}$
- (iii) given any nondecreasing sequence  $r_n \in (0, 1]$  with  $\lim_{n \rightarrow \infty} r_n = r$  it is  $U_r = \bigcap_{n=1}^\infty U_{r_n}$

Then there exists a unique fuzzy number  $u \in \mathbf{R}_F$  such that  $[u]^r = U_r$  for all  $r \in I$  and  $[u]^0 = cl(\bigcup_{r \in (0,1]} U_r)$ .

Let us consider the following fractional integral equation:

$$U(t) + b(t) I_{0+}^{\lambda_1} U(t) + c(t) I_{0+}^{\lambda_2} U(t) = V(t), \quad t \in I, \tag{14}$$

where  $V, b, c \in C(I), \lambda_1, \lambda_2 > 0$ .

**Lemma 12.** There exists a unique solution of (14) in  $C(I)$  and a positive real number  $k_*$  for the solution to satisfy the following inequality:

$$\|U\|_{C(I)} \leq \frac{e^{k_*}}{1-q} \|V\|_{C(I)}, \tag{15}$$

where  $k_*$  is any positive number that satisfies  $q := (\|b\|_{C(I)} / k_*^{\lambda_1} + \|c\|_{C(I)} / k_*^{\lambda_2}) < 1$ .

*Proof.* Let  $F(U)(t) := V(t) - b(t)I_{0+}^{\lambda_1}U(t) - c(t)I_{0+}^{\lambda_2}U(t)$ . Then equation (14) is as follows:

$$U(t) = F(U)(t). \tag{16}$$

Now, we use the following  $k$ -norm equivalent to the norm  $\|x\|_{C(I)} := \max\{|x(t)|, t \in I\}$ .

That is

$$\|x\|_k := \max\{e^{-kt} |x(t)|\}, \quad x \in C(I). \tag{17}$$

For all  $X, Y \in C(I)$ , we have the following inequalities:

$$\begin{aligned} & |F(X)(t) - F(Y)(t)| \\ & \leq \|b\|_{C(I)} \left| I_{0+}^{\lambda_1} X(t) - I_{0+}^{\lambda_1} Y(t) \right| \\ & \quad + \|c\|_{C(I)} \left| I_{0+}^{\lambda_2} X(t) - I_{0+}^{\lambda_2} Y(t) \right| \\ & \leq \left( \frac{\|b\|_{C(I)}}{k^{\lambda_1}} + \frac{\|c\|_{C(I)}}{k^{\lambda_2}} \right) e^{kt} \|X - Y\|_k. \end{aligned} \tag{18}$$

Therefore, we get

$$\begin{aligned} & e^{-kt} |F(X)(t) - F(Y)(t)| \\ & \leq \left( \frac{\|b\|_{C(I)}}{k^{\lambda_1}} + \frac{\|c\|_{C(I)}}{k^{\lambda_2}} \right) \|X - Y\|_k. \end{aligned} \tag{19}$$

Thus, we obtain

$$\begin{aligned} & \|F(X)(t) - F(Y)(t)\|_k \\ & \leq \left( \frac{\|b\|_{C(I)}}{k^{\lambda_1}} + \frac{\|c\|_{C(I)}}{k^{\lambda_2}} \right) \|X - Y\|_k. \end{aligned} \tag{20}$$

Since the number  $k_*$  satisfies

$$q := \frac{\|b\|_{C(I)}}{k_*^{\lambda_1}} + \frac{\|c\|_{C(I)}}{k_*^{\lambda_2}} < 1, \tag{21}$$

we obtain the following inequality:

$$\|F(X) - F(Y)\|_{k_*} \leq q \|X - Y\|_{k_*}. \tag{22}$$

Since  $C(I)$  is complete, we have

$$\exists \bar{Y} \in C(I); \quad \bar{Y} = F(\bar{Y}). \tag{23}$$

That is, (14) has unique solution  $U(t) \in C(I)$ . Then the following equation holds:

$$U(t) + b(t) I_{0+}^{\lambda_1} U(t) + c(t) I_{0+}^{\lambda_2} U(t) = V(t), \quad t \in I, \tag{24}$$

and we obtain the following inequality:

$$\begin{aligned}
 |U(t)| &\leq \|b\| I_{0+}^{\lambda_1} |U(t)| + \|c\| I_{0+}^{\lambda_2} |U(t)| + |V(t)| \\
 &\leq \|b\| \frac{e^{kt}}{k^{\lambda_1}} \|U\|_k + \|c\| \frac{e^{kt}}{k^{\lambda_2}} \|U\|_k + |V(t)|,
 \end{aligned} \tag{25}$$

so we have

$$\begin{aligned}
 e^{-kt} |U(t)| &\leq \|b\| \frac{1}{k^{\lambda_1}} \|U\|_k + \|c\| \frac{1}{k^{\lambda_2}} \|U\|_k \\
 &\quad + e^{-kt} |V(t)|.
 \end{aligned} \tag{26}$$

Therefore, we get

$$\begin{aligned}
 \|U\|_k &\leq \|b\| \frac{1}{k^{\lambda_1}} \|U\|_k + \|c\| \frac{1}{k^{\lambda_2}} \|U\|_k + \|V\|_k \\
 &= \left( \|b\| \frac{1}{k^{\lambda_1}} + \|c\| \frac{1}{k^{\lambda_2}} \right) \|U\|_k + \|V\|_k.
 \end{aligned} \tag{27}$$

So if there is a positive number  $k_*$  satisfying

$$q := \frac{\|b\|_{C(I)}}{k_*^{\lambda_1}} + \frac{\|c\|_{C(I)}}{k_*^{\lambda_2}} < 1, \tag{28}$$

then we get

$$\|U\|_{k_*} \leq \frac{1}{1-q} \|V\|_{k_*}, \tag{29}$$

and so we obtain

$$e^{-k_*} \|U\|_{C(I)} \leq \|U\|_{k_*} \leq \frac{1}{1-q} \|V\|_{k_*} \leq \frac{1}{1-q} \|V\|_{C(I)}. \tag{30}$$

Thus we have

$$\|U\|_{C(I)} \leq \frac{e^{k_*}}{1-q} \|V\|_{C(I)}. \tag{31}$$

□

Consider the integral equation

$$U(t) + bI_{0+}^{\alpha-\beta}U(t) + cI_{0+}^{\alpha}U(t) = h(t), \quad t \in [0, T], \tag{32}$$

where  $b > 0, c \geq 0, 1 < \beta < \alpha \leq 2, 0 < \alpha - \beta < 1$ .

We define an operator  $A$  by  $A(U)(t) := bI_{0+}^{\alpha-\beta}U(t) + cI_{0+}^{\alpha}U(t)$ .

**Lemma 13.** *If  $(I - A)h(t) \geq 0$ , then the integral equation (32) has one nonnegative solution, where  $I$  is the identity operator.*

*Proof.* By Lemma 12, (32) has a unique solution. If there is a positive number  $k_*$  satisfying

$$q := \frac{b}{k_*^{\alpha-\beta}} + \frac{c}{k_*^{\alpha}} < 1, \tag{33}$$

then  $\|A\|_{k_*} < 1$ .

Therefore we can know that

$$(I + A)^{-1} = I - A + A^2 - \dots + (-1)^n A^n + \dots, \tag{34}$$

so we have

$$U(t) = (I + A)^{-1} h(t). \tag{35}$$

Since  $(I - A)h(t) \geq 0$ , we obtain

$$\begin{aligned}
 U(t) &= (I - A)h(t) + A^2(I - A)h(t) \\
 &\quad + A^4(I - A)h(t) \dots \geq 0.
 \end{aligned} \tag{36}$$

□

**Lemma 14.**  *$(I - A)h(t) \leq 0$ , and then the integral equation*

$$U(t) + bI_{0+}^{\alpha-\beta}U(t) + cI_{0+}^{\alpha}U(t) = h(t), \quad t \in I \tag{37}$$

*has one nonpositive solution.*

### 3. Constructive Existence of (1,1)-Solution of Two-Point Value Problem for the Fuzzy Linear Multiterm Fractional Differential Equation

Let us consider the following fuzzy boundary value problem:

$$\left( {}^c D_{1,1}^{\alpha} y \right) (t) \oplus b(t) \otimes \left( {}^c D_{1,1}^{\beta} y \right) (t) \oplus c(t) \otimes y(t) \tag{38}$$

$$= f(t), \quad t \in (0, 1),$$

$$y(0) = y_0, \tag{39}$$

$$y(1) = y_1,$$

where  $b, c \in C(I), b(t), c(t) \geq 0, y, f \in C(I, \mathbf{R}_F), I = [0, 1], y_0, y_1 \in \mathbf{R}_F, 1 < \beta < \alpha \leq 2$ .

*Definition 15.* A fuzzy valued function  $y(t)$  is called a (1,1)-solution of the problem (38), (39) if it satisfies (38), (39) and  ${}^c D_{1,1}^{\alpha} y \in C(I, \mathbf{R}_F)$ .

By using r-cuts, we can obtain

$$\left[ \left( {}^c D_{1,1}^{\alpha} y \right) (t) \oplus b(t) \otimes \left( {}^c D_{1,1}^{\beta} y \right) (t) \oplus c(t) \otimes y(t) \right]^r \tag{40}$$

$$= [f(t)]^r, \quad r \in [0, 1]$$

and

$$\left[ \left( {}^c D_{1,1}^{\alpha} y \right) (t) \right]^r + b(t) \left[ \left( {}^c D_{1,1}^{\beta} y \right) (t) \right]^r$$

$$+ c(t) [y(t)]^r = [f(t)]^r.$$

$$\left[ {}^c D_{0+}^{\alpha} y_1(t, r), {}^c D_{0+}^{\alpha} y_2(t, r) \right] \tag{41}$$

$$+ b(t) \left[ {}^c D_{0+}^{\beta} y_1(t, r), {}^c D_{0+}^{\beta} y_2(t, r) \right]$$

$$+ c(t) [y_1(t, r), y_2(t, r)] = [f_1(t, r), f_2(t, r)].$$

By the operation of intervals, we can have

$$\begin{aligned} & {}^c D_{0+}^\alpha y_1(t, r) + b(t) {}^c D_{0+}^\beta y_1(t, r) + c y_1(t, r) \\ &= f_1(t, r), \\ & {}^c D_{0+}^\alpha y_2(t, r) + b(t) {}^c D_{0+}^\beta y_2(t, r) + c y_2(t, r) \\ &= f_2(t, r). \end{aligned} \tag{42}$$

In particular, the inequalities

$$\begin{aligned} & y_1(t, r) \leq y_2(t, r), \\ & {}^c D_{0+}^\alpha y_1(t, r) \leq {}^c D_{0+}^\alpha y_2(t, r), \\ & {}^c D_{0+}^\beta y_1(t, r) \leq {}^c D_{0+}^\beta y_2(t, r) \end{aligned} \tag{43}$$

imply that  $y_1(t, r)$  and  $y_2(t, r)$  constitute interval and it is the same for  ${}^c D_{0+}^\beta y_1(t, r)$ ,  ${}^c D_{0+}^\beta y_2(t, r)$  and  ${}^c D_{0+}^\alpha y_1(t, r)$ ,  ${}^c D_{0+}^\alpha y_2(t, r)$ .

Let us denote the  $r$ -cut representation of  $y_0, y_1$  by  $[y_0]^r := [y_{0,1}(r), y_{0,2}(r)]$ ,  $[y_1]^r := [y_{1,1}(r), y_{1,2}(r)]$ . Then the cut forms of boundary conditions (39) are

$$\begin{aligned} & y_1(0, r) = y_{0,1}(r), \\ & y_1(1, r) = y_{1,1}(r), \\ & y_2(0, r) = y_{0,2}(r), \\ & y_2(1, r) = y_{1,2}(r). \end{aligned} \tag{44}$$

Now the expression (42)-(44) is called cut problem of (38)-(39).

**Definition 16.**  $(y_1(t, r), y_2(t, r))$  is called a solution of (42)-(44) if it satisfies (42)-(44) and  ${}^c D_{0+}^\alpha y_1(\cdot, r), {}^c D_{0+}^\alpha y_2(\cdot, r) \in C(I)$ .

**Theorem 17.** (i) If  $(y_1(t, r), y_2(t, r))$  is a solution of the problem (42)-(44), then  $(u_1(t, r), u_2(t, r))$  satisfies

$$\begin{aligned} & u_1(t, r) + b(t) I_{0+}^{\alpha-\beta} u_1(t, r) \\ &+ c(t) ((1-t) y_{0,1}(r) + t y_{1,1}(r)) \\ &- c(t) \int_0^1 G(t, s) u_1(s, r) ds = f_1(t, r), \\ & u_2(t, r) + b(t) I_{0+}^{\alpha-\beta} u_2(t, r) \\ &+ c(t) ((1-t) y_{0,2}(r) + t y_{1,2}(r)) \\ &- c(t) \int_0^1 G(t, s) u_2(s, r) ds = f_2(t, r), \\ & u_1(t, r) \leq u_2(t, r), \\ & (1-t)(y_{0,2}(r) - y_{0,1}(r)) + t(y_{1,2}(r) - y_{1,1}(r)) \\ &+ I_{0+}^\alpha (u_2(t, r) - u_1(t, r)) \\ &- I_{0+}^\alpha (u_2(t, r) - u_1(t, r))|_{t=1} \cdot t \geq 0, \end{aligned} \tag{45}$$

where  ${}^c D_{0+}^\alpha y_1(t, r) = u_1(t, r)$ ,  ${}^c D_{0+}^\alpha y_2(t, r) = u_2(t, r)$ .

(ii) If  $(u_1(t, r), u_2(t, r))$  satisfies (45), then  $(y_1(t, r), y_2(t, r))$  is a solution of the problem (42)-(44), where

$$\begin{aligned} & y_1(t, r) \\ &= (1-t) y_{0,1}(r) + t y_{1,1}(r) \\ &- \int_0^1 G(t, s) u_1(s, r) ds, \end{aligned} \tag{46}$$

$$\begin{aligned} & y_2(t, r) \\ &= (1-t) y_{0,2}(r) + t y_{1,2}(r) \\ &- \int_0^1 G(t, s) u_2(s, r) ds, \end{aligned}$$

$$\begin{aligned} & G(t, s) \\ &:= \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{t}{(1-s)^{1-\alpha}}, & s \geq t, \\ \left( \frac{t}{(1-s)^{1-\alpha}} - \frac{1}{(t-s)^{1-\alpha}} \right), & s < t. \end{cases} \end{aligned} \tag{47}$$

*Proof.* Suppose  $(y_1(t, r), y_2(t, r))$  is a solution of the problem (42)-(44).

Then, we can get

$$\begin{aligned} & I_{0+}^\alpha {}^c D_{0+}^\alpha y_1(t, r) = I_{0+}^\alpha u_1(t, r), \\ & I_{0+}^\alpha {}^c D_{0+}^\alpha y_2(t, r) = I_{0+}^\alpha u_2(t, r). \end{aligned} \tag{48}$$

From the boundary condition (44), we have

$$\begin{aligned} & y_1(t, r) = y_{0,1}(r) + c_1 \cdot t + I_{0+}^\alpha u_1(t, r), \\ & y_2(t, r) = y_{0,2}(r) + d_1 \cdot t + I_{0+}^\alpha u_2(t, r), \\ & y_1(1, r) = y_{0,1}(r) + c_1 + I_{0+}^\alpha u_1(t, r)|_{t=1}, \\ & y_2(1, r) = y_{0,2}(r) + d_1 + I_{0+}^\alpha u_2(t, r)|_{t=1}, \\ & y_{1,1}(r) = y_{0,1}(r) + c_1 + I_{0+}^\alpha u_1(t, r)|_{t=1}, \\ & y_{1,2}(r) = y_{0,2}(r) + d_1 + I_{0+}^\alpha u_2(t, r)|_{t=1}, \\ & y_{1,1}(r) - y_{0,1}(r) - I_{0+}^\alpha u_1(t, r)|_{t=1} = c_1, \\ & y_{1,2}(r) - y_{0,2}(r) - I_{0+}^\alpha u_2(t, r)|_{t=1} = d_1. \end{aligned} \tag{49}$$

Thus, we obtain

$$\begin{aligned} & y_1(t, r) = y_{0,1}(r) \\ &+ (y_{1,1}(r) - y_{0,1}(r) - I_{0+}^\alpha u_1(t, r)|_{t=1}) \\ &\cdot t + I_{0+}^\alpha u_1(t, r), \\ & y_2(t, r) = y_{0,2}(r) \\ &+ (y_{1,2}(r) - y_{0,2}(r) - I_{0+}^\alpha u_2(t, r)|_{t=1}) \\ &\cdot t + I_{0+}^\alpha u_2(t, r), \end{aligned} \tag{50}$$

and so we have

$$\begin{aligned} {}^c D_{0+}^\beta y_1(t, r) &= I_{0+}^{\alpha-\beta} u_1(t, r), \\ {}^c D_{0+}^\beta y_2(t, r) &= I_{0+}^{\alpha-\beta} u_2(t, r). \end{aligned} \quad (51)$$

Using Green's function, (50) can be expressed as follows:

$$\begin{aligned} y_1(t, r) &= (1-t) y_{0,1}(r) + t y_{1,1}(r) \\ &\quad - \int_0^1 G(t, s) u_1(s, r) ds, \\ y_2(t, r) &= (1-t) y_{0,2}(r) + t y_{1,2}(r) \\ &\quad - \int_0^1 G(t, s) u_2(s, r) ds. \end{aligned} \quad (52)$$

From (43), (51), and (52), we can obtain (45).

On the other hand, since  ${}^c D_{0+}^\alpha y_1(t, r) \leq {}^c D_{0+}^\alpha y_2(t, r)$ , we have

$$u_1(t, r) \leq u_2(t, r). \quad (53)$$

At last, from the inequality  $y_1(t, r) \leq y_2(t, r)$ , we can have

$$\begin{aligned} (1-t)(y_{0,2}(r) - y_{0,1}(r)) + t(y_{1,2}(r) - y_{1,1}(r)) \\ + I_{0+}^\alpha (u_2(t, r) - u_1(t, r)) \\ - I_{0+}^\alpha (u_2(t, r) - u_1(t, r)) \Big|_{t=1} \cdot t \geq 0. \end{aligned} \quad (54)$$

Conversely, let us suppose that the pair  $(u_1(t, r), u_1(t, r))$  satisfies (45). From (46), we can get

$$\begin{aligned} {}^c D_{0+}^\alpha y_1(t, r) &= u_1(t, r), \\ {}^c D_{0+}^\alpha y_2(t, r) &= u_2(t, r). \end{aligned} \quad (55)$$

Therefore, we have  ${}^c D_{0+}^\alpha y_1(t, r), {}^c D_{0+}^\alpha y_2(t, r) \in C(I)$  and so we can get

$$\begin{aligned} {}^c D_{0+}^\beta y_1(t, r) &= I_{0+}^{\alpha-\beta} u_1(t, r), \\ {}^c D_{0+}^\beta y_2(t, r) &= I_{0+}^{\alpha-\beta} u_2(t, r), \end{aligned} \quad (56)$$

and

$$\begin{aligned} {}^c D_{0+}^\alpha y_1(t, r) + b(t) {}^c D_{0+}^\beta y_1(t, r) + c(t) y_1(t, r) \\ = f_1(t, r), \\ {}^c D_{0+}^\alpha y_2(t, r) + b(t) {}^c D_{0+}^\beta y_2(t, r) + c(t) y_2(t, r) \\ = f_2(t, r). \end{aligned} \quad (57)$$

We can easily obtain the other inequalities and prove the boundary condition (44).  $\square$

By this theorem, the existence of solutions of (42)-(44) is equivalent to the existence of solutions of (45).

Now, let us consider the integral equation

$$\begin{aligned} u_1(t, r) + b(t) I_{0+}^{\alpha-\beta} u_1(t, r) \\ + c(t) ((1-t) y_{0,1}(r) + t y_{1,1}(r)) + c(t) I_{0+}^\alpha u_1(t, r) \\ - t c(t) I_{0+}^\alpha u_1(t, r) \Big|_{t=1} = f_1(t, r), \\ u_2(t, r) + b(t) I_{0+}^{\alpha-\beta} u_2(t, r) \\ + c(t) ((1-t) y_{0,2}(r) + t y_{1,2}(r)) + c(t) I_{0+}^\alpha u_2(t, r) \\ - t c(t) I_{0+}^\alpha u_2(t, r) \Big|_{t=1} = f_2(t, r). \end{aligned} \quad (58)$$

*Assumption 1.*  $q := (\|b\|/\Gamma(\alpha - \beta + 1) + 2\|c\|/\Gamma(\alpha + 1)) < 1$ .

Denote the following:

$$\begin{aligned} g_1(t, r) &:= f_1(t, r) - c(t) ((1-t) y_{0,1}(r) + t y_{1,1}(r)), \\ g_2(t, r) &:= f_2(t, r) - c(t) ((1-t) y_{0,2}(r) + t y_{1,2}(r)), \\ u_1^0(t, r) &= 0, \\ u_2^0(t, r) &= 0. \end{aligned} \quad (59)$$

We are going to consider a the scheme of successive approximation

$$\begin{aligned} u_1^{n+1}(t, r) + b(t) I_{0+}^{\alpha-\beta} u_1^{n+1}(t, r) + c(t) I_{0+}^\alpha u_1^{n+1}(t, r) \\ = g_1(t, r) + t c(t) I_{0+}^\alpha u_1^n(t, r) \Big|_{t=1}, \\ u_2^{n+1}(t, r) + b(t) I_{0+}^{\alpha-\beta} u_2^{n+1}(t, r) + c(t) I_{0+}^\alpha u_2^{n+1}(t, r) \\ = g_2(t, r) + t c(t) I_{0+}^\alpha u_2^n(t, r) \Big|_{t=1}. \end{aligned} \quad (60)$$

**Lemma 18.** *The sequence  $(u_1^n(t, r), u_2^n(t, r))$  that satisfies (60) is a Cauchy's sequence in  $C(I) \times C(I)$ .*

*Proof.* We can have the following equations from  $n^{\text{th}}, (n+1)^{\text{th}}$  terms of the scheme of successive approximation

$$\begin{aligned} (u_1^{n+1}(t, r) - u_1^n(t, r)) \\ + b(t) I_{0+}^{\alpha-\beta} (u_1^{n+1}(t, r) - u_1^n(t, r)) \\ + c(t) I_{0+}^\alpha (u_1^{n+1}(t, r) - u_1^n(t, r)) \\ = t c(t) I_{0+}^\alpha (u_1^n(t, r) - u_1^{n-1}(t, r)) \Big|_{t=1}, \\ (u_2^{n+1}(t, r) - u_2^n(t, r)) \\ + b(t) I_{0+}^{\alpha-\beta} (u_2^{n+1}(t, r) - u_2^n(t, r)) \\ + c(t) I_{0+}^\alpha (u_2^{n+1}(t, r) - u_2^n(t, r)) \\ = t c(t) I_{0+}^\alpha (u_2^n(t, r) - u_2^{n-1}(t, r)) \Big|_{t=1}. \end{aligned} \quad (61)$$

For the first equation, we can get

$$\begin{aligned}
 & \left| u_1^{n+1}(t, r) - u_1^n(t, r) \right| \\
 & \leq \|b\| I_{0+}^{\alpha-\beta} \left| u_1^{n+1}(t, r) - u_1^n(t, r) \right| \\
 & \quad + \|c\| I_{0+}^\alpha \left| u_1^{n+1}(t, r) - u_1^n(t, r) \right| \\
 & \quad + \|tc\| I_{0+}^\alpha \left| u_1^n(t, r) - u_1^{n-1}(t, r) \right|_{t=1} \\
 & \leq \left( \frac{\|b\|}{\Gamma(\alpha - \beta + 1)} + \frac{\|c\|}{\Gamma(\alpha + 1)} \right) \\
 & \quad \cdot \left\| u_1^{n+1}(\cdot, r) - u_1^n(\cdot, r) \right\| + \frac{\|c\|}{\Gamma(\alpha + 1)} \\
 & \quad \cdot \left\| u_1^n(\cdot, r) - u_1^{n-1}(\cdot, r) \right\|
 \end{aligned} \tag{62}$$

and so we obtain

$$\begin{aligned}
 \left\| u_1^{n+1}(\cdot, r) - u_1^n(\cdot, r) \right\| & \leq \left( \frac{\|b\|}{\Gamma(\alpha - \beta + 1)} + \frac{\|c\|}{\Gamma(\alpha + 1)} \right) \\
 & \quad \cdot \left\| u_1^{n+1}(\cdot, r) - u_1^n(\cdot, r) \right\| \\
 & \quad + \frac{\|c\|}{\Gamma(\alpha + 1)} \\
 & \quad \cdot \left\| u_1^n(\cdot, r) - u_1^{n-1}(\cdot, r) \right\|.
 \end{aligned} \tag{63}$$

If

$$\gamma := \frac{1}{1 - q + \|c\|/\Gamma(\alpha + 1)} \frac{\|c\|}{\Gamma(\alpha + 1)} < 1, \tag{64}$$

we can get

$$\begin{aligned}
 & \left\| u_1^{n+1}(\cdot, r) - u_1^n(\cdot, r) \right\| \\
 & \leq \frac{1}{1 - (\|b\|/\Gamma(\alpha - \beta + 1) + \|c\|/\Gamma(\alpha + 1))} \frac{\|c\|}{\Gamma(\alpha + 1)} \\
 & \quad \cdot \left\| u_1^n(\cdot, r) - u_1^{n-1}(\cdot, r) \right\| \\
 & = \frac{1}{1 - q + \|c\|/\Gamma(\alpha + 1)} \frac{\|c\|}{\Gamma(\alpha + 1)} \\
 & \quad \cdot \left\| u_1^n(\cdot, r) - u_1^{n-1}(\cdot, r) \right\| \\
 & \leq \gamma \left\| u_1^n(\cdot, r) - u_1^{n-1}(\cdot, r) \right\|.
 \end{aligned} \tag{65}$$

As the same way, we can prove for  $u_2(t, r)$ . □

Since the space  $C(I)$  is complete, we can say that

$$\begin{aligned}
 & \exists \bar{u}_1(\cdot, r) \in C(I); \\
 & u_1^{(n)}(\cdot, r) \longrightarrow \bar{u}_1(\cdot, r), \\
 & \exists \bar{u}_2(\cdot, r) \in C(I); \\
 & u_2^{(n)}(\cdot, r) \longrightarrow \bar{u}_2(\cdot, r).
 \end{aligned} \tag{66}$$

Thus, we have

$$\begin{aligned}
 & \bar{u}_1(t, r) + b(t) I_{0+}^{\alpha-\beta} \bar{u}_1(t, r) + c(t) I_{0+}^\alpha \bar{u}_1(t, r) \\
 & = g_1(t, r) + tc(t) I_{0+}^\alpha \bar{u}_1(t, r)|_{t=1}, \\
 & \bar{u}_2(t, r) + b(t) I_{0+}^{\alpha-\beta} \bar{u}_2(t, r) + c(t) I_{0+}^\alpha \bar{u}_2(t, r) \\
 & = g_2(t, r) + tc(t) I_{0+}^\alpha \bar{u}_2(t, r)|_{t=1}.
 \end{aligned} \tag{67}$$

Let us denote the following:

$$\begin{aligned}
 \text{len}(g)(t, r) & := g_2(t, r) - g_1(t, r), \\
 \text{len}(u^n)(t, r) & := u_2^n(t, r) - u_1^n(t, r), \\
 g_1(t, r) & := f_1(t, r) \\
 & \quad - c(t) ((1-t) y_{0,1}(r) + t y_{1,1}(r)), \\
 g_2(t, r) & := f_2(t, r) \\
 & \quad - c(t) ((1-t) y_{0,2}(r) + t y_{1,2}(r)), \\
 \text{len}(g)(t, r) & = \text{len}(f)(t, r) \\
 & \quad - c(t) ((1-t) \text{len}(y_0) + t \text{len}(y_1)).
 \end{aligned} \tag{68}$$

*Assumption 2.*  $(I - A)(\text{len}(g)) \geq 0$ .

*Assumption 3.*  $(I - A)(tc(t)) \geq 0$ .

**Theorem 19.** *If  $\bar{u}_1(t, r), \bar{u}_2(t, r)$  satisfy (67), then they satisfy*

$$\bar{u}_1(t, r) \leq \bar{u}_2(t, r), \quad t \in I. \tag{69}$$

*Proof.* Let us consider a the scheme of successive approximation

$$\begin{aligned}
 \text{len}(u^0)(t, r) & = 0, \\
 \text{len}(u^{n+1})(t, r) + b(t) I_{0+}^{\alpha-\beta} \text{len}(u^{n+1})(t, r) \\
 & + c(t) I_{0+}^\alpha \text{len}(u^{n+1})(t, r) = \text{len}(g)(t, r) \\
 & + tc(t) I_{0+}^\alpha \text{len}(u^n)(t, r)|_{t=1}.
 \end{aligned} \tag{70}$$

When  $n = 0$ , we can have the inequality  $\text{len}(u^1)(t, r) \geq 0$  from  $\text{len}(u^0)(t, r) = 0$ , Assumption 2, and Lemma 13.

For any  $n \geq 1$ , let us suppose that  $\text{len}(u^n)(t, r) \geq 0$ .

Then, we can obtain

$$\begin{aligned}
 & (I - A) (\text{len}(g)(t, r) + tc(t) I_{0+}^\alpha \text{len}(u^n)(t, r)|_{t=1}) \\
 & = (I - A) (\text{len}(g))(t, r) \\
 & \quad + I_{0+}^\alpha \text{len}(u^n)(t, r)|_{t=1} (I - A) (tc(t)) \geq 0
 \end{aligned} \tag{71}$$

and so we have

$$\text{len}(u^{n+1})(t, r) \geq 0. \tag{72}$$

By the limit of inequality (72), the proof is completed. □

Let us use the following notations:

$$\begin{aligned} \text{len}(y_0)(t, r) &:= y_{0,2}(r) - y_{0,1}(r), \\ \text{len}(y_1)(t, r) &:= y_{1,2}(r) - y_{1,1}(r), \\ \text{len}(\bar{u})(t, r) &:= \bar{u}_2(t, r) - \bar{u}_1(t, r). \end{aligned} \tag{73}$$

*Assumption 4.*  $\text{len}(f)(t, r) - (\Gamma(\alpha + 2)(b(t) + \Gamma(\alpha - \beta + 1)) / (\Gamma(\alpha + 2)\Gamma(\alpha - \beta + 1) - \|c\|\Gamma(\alpha - \beta + 1) - \|b\|\Gamma(\alpha + 2))) \|\text{len}(g)\| \geq 0$ .

**Theorem 20.**  $(1 - t) \cdot \text{len}(y_0)(t, r) + t \cdot \text{len}(y_1)(t, r) + I_{0+}^\alpha \text{len}(\bar{u})(t, r) - I_{0+}^\alpha \text{len}(\bar{u})(t, r)|_{t=1} \cdot t \geq 0$ .

*Proof.* From equations (67), we can get

$$\begin{aligned} \text{len}(\bar{u}) + b(t) I_{0+}^{\alpha-\beta} \text{len}(\bar{u}) + c(t) I_{0+}^\alpha \text{len}(\bar{u}) \\ = \text{len}(g) + tc(t) I_{0+}^\alpha \text{len}(\bar{u})|_{t=1}, \end{aligned} \tag{74}$$

$$\begin{aligned} \text{len}(g)(t, r) \\ = \text{len}(f)(t, r) \\ - c(t) ((1 - t) \text{len}(y_0) + t \text{len}(y_1)). \end{aligned} \tag{75}$$

Since  $\text{len}(\bar{u})$  is the solution of the equation

$$\begin{aligned} \text{len}(\bar{u}) = \text{len}(g) + tc(t) I_{0+}^\alpha \text{len}(\bar{u})|_{t=1} \\ - b(t) I_{0+}^{\alpha-\beta} \text{len}(\bar{u}) - c(t) I_{0+}^\alpha \text{len}(\bar{u}) \end{aligned} \tag{76}$$

and the inequality

$$G(t, s) \leq \frac{s(1 - s)^{\alpha-1}}{\Gamma(\alpha)}, \tag{77}$$

we have

$$\begin{aligned} \|\text{len}(\bar{u})\| &\leq \|\text{len}(g)\| + \|c\| \\ &\cdot \|t I_{0+}^\alpha \text{len}(\bar{u})|_{t=1} - I_{0+}^\alpha \text{len}(\bar{u})\| + \|b\| \\ &\cdot \|I_{0+}^{\alpha-\beta} \text{len}(\bar{u})\| \\ &\leq \|\text{len}(g)\| + \|c\| \cdot \int_0^1 \frac{s(1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &\cdot \|\text{len}(\bar{u})\| + \frac{\|b\|}{\Gamma(\alpha - \beta + 1)} \cdot \|\text{len}(\bar{u})\|. \end{aligned} \tag{78}$$

We can get

$$\int_0^1 s(1 - s)^{\alpha-1} ds = \frac{1}{\alpha + \alpha^2} \tag{79}$$

and so we obtain

$$\begin{aligned} \|\text{len}(\bar{u})\| &\leq \|\text{len}(g)\| + \frac{\|c\|}{\Gamma(\alpha + 2)} \cdot \|\text{len}(\bar{u})\| \\ &+ \frac{\|b\|}{\Gamma(\alpha - \beta + 1)} \cdot \|\text{len}(\bar{u})\|. \end{aligned} \tag{80}$$

By Assumption 1, we can have the inequalities

$$Q := \frac{\|c\|}{\Gamma(\alpha + 2)} + \frac{\|b\|}{\Gamma(\alpha - \beta + 1)} \leq q < 1, \tag{81}$$

$$\|\text{len}(\bar{u})\| \leq \frac{1}{(1 - Q)} \|\text{len}(g)\|.$$

We can get

$$\begin{aligned} c(t) I_{0+}^\alpha \text{len}(\bar{u}) - tc(t) I_{0+}^\alpha \text{len}(\bar{u})|_{t=1} \\ = \text{len}(g) - \text{len}(\bar{u}) - b(t) I_{0+}^{\alpha-\beta} \text{len}(\bar{u}) \end{aligned} \tag{82}$$

and since  $c(t) \geq 0$ , we have

$$\begin{aligned} c(t) ((1 - t) \cdot \text{len}(y_0)(t, r) + t \cdot \text{len}(y_1)(t, r) \\ + I_{0+}^\alpha \text{len}(\bar{u})(t, r) - I_{0+}^\alpha \text{len}(\bar{u})(t, r)|_{t=1} \cdot t) \geq 0 \end{aligned} \tag{83}$$

and

$$\begin{aligned} c(t) ((1 - t) - \text{len}(y_0)(t, r) + t \cdot \text{len}(y_1)(t, r) \\ + \text{len}(g) - \text{len}(\bar{u}) - b(t) I_{0+}^{\alpha-\beta} \text{len}(\bar{u})) \geq 0. \end{aligned} \tag{84}$$

Since  $c(t)((1 - t) \cdot \text{len}(y_0)(t, r) + t \cdot \text{len}(y_1)(t, r)) + \text{len}(g) = \text{len}(f)$ , we have

$$\text{len}(f) \geq \text{len}(\bar{u}) + b(t) I_{0+}^{\alpha-\beta} \text{len}(\bar{u}). \tag{85}$$

By equation (81), we have a condition

$$\text{len}(f) \geq \frac{\|\text{len}(g)\|}{1 - Q} \left( 1 + \frac{b(t)}{\Gamma(\alpha - \beta + 1)} \right), \tag{86}$$

that is,

$$\begin{aligned} \text{len}(f)(t, r) \\ - \frac{\Gamma(\alpha + 2)(b(t) + \Gamma(\alpha - \beta + 1))}{\Gamma(\alpha + 2)\Gamma(\alpha - \beta + 1) - \|c\|\Gamma(\alpha - \beta + 1) - \|b\|\Gamma(\alpha + 2)} \|\text{len}(g)\| \geq 0. \end{aligned} \tag{87}$$

□

Now let us consider if the solutions of (42)-(44) generate a fuzzy valued function and if the generated fuzzy valued function satisfies (38), (39).

From Theorem 17 and equation (43), we can have the following inequalities:

$$\begin{aligned} {}^c D_{0+}^\alpha \gamma_1(t, r) &\leq {}^c D_{0+}^\alpha \gamma_2(t, r), \quad t \in I, r \in [0, 1], \\ \gamma_1(t, r) &\leq \gamma_2(t, r), \quad t \in I, r \in [0, 1], \end{aligned} \tag{88}$$

$${}^c D_{0+}^\beta \gamma_1(t, r) \leq {}^c D_{0+}^\beta \gamma_2(t, r), \quad t \in I, r \in [0, 1].$$

Hence, we can consider the sets of intervals

$$\{U_\alpha(t, r) := [{}^c D_{0+}^\alpha \gamma_1(t, r), {}^c D_{0+}^\alpha \gamma_2(t, r)], r \in [0, 1]\}, \tag{89}$$

$$\{U_\beta(t, r) := [{}^c D_{0+}^\beta \gamma_1(t, r), {}^c D_{0+}^\beta \gamma_2(t, r)], r \in [0, 1]\}, \tag{90}$$

$$\{U_0(t, r) := [\gamma_1(t, r), \gamma_2(t, r)], r \in [0, 1]\}. \tag{91}$$



**Theorem 21.** *The sets of intervals (89)-(91) generate continuous fuzzy valued functions.*

*Proof.* Consider again the notion  ${}^c D_{0+}^\alpha y_1(t, r) = u_1(t, r)$ ,  ${}^c D_{0+}^\alpha y_2(t, r) = u_2(t, r)$ .

Since  $u_1(t, r), u_2(t, r)$  satisfy (45), then we can have  $u_2(t, r) \geq u_1(t, r)$ .

Let us prove  $[u_1(t, r_1), u_2(t, r_1)] \supset [u_1(t, r_2), u_2(t, r_2)]$ , where  $0 < r_1 < r_2 \leq 1$ .

The proof is equivalent to

$$\begin{aligned} u_1(t, r_1) &\leq u_1(t, r_2), \\ u_2(t, r_2) &\leq u_2(t, r_1). \end{aligned} \tag{92}$$

First, let us prove  $u_1(t, r_1) \leq u_1(t, r_2)$ . From the first equation of (45), we can have

$$\begin{aligned} u_1(t, r) + b(t) I_{0+}^{\alpha-\beta} u_1(t, r) + c(t) I_{0+}^\alpha u_1(t, r) \\ = g_1(t, r) + tc(t) I_{0+}^\alpha u_1(t, r)|_{t=1}, \\ u_2(t, r) + b(t) I_{0+}^{\alpha-\beta} u_2(t, r) + c(t) I_{0+}^\alpha u_2(t, r) \\ = g_2(t, r) + tc(t) I_{0+}^\alpha u_2(t, r)|_{t=1}. \end{aligned} \tag{93}$$

We need the following notions:

$$\begin{aligned} g_1(t, r) &:= f_1(t, r) - c(t) ((1-t) y_{0,1}(r) + ty_{1,1}(r)), \\ g_2(t, r) &:= f_2(t, r) - c(t) ((1-t) y_{0,2}(r) + ty_{1,2}(r)), \\ \Delta u_1 &:= u_1(t, r_2) - u_1(t, r_1), \\ \Delta u_2 &:= u_2(t, r_2) - u_2(t, r_1), \\ \Delta g_1 &:= g_1(t, r_2) - g_1(t, r_1), \\ \Delta g_2 &:= g_2(t, r_2) - g_2(t, r_1). \end{aligned} \tag{94}$$

From (93), we can have

$$\begin{aligned} \Delta u_1 + b(t) I_{0+}^{\alpha-\beta} \Delta u_1 + c(t) I_{0+}^\alpha \Delta u_1 \\ = \Delta g_1 + tc(t) I_{0+}^\alpha \Delta u_1|_{t=1}. \end{aligned} \tag{95}$$

*Assumption 5.*  $((I - A)\Delta g_1 \geq 0, (I - A)\Delta g_2 \leq 0)$ .

Let us consider a the scheme of successive approximation

$$\begin{aligned} \Delta u_1^0 &= 0, \\ \Delta u_1^{n+1} + b(t) I_{0+}^{\alpha-\beta} \Delta u_1^{n+1} + c(t) I_{0+}^\alpha \Delta u_1^{n+1} \\ &= \Delta g_1 + tc(t) I_{0+}^\alpha \Delta u_1^n|_{t=1}. \end{aligned} \tag{96}$$

If  $n = 0$ , then we can have

$$\Delta u_1^1 + b(t) I_{0+}^{\alpha-\beta} \Delta u_1^1 + c(t) I_{0+}^\alpha \Delta u_1^1 = \Delta g_1, \tag{97}$$

and so from Assumption 5 and Lemma 13, we can obtain  $\Delta u_1^1 \geq 0$ .

Suppose  $\Delta u_1^n \geq 0$  for any  $n \geq 1$ .

Let us consider the equation

$$\begin{aligned} \Delta u_1^{n+1} + b(t) I_{0+}^{\alpha-\beta} \Delta u_1^{n+1} + c(t) I_{0+}^\alpha \Delta u_1^{n+1} \\ = \Delta g_1 + tc(t) I_{0+}^\alpha \Delta u_1^n|_{t=1}. \end{aligned} \tag{98}$$

By Assumptions 3, 5 and the induction, we can obtain  $(I - A)(\Delta g_1 + tc(t) I_{0+}^\alpha \Delta u_1^{(n)}|_{t=1}) \geq 0$  and so we have  $\Delta u_1^{(n+1)} \geq 0$  by Lemma 10.

Finally, we have  $\Delta u_1 \geq 0$ . We can also prove  $\Delta u_2 \leq 0$  as the same way by Lemma 14.

Then, let us consider  $u(t, r) = \bigcap_{n=1}^\infty [u_1(t, r_k), u_2(t, r_k)]$  for any nondecreasing sequence  $r_k \in (0, 1]$  which satisfies  $\lim_{k \rightarrow \infty} r_k = r$ , i.e.,  $\lim_{n \rightarrow \infty} u_1(t, r_k) = u_1(t, r)$ ,  $\lim_{n \rightarrow \infty} u_2(t, r_k) = u_2(t, r)$ .

From the first equation of (45), we have

$$\begin{aligned} u_1(t, r) + b(t) I_{0+}^{\alpha-\beta} u_1(t, r) \\ + c(t) ((1-t) y_{0,1}(r) + ty_{1,1}(r)) \\ + c(t) I_{0+}^\alpha u_1(t, r) \\ - tc(t) I_{0+}^\alpha u_1(t, r)|_{t=1} = f_1(t, r), \\ u_1(t, r_k) + b(t) I_{0+}^{\alpha-\beta} u_1(t, r_k) \\ + c(t) ((1-t) y_{0,1}(r_k) + ty_{1,1}(r)) \\ + c(t) I_{0+}^\alpha u_1(t, r_k) \\ - tc(t) I_{0+}^\alpha u_1(t, r_k)|_{t=1} = f_1(t, r_k). \end{aligned} \tag{99}$$

and so we can get

$$\begin{aligned} u_1(t, r) - u_1(t, r_k) \\ = h(t, r_k, r) - b(t) I_{0+}^{\alpha-\beta} (u_1(t, r) - u_1(t, r_k)) \\ - c(t) I_{0+}^\alpha (u_1(t, r) - u_1(t, r_k)) \\ + tc(t) I_{0+}^\alpha (u_1(t, r) - u_1(t, r_k))|_{t=1}. \end{aligned} \tag{100}$$

Let us denote the following:

$$\begin{aligned} \Delta_k u_1 &:= u_1(t, r) - u_1(t, r_k), \\ h(t, r_k, r) &:= f_1(t, r) - f_1(t, r_k) - c(t) \\ &\cdot ((1-t)(y_{0,1}(r) - y_{0,1}(r_k)) \\ &+ t(y_{1,1}(r) - y_{1,1}(r_k))). \end{aligned} \tag{101}$$

Using the above notions, the equation can be written as

$$\begin{aligned} \Delta_k u_1 + b(t) I_{0+}^{\alpha-\beta} \Delta_k u_1 + c(t) I_{0+}^\alpha \Delta_k u_1 \\ = h(t, r_k, r) + tc(t) I_{0+}^\alpha \Delta_k u_1|_{t=1}. \end{aligned} \tag{102}$$

Now, we use the following scheme of successive approximation

$$\begin{aligned} \Delta_k u_1^0 &= 0, \\ \Delta_k u_1^{n+1} + b(t) I_{0+}^{\alpha-\beta} \Delta_k u_1^{n+1} + c(t) I_{0+}^\alpha \Delta_k u_1^{n+1} \\ &= h(t, r_k, r) + tc(t) I_{0+}^\alpha \Delta_k u_1^n|_{t=1}. \end{aligned} \tag{103}$$

When  $n = 0$ , we have

$$\begin{aligned} \Delta_k u_1^1 + b(t) I_{0+}^{\alpha-\beta} \Delta_k u_1^1 + c(t) I_{0+}^\alpha \Delta_k u_1^1 \\ = h(t, r_k, r), \end{aligned} \tag{104}$$

and then from Lemma 12, we can obtain inequality  $\|\Delta_k u_1^1\| \leq (e^{k_*}/(1-q))\|h(\cdot, r_k, r)\|$ .

Since  $\lim_{r_k \rightarrow r} \|h(\cdot, r_k, r)\| = 0$ , we can get

$$\lim_{r_k \rightarrow r} \Delta_k u_1^1 = 0. \tag{105}$$

For any  $n \in N$ , suppose  $\lim_{r_k \rightarrow r} \Delta_k u_1^n = 0$  and let us consider when  $n + 1$ .

We can have

$$\begin{aligned} \Delta_k u_1^{n+1} + b(t) I_{0+}^{\alpha-\beta} \Delta_k u_1^{n+1} + c(t) I_{0+}^\alpha \Delta_k u_1^{n+1} \\ = h(t, r_k, r) + tc(t) I_{0+}^\alpha \Delta_k u_1^n|_{t=1}, \end{aligned} \tag{106}$$

and so we can obtain

$$\begin{aligned} \lim_{r_k \rightarrow r} (h(t, r_k, r) + tc(t) I_{0+}^\alpha \Delta_k u_1^n|_{t=1}) \\ = \lim_{r_k \rightarrow r} h(t, r_k, r) + \lim_{r_k \rightarrow r} (tc(t) I_{0+}^\alpha \Delta_k u_1^n|_{t=1}) \\ = \lim_{r_k \rightarrow r} h(t, r_k, r) \\ + tc(t) \lim_{r_k \rightarrow r} \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{\Delta_k u_1^n(s, r_k)}{(1-s)^{1-\alpha}} ds \\ = \lim_{r_k \rightarrow r} h(t, r_k, r) \\ + tc(t) \frac{1}{\Gamma(\alpha)} \int_0^1 \lim_{r_k \rightarrow r} \frac{\Delta_k u_1^n(s, r_k)}{(1-s)^{1-\alpha}} ds = 0. \end{aligned} \tag{107}$$

We can get

$$\begin{aligned} \|\Delta_k u_1^{n+1}\| \\ \leq \frac{e^{k_*}}{1-q} \max_{0 \leq t \leq 1} |h(t, r_k, r) + tc(t) I_{0+}^\alpha \Delta_k u_1^n|_{t=1}|, \end{aligned} \tag{108}$$

and so we can have  $\lim_{r_k \rightarrow r} \Delta_k u_1^{n+1} = 0$ .

Hence, we have  $\lim_{r_k \rightarrow r} \Delta_k u_1 = 0$  and as the same way, we obtain  $\lim_{r_k \rightarrow r} \Delta_k u_2 = 0$ , where  $\Delta_k u_2 := u_2(t, r) - u_2(t, r_k)$ .

Finally, we proved that the set of intervals  $\{[u_1(t, r), u_2(t, r)]\}_{r \in (0,1]}$  generates a fuzzy valued function.

Denote the fuzzy valued function generated by the set of intervals  $\{[u_1(t, r), u_2(t, r)]\}_{r \in (0,1]}$  by a fuzzy valued function  $\tilde{y}_\alpha(t)$ .

For any  $t_* \in I$ , we can get

$$\begin{aligned} D(\tilde{y}_\alpha(t), \tilde{y}_\alpha(t_*)) \\ = \sup_{r \in [0,1]} \max \left\{ |{}^c D_{0+}^\alpha \gamma_1(t, r) - {}^c D_{0+}^\alpha \gamma_1(t_*, r)|, \right. \\ \left. |{}^c D_{0+}^\alpha \gamma_2(t, r) - {}^c D_{0+}^\alpha \gamma_2(t_*, r)| \right\}. \end{aligned} \tag{109}$$

Real-valued functions  ${}^c D_{0+}^\alpha \gamma_1(t, r), {}^c D_{0+}^\alpha \gamma_2(t, r)$  are continuous since  $\gamma_1(t, r), \gamma_2(t, r)$  are solutions of (42)-(44). So we have  $D(\tilde{y}_\alpha(t), \tilde{y}_\alpha(t_*)) \rightarrow 0(t \rightarrow t_*)$ .

Therefore, the fuzzy valued function  $\tilde{y}_\alpha(t)$  is continuous.

As the same way, we can easily prove that the set of intervals  $\{U_\beta(t, r)\}_r$  generates a fuzzy valued function.

Now let us consider that the set of intervals  $\{U_0(t, r)\}_r$  generates a fuzzy valued function. The equation (46) can be expressed as follows:

$$\begin{aligned} y_1(t, r) &= (1-t) y_{0,1}(r) + t y_{1,1}(r) \\ &+ \int_0^1 (-G(t, s)) u_1(s, r) ds, \\ y_2(t, r) &= (1-t) y_{0,2}(r) + t y_{1,2}(r) \\ &+ \int_0^1 (-G(t, s)) u_2(s, r) ds. \end{aligned} \tag{110}$$

Because Green's function is continuous and the set of intervals  $\{[u_1(t, r), u_2(t, r)]\}_{r \in (0,1]}$  generates a continuous fuzzy valued function, the set of intervals that is composed of right sides of these equations generates a fuzzy valued function.

Hence, the set of intervals  $\{[y_1(t, r), y_2(t, r)]\}_r$  generates a fuzzy valued function.

Let us denote the fuzzy valued functions generated by (89), (90) by  $\tilde{y}_*(t), \tilde{y}_\beta(t)$ .

Then, we can easily prove that fuzzy valued functions  $\tilde{y}_*(t), \tilde{y}_\beta(t)$  are continuous.  $\square$

From equation (50), we can have

$$\begin{aligned} y_1(t, r) + y_{0,1}(r) \cdot t + I_{0+}^\alpha u_1(t, r)|_{t=1} \cdot t \\ = y_{0,1}(r) + y_{1,1}(r) \cdot t + I_{0+}^\alpha u_1(t, r), \\ y_2(t, r) + y_{0,2}(r) \cdot t + I_{0+}^\alpha u_2(t, r)|_{t=1} \cdot t \\ = y_{0,2}(r) + y_{1,2}(r) \cdot t + I_{0+}^\alpha u_2(t, r), \end{aligned} \tag{111}$$

and we can get

$$\tilde{y}_* \oplus \tilde{y}_0 \otimes t \oplus I_{0+}^\alpha \tilde{y}_\alpha|_{t=1} \otimes t = \tilde{y}_0 \oplus \tilde{y}_1 \otimes t \oplus I_{0+}^\alpha \tilde{y}_\alpha. \tag{112}$$

**Theorem 22.**  ${}^c D_{1,1}^\alpha \tilde{y}_* = \tilde{y}_\alpha, {}^c D_{1,1}^\beta \tilde{y}_* = \tilde{y}_\beta$ .

*Proof.* Let us consider again that the set of intervals,

$$\begin{aligned} \left\{ \left[ {}^c D_{0+}^\alpha I_{0+}^\alpha \gamma_1(t, r), {}^c D_{0+}^\alpha I_{0+}^\alpha \gamma_2(t, r) \right] \right\}_r \\ = \left\{ [y_1(t, r), y_2(t, r)] \right\}_r = \{U_0(t, r)\}_r, \end{aligned} \tag{113}$$

generates a fuzzy valued function and the generated fuzzy valued function is  $\tilde{y}_\alpha$ . So  $I_{0+}^\alpha \tilde{y}_\alpha$  is  ${}^c[(1, 1) - \alpha]$ -differentiable and  ${}^c D_{1,1}^\alpha I_{0+}^\alpha \tilde{y}_\alpha = \tilde{y}_\alpha$ .

This shows that the left side of equation (112) is  ${}^c[(1, 1) - \alpha]$ -differentiable.

Then, we can have

$$\begin{aligned} {}^c D_{1,1}^\alpha \tilde{y}_* \oplus {}^c D_{1,1}^\alpha (\tilde{y}_0 \otimes t) \oplus {}^c D_{1,1}^\alpha (I_{0+}^\alpha \tilde{y}_\alpha|_{t=1} \otimes t) \\ = {}^c D_{1,1}^\alpha \tilde{y}_0 \oplus {}^c D_{1,1}^\alpha (\tilde{y}_1 \otimes t) \oplus {}^c D_{1,1}^\alpha I_{0+}^\alpha \tilde{y}_\alpha. \end{aligned} \tag{114}$$

Furthermore, we can have  ${}^c D_{1,1}^\alpha(\tilde{y}_0 \otimes t) = 0$ ,  ${}^c D_{1,1}^\alpha(I_{0+}^\alpha \tilde{y}_\alpha|_{t=1} \otimes t) = 0$ , so we obtain

$${}^c D_{1,1}^\alpha \tilde{y}_* = {}^c D_{1,1}^\alpha I_{0+}^\alpha \tilde{y}_\alpha = \tilde{y}_\alpha. \tag{115}$$

As the same way, we can prove that  ${}^c D_{1,1}^\beta \tilde{y}_* = \tilde{y}_\beta$ .  $\square$

Through the above investigations, we can see that  $\tilde{y}_*$  is the solution of (38) and (39) by the sense of Definition 15.

In fact, from equation (112), we can see that

$$\begin{aligned} \tilde{y}_* &= (\tilde{y}_0 \oplus \tilde{y}_1 \otimes t \oplus I_{0+}^\alpha \tilde{y}_\alpha) \\ &\oplus_H (\tilde{y}_0 \otimes t \oplus I_{0+}^\alpha \tilde{y}_\alpha|_{t=1} \otimes t), \end{aligned} \tag{116}$$

and we can see that the boundary condition is concluded for the fuzzy valued function  $\tilde{y}_*$ .

At last, we have the fuzzy equation

$$\tilde{y}_\alpha(t) \oplus b(t) \otimes \tilde{y}_\beta(t) \oplus c(t) \otimes \tilde{y}_*(t) = f(t). \tag{117}$$

Thus, the fuzzy valued function  $\tilde{y}_*$  is a solution of the problem (38) and (39).

#### 4. The Numerical Solution of the Fuzzy Fractional Differential Equations by Using Haar Wavelet Operational Matrix

*Definition 23.* The orthogonal set of Haar functions are defined in the interval  $[0, 1)$  by

$$\begin{aligned} h_0(t) &= \frac{1}{\sqrt{m}}, \\ h_i(t) &= \frac{1}{\sqrt{m}} \begin{cases} 2^{j/2}, & \frac{k-1}{2^j} < t \leq \frac{k-1/2}{2^j}, \\ 2^{j/2}, & \frac{k-1/2}{2^j} < t \leq \frac{k}{2^j}, \\ 0, & \text{else,} \end{cases} \end{aligned} \tag{118}$$

where  $i = 0, 1, \dots, m-1$ ,  $m = 2^M$ ,  $M$  is a positive integer, and  $j, k$  represent the integer decomposition of the index  $i$ .

We need the following notations:

$$\begin{aligned} H(t) &:= (h_0(t), h_1(t), \dots, h_{m-1}(t))^T, \\ C^T &:= (c_0, c_1, \dots, c_{m-1}), \\ \Delta t &:= \frac{1}{m} = \frac{1}{2^r}, \\ t_k &:= (k-0.5)\Delta t, \quad k = \overline{1, m}. \end{aligned} \tag{119}$$

The following matrix is called Haar wavelet matrix on the set of collocation points  $\{t_k\}$ :

$$H_{mat} := \begin{pmatrix} h_0(t_1) & h_0(t_2) & \cdots & h_0(t_m) \\ h_1(t_1) & h_1(t_2) & \cdots & h_1(t_m) \\ \vdots & \vdots & \ddots & \vdots \\ h_{m-1}(t_1) & h_{m-1}(t_2) & \cdots & h_{m-1}(t_m) \end{pmatrix}. \tag{120}$$

This is an orthogonal matrix; namely,  $H_{mat}^{-1} = H_{mat}^T$ . Put  $(I_{0+}^\alpha H)(t) := (I_{0+}^\alpha h_0(t), I_{0+}^\alpha h_1(t), \dots, I_{0+}^\alpha h_{m-1}(t))^T$ .

*Definition 24.*  $m \times m$  matrix  $F_H^\alpha$  determined by  $(I_{0+}^\alpha H)(t) \approx F_H^\alpha \cdot H(t)$  is called the operational matrix of the fractional integral of order  $\alpha$ .

If we use the matrix  $F_B^\alpha$  determined by

$$F_B^\alpha := \frac{1}{m^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{121}$$

the following equation holds:

$$F_H^\alpha = H_{mat} F_B^\alpha H_{mat}^T, \tag{122}$$

where  $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$ .

To solve the problem (38), (39), first, we need to solve the fractional integral equation sequence

$$\begin{aligned} g_1(t, r) &:= f_1(t, r) - c(t) \left( (1-t)y_{0,1}(r) + ty_{1,1}(r) \right), \\ g_2(t, r) &:= f_2(t, r) - c(t) \left( (1-t)y_{0,2}(r) + ty_{1,2}(r) \right), \\ u_1^0(t, r) &= 0, \end{aligned} \tag{123}$$

$$\begin{aligned} u_1^0(t, r) &= 0, \\ u_1^{n+1}(t, r) &+ b(t) I_{0+}^{\alpha-\beta} u_1^{n+1}(t, r) + c(t) I_{0+}^\alpha u_1^{n+1}(t, r) \\ &= g_1(t, r) + tc(t) I_{0+}^\alpha u_1^n(t, r)|_{t=1}, \\ u_2^{n+1}(t, r) &+ b(t) I_{0+}^{\alpha-\beta} u_2^{n+1}(t, r) + c(t) I_{0+}^\alpha u_2^{n+1}(t, r) \\ &= g_2(t, r) + tc(t) I_{0+}^\alpha u_2^n(t, r)|_{t=1} \end{aligned} \tag{124}$$

by using operational matrix and get  $u_1(t, r), u_2(t, r)$  approximately and then have  $y_1(t, r), y_2(t, r)$  by the equations

$$\begin{aligned} y_1(t, r) &= y_{0,1}(r) \\ &+ (y_{1,1}(r) - y_{0,1}(r) - I_{0+}^\alpha u_1(t, r)|_{t=1}) \\ &\cdot t + I_{0+}^\alpha u_1(t, r), \\ y_2(t, r) &= y_{0,2}(r) \\ &+ (y_{1,2}(r) - y_{0,2}(r) - I_{0+}^\alpha u_2(t, r)|_{t=1}) \\ &\cdot t + I_{0+}^\alpha u_2(t, r). \end{aligned} \tag{125}$$

We first calculate (124)

$$\begin{aligned} C_{g_1}^T &= (g_1(t_1, r), g_1(t_2, r), \dots, g_1(t_m, r)) \cdot H_{mat}^T, \\ C_{g_2}^T &= (g_2(t_1, r), g_2(t_2, r), \dots, g_2(t_m, r)) \cdot H_{mat}^T, \end{aligned} \tag{126}$$

to solve (124) by using operational matrix. Using the notion

$$\begin{aligned}
 u_1^n(t, r) &: \approx C_{u_1}^n H(t), \\
 u_2^n(t, r) &: \approx C_{u_2}^n H(t), \\
 g_1(t, r) &: \approx C_{g_1}^T H(t), \\
 g_2(t, r) &: \approx C_{g_2}^T H(t),
 \end{aligned}
 \tag{127}$$

we have the following repeat equations from (124).

$$\begin{aligned}
 &u_1^{n+1}(t, r) + b(t) I_{0+}^{\alpha-\beta} u_1^{n+1}(t, r) + c(t) I_{0+}^\alpha u_1^{n+1}(t, r) \\
 &= g_1(t, r) + tc(t) I_{0+}^\alpha u_1^n(t, r) \Big|_{t=1}, \\
 &u_2^{n+1}(t, r) + b(t) I_{0+}^{\alpha-\beta} u_2^{n+1}(t, r) + c(t) I_{0+}^\alpha u_2^{n+1}(t, r) \\
 &= g_2(t, r) + tc(t) I_{0+}^\alpha u_2^n(t, r) \Big|_{t=1}. \\
 &C_{u_1}^{n+1} H(t) + b(t) I_{0+}^{\alpha-\beta} C_{u_1}^{n+1} H(t) + c(t) I_{0+}^\alpha C_{u_1}^{n+1} H(t) \\
 &= C_{g_1}^T H(t) + tc(t) I_{0+}^\alpha C_{u_1}^n H(t) \Big|_{t=1}, \\
 &C_{u_2}^{n+1} H(t) + b(t) I_{0+}^{\alpha-\beta} C_{u_2}^{n+1} H(t) + c(t) I_{0+}^\alpha C_{u_2}^{n+1} H(t) \\
 &= C_{g_2}^T H(t) + tc(t) I_{0+}^\alpha C_{u_2}^n H(t) \Big|_{t=1}, \\
 &C_{u_1}^{n+1} H(t) + b(t) C_{u_1}^{n+1} I_{0+}^{\alpha-\beta} H(t) + c(t) C_{u_1}^{n+1} I_{0+}^\alpha H(t) \\
 &= C_{g_1}^T H(t) + tc(t) C_{u_1}^n I_{0+}^\alpha H(t) \Big|_{t=1}, \\
 &C_{u_2}^{n+1} H(t) + b(t) C_{u_2}^{n+1} I_{0+}^{\alpha-\beta} H(t) + c(t) C_{u_2}^{n+1} I_{0+}^\alpha H(t) \\
 &= C_{g_2}^T H(t) + tc(t) C_{u_2}^n I_{0+}^\alpha H(t) \Big|_{t=1}, \\
 &C_{u_1}^{n+1} H(t) + b(t) C_{u_1}^{n+1} F_H^{\alpha-\beta} H(t) + c(t) C_{u_1}^{n+1} F_H^\alpha H(t) \\
 &= C_{g_1}^T H(t) + tc(t) C_{u_1}^n F_H^\alpha H(t) \Big|_{t=1}, \\
 &C_{u_2}^{n+1} H(t) + b(t) C_{u_2}^{n+1} F_H^{\alpha-\beta} H(t) + c(t) C_{u_2}^{n+1} F_H^\alpha H(t) \\
 &= C_{g_2}^T H(t) + tc(t) C_{u_2}^n F_H^\alpha H(t) \Big|_{t=1}.
 \end{aligned}
 \tag{128}$$

This is a point equation.

$$\begin{aligned}
 &C_{u_1}^{n+1} H(t_k) + b(t_k) C_{u_1}^{n+1} F_H^{\alpha-\beta} H(t_k) \\
 &+ c(t_k) C_{u_1}^{n+1} F_H^\alpha H(t_k) \\
 &= C_{g_1}^T H(t_k) + t_k c(t_k) C_{u_1}^n F_H^\alpha H(t_M), \\
 &k = \overline{1, M}.
 \end{aligned}
 \tag{129}$$

On the other hand, we can have

$$\begin{aligned}
 &C_{u_1}^{n+1} H(t_k) + b(t_k) C_{u_1}^{n+1} F_H^{\alpha-\beta} H(t_k) \\
 &+ c(t_k) C_{u_1}^{n+1} F_H^\alpha H(t_k) \\
 &= C_{u_1}^{n+1} (I + b(t_k) F_H^{\alpha-\beta} + c(t_k) F_H^\alpha) H(t_k).
 \end{aligned}
 \tag{130}$$

Denote the following:

$$\begin{aligned}
 A &:= \left( (I + b(t_1) F_H^{\alpha-\beta} + c(t_1) F_H^\alpha) \right. \\
 &\quad \cdot H(t_1), \dots, (I + b(t_M) F_H^{\alpha-\beta} + c(t_M) F_H^\alpha) H(t_M) \Big), \\
 B &:= \begin{cases} C_{g_1}^T H(t_1) + t_1 c(t) C_{u_1}^n F_H^\alpha H(t_M) \\ C_{g_1}^T H(t_2) + t_2 c(t) C_{u_1}^n F_H^\alpha H(t_M) \\ \vdots \\ C_{g_1}^T H(t_M) + t_M c(t) C_{u_1}^n F_H^\alpha H(t_M). \end{cases}
 \end{aligned}
 \tag{131}$$

Then, the simultaneous equation to determine  $C_{u_1}^{n+1}$  is as follows:

$$AC_{u_1}^{n+1} = B.
 \tag{132}$$

**Lemma 25.** *If  $\|b\|_{C(I)} + \|c\|_{C(I)} < 1$ , equation (132) has a unique solution.*

*Proof.* Prove that the coefficient matrix  $A$  is nonnullity. We can have

$$\begin{aligned}
 A &= \left( (I + b(t_1) F_H^{\alpha-\beta} + c(t_1) F_H^\alpha) \right. \\
 &\quad \cdot H(t_1), \dots, (I + b(t_M) F_H^{\alpha-\beta} + c(t_M) F_H^\alpha) H(t_M) \Big) \\
 &= (H(t_1) + (b(t_1) F_H^{\alpha-\beta} + c(t_1) F_H^\alpha) \\
 &\quad \cdot H(t_1), \dots, H(t_M) + (b(t_M) F_H^{\alpha-\beta} + c(t_M) F_H^\alpha) \\
 &\quad \cdot H(t_M)) = (H(t_1), \dots, H(t_M)) \\
 &\quad + ((b(t_1) F_H^{\alpha-\beta} + c(t_1) F_H^\alpha) \\
 &\quad \cdot H(t_1), \dots, (b(t_M) F_H^{\alpha-\beta} + c(t_M) F_H^\alpha) H(t_M)) \\
 &= H_{mat} + \tilde{A}, \\
 \tilde{A} &:= ((b(t_1) F_H^{\alpha-\beta} + c(t_1) F_H^\alpha) \\
 &\quad \cdot H(t_1), \dots, (b(t_M) F_H^{\alpha-\beta} + c(t_M) F_H^\alpha) H(t_M)), \\
 A &= H_{mat} + \tilde{A} = (I + \tilde{A} H_{mat}^T) H_{mat}.
 \end{aligned}
 \tag{133}$$

Thus, the nonnullity of the matrix  $A$  is equivalent to the nonnullity of the matrix  $(I + \tilde{A} H_{mat}^T)$ . On the other hand, the inequality  $\|\tilde{A} H_{mat}^T\| < 1$  guarantees the nonnullity of the matrix  $(I + \tilde{A} H_{mat}^T)$ .

Since the inequality

$$\|\tilde{A} H_{mat}^T\|_2 \leq \|\tilde{A}\|_2 \|H_{mat}^T\|_2 = \|\tilde{A}\|_2
 \tag{134}$$

holds, the inequality  $\|\tilde{A}\|_2 < 1$  is a sufficient condition to  $\|\tilde{A} H_{mat}^T\| < 1$ . And by the notion

$$\begin{aligned}
 \tilde{A} &= \left( (b(t_1) F_H^{\alpha-\beta} + c(t_1) F_H^\alpha) \right. \\
 &\quad \cdot H(t_1), \dots, (b(t_M) F_H^{\alpha-\beta} + c(t_M) F_H^\alpha) H(t_M) \Big),
 \end{aligned}
 \tag{135}$$

we can obtain

$$\begin{aligned} \bar{A}^T &= \begin{pmatrix} H^T(t_1) \left( b(t_1) (F_H^{\alpha-\beta})^T + c(t_1) (F_H^\alpha)^T \right) \\ H^T(t_2) \left( b(t_2) (F_H^{\alpha-\beta})^T + c(t_2) (F_H^\alpha)^T \right) \\ \vdots \\ H^T(t_M) \left( b(t_M) (F_H^{\alpha-\beta})^T + c(t_M) (F_H^\alpha)^T \right) \end{pmatrix} \\ &= \begin{pmatrix} \left( b(t_1) H^T(t_1) (F_H^{\alpha-\beta})^T + c(t_1) H^T(t_1) (F_H^\alpha)^T \right) \\ \left( b(t_2) H^T(t_2) (F_H^{\alpha-\beta})^T + c(t_2) H^T(t_2) (F_H^\alpha)^T \right) \\ \vdots \\ \left( b(t_M) H^T(t_M) (F_H^{\alpha-\beta})^T + c(t_M) H^T(t_M) (F_H^\alpha)^T \right) \end{pmatrix}. \end{aligned} \tag{136}$$

By using the notion of

$$D_1 := \begin{pmatrix} b(t_1) H^T(t_1) (F_H^{\alpha-\beta})^T \\ b(t_2) H^T(t_2) (F_H^{\alpha-\beta})^T \\ \vdots \\ b(t_M) H^T(t_M) (F_H^{\alpha-\beta})^T \end{pmatrix},$$

$$D_2 := \begin{pmatrix} c(t_1) H^T(t_1) (F_H^\alpha)^T \\ c(t_2) H^T(t_2) (F_H^\alpha)^T \\ \vdots \\ c(t_M) H^T(t_M) (F_H^\alpha)^T \end{pmatrix}, \tag{137}$$

we have

$$\begin{aligned} \bar{A}^T &= D_1 + D_2, \\ \|\bar{A}^T\|_2 &= \|D_1 + D_2\|_2 \leq \|D_1\|_2 + \|D_2\|_2. \end{aligned} \tag{138}$$

Now let us consider that

$$\|D_1\|_2 := \left\| \begin{pmatrix} b(t_1) H^T(t_1) (F_H^{\alpha-\beta})^T \\ b(t_2) H^T(t_2) (F_H^{\alpha-\beta})^T \\ \vdots \\ b(t_M) H^T(t_M) (F_H^{\alpha-\beta})^T \end{pmatrix} \right\|_2. \tag{139}$$

We already knew that

$$\begin{aligned} &\begin{pmatrix} b(t_1) H^T(t_1) (F_H^{\alpha-\beta})^T \\ b(t_2) H^T(t_2) (F_H^{\alpha-\beta})^T \\ \vdots \\ b(t_M) H^T(t_M) (F_H^{\alpha-\beta})^T \end{pmatrix} \begin{pmatrix} b(t_1) H^T(t_1) (F_H^{\alpha-\beta})^T \\ b(t_2) H^T(t_2) (F_H^{\alpha-\beta})^T \\ \vdots \\ b(t_M) H^T(t_M) (F_H^{\alpha-\beta})^T \end{pmatrix}^T \\ &= \begin{pmatrix} b(t_1) H^T(t_1) (F_H^{\alpha-\beta})^T \\ b(t_2) H^T(t_2) (F_H^{\alpha-\beta})^T \\ \vdots \\ b(t_M) H^T(t_M) (F_H^{\alpha-\beta})^T \end{pmatrix} (b(t_1) F_H^{\alpha-\beta} H(t_1), b(t_2) F_H^{\alpha-\beta} H(t_2), \dots, b(t_M) F_H^{\alpha-\beta} H(t_M)) \end{aligned} \tag{140}$$

and because

$$\begin{aligned} &\langle b(t_1) H^T(t_1) (F_H^{\alpha-\beta})^T, b(t_1) F_H^{\alpha-\beta} H(t_1) \rangle \\ &= (b(t_1))^2 H^T(t_1) (F_H^{\alpha-\beta})^T F_H^{\alpha-\beta} H(t_1) \\ &= (b(t_1))^2, \\ &\langle b(t_1) H^T(t_1) (F_H^{\alpha-\beta})^T, b(t_2) F_H^{\alpha-\beta} H(t_2) \rangle \\ &= b(t_1) b(t_2) H^T(t_1) (F_H^{\alpha-\beta})^T F_H^{\alpha-\beta} H(t_2) = 0, \end{aligned} \tag{141}$$

we obtain

$$\begin{aligned} &\begin{pmatrix} b(t_1) H^T(t_1) (F_H^{\alpha-\beta})^T \\ b(t_2) H^T(t_2) (F_H^{\alpha-\beta})^T \\ \vdots \\ b(t_M) H^T(t_M) (F_H^{\alpha-\beta})^T \end{pmatrix} \begin{pmatrix} b(t_1) H^T(t_1) (F_H^{\alpha-\beta})^T \\ b(t_2) H^T(t_2) (F_H^{\alpha-\beta})^T \\ \vdots \\ b(t_M) H^T(t_M) (F_H^{\alpha-\beta})^T \end{pmatrix}^T \\ &= \text{diag}((b(t_1))^2, (b(t_2))^2, \dots, (b(t_M))^2). \end{aligned} \tag{142}$$

Thus, we have

$$\|D_1\|_2 \leq \sqrt{\max_k (b(t_k))^2} \leq \|b\|_{C(I)}. \tag{143}$$

As the same way, we can obtain

$$\|D_2\|_2 \leq \sqrt{\max_k (c(t_k))^2} \leq \|c\|_{C(I)}. \tag{144}$$

Finally, we can see that

$$\|\tilde{A}\|_2 = \|\tilde{A}^T\|_2 \leq \|b\|_{C(I)} + \|c\|_{C(I)}. \tag{145}$$

□

We compute the solutions  $u_1(t, r), u_2(t, r)$  of (124) by using the limit  $C_{u_1}^T, C_{u_2}^T$  in the repeat equation (132).

Now, we consider the algorithm to compute (125) by using the operational matrix. We can rewrite the equation

$$\begin{aligned} y_1(t, r) &= y_{0,1}(r) \\ &+ (y_{1,1}(r) - y_{0,1}(r) - I_{0+}^\alpha u_1(t, r)|_{t=1}) \\ &\cdot t + I_{0+}^\alpha u_1(t, r) \end{aligned} \tag{146}$$

$$\begin{aligned} y_2(t, r) &= y_{0,2}(r) \\ &+ (y_{1,2}(r) - y_{0,2}(r) - I_{0+}^\alpha u_2(t, r)|_{t=1}) \\ &\cdot t + I_{0+}^\alpha u_2(t, r) \end{aligned}$$

by using the notion

$$\begin{aligned} h_1(t, r) &= (1-t)y_{0,1}(r) + ty_{1,1}(r) \\ h_2(t, r) &= (1-t)y_{0,2}(r) + ty_{1,2}(r) \end{aligned} \tag{147}$$

as follows:

$$\begin{aligned} y_1(t, r) &= h_1(t, r) - I_{0+}^\alpha u_1(t, r)|_{t=1} \cdot t + I_{0+}^\alpha u_1(t, r) \\ y_2(t, r) &= h_2(t, r) - I_{0+}^\alpha u_2(t, r)|_{t=1} \cdot t + I_{0+}^\alpha u_2(t, r). \end{aligned} \tag{148}$$

First, we denote the following as

$$\begin{aligned} h_1(t, r) &: \approx C_{h_1}^T H(t), \\ h_2(t, r) &: \approx C_{h_2}^T H(t) \end{aligned} \tag{149}$$

and compute

$$\begin{aligned} C_{h_1}^T &= (h_1(t_1, r), h_1(t_2, r), \dots, h_1(t_m, r)) \cdot H_{mat}^T, \\ C_{h_2}^T &= (h_2(t_1, r), h_2(t_2, r), \dots, h_2(t_m, r)) \cdot H_{mat}^T. \end{aligned} \tag{150}$$

Then,  $y_1(t, r), y_2(t, r)$  are computed by

$$\begin{aligned} y_1(t, r) &\approx C_{h_1}^T H(t) - tC_{u_1}^T F_H^\alpha H(t_M) + C_{u_1}^T F_H^\alpha H(t) \\ y_2(t, r) &\approx C_{h_2}^T H(t) - tC_{u_2}^T F_H^\alpha H(t_M) + C_{u_2}^T F_H^\alpha H(t) \end{aligned} \tag{151}$$

from (132).

### 5. Numerical Examples

Example 1. Consider a FVBP

$$\begin{aligned} D_{1,1}^{(2)} y(t) \oplus b \otimes ({}^c D_{1,1}^{1.5} y)(t) \oplus c \otimes y(t) \\ = (1+t) \otimes (7.9, 8, 8.1), \quad t \in (0, 1), \\ y(0) = (-0.1, 0, 0.1), \\ y(1) = (-0.1, 0, 0.1), \end{aligned} \tag{152}$$

where  $b, c \in \mathbf{R}_+, y \in C([0, 1], \mathbf{R}_F)$  and the boundary values  $(-0.1, 0, 0.1), (-0.1, 0, 0.1)$ , and  $(7.9, 8, 8.1)$  are triangular fuzzy numbers. Now, let us consider the conditions for Assumptions 1-5.

A condition for Assumption 1:

$$\begin{aligned} q := \left( \frac{\|b\|}{\Gamma(\alpha - \beta + 1)} + \frac{2\|c\|}{\Gamma(\alpha + 1)} \right) = 1.12838b + c \\ < 1. \end{aligned} \tag{153}$$

A condition for Assumption 2: we can have

$$\begin{aligned} len(f)(t, r) &= f_2(t, r) - f_1(t, r) \\ &= 0.2(1-r)(1+t), \\ len(g)(t, r) &= 0.2(1-r)(1+t) - c \cdot 0.2(1-r) \\ &= 0.2(1-r)(1+t-c) \end{aligned} \tag{154}$$

and so we can obtain an inequality

$$(I - A)(1+t-c) \geq 0 \tag{155}$$

for a condition for Assumption 2.

Consequently, we can have

$$\begin{aligned} (I - A)(1+t-c) &= (1+t-c) - bI_{0+}^{\alpha-\beta}(1+t-c) \\ &- cI_{0+}^\alpha(1+t-c) = (1+t-c) \\ &- \frac{b}{\Gamma(1.5)} \left( (1-c)t^{0.5} + \frac{t^{1.5}}{1.5} \right) \\ &- \frac{c}{2} \left( (1-c)t^2 + \frac{t^3}{3} \right) \geq (1-c) \\ &- \left( 1.12838b \left( (1-c) + \frac{1}{1.5} \right) - \frac{c}{2} \left( (1-c) + \frac{1}{3} \right) \right) \\ &= 0.5(-2.54858+c)(-0.78475+c) \\ &+ b(-1.88063+1.12838c) \end{aligned} \tag{156}$$

and so the condition for Assumption 2 is

$$\begin{aligned} 0.5(-2.54858+c)(-0.78475+c) \\ + b(-1.88063+1.12838c) \geq 0. \end{aligned} \tag{157}$$

A condition for Assumption 3: we can easily prove that the equation

$$(I - A)(t) = t - \frac{t^{\alpha-\beta}}{(\alpha - \beta)(\alpha - \beta + 1)} - \frac{t^\alpha}{\alpha(\alpha + 1)} \tag{158}$$

$$= t - \frac{4t^{0.5}}{3} - \frac{t^2}{6}$$

is nonnegative.

A condition for Assumption 4: we can have

$$\frac{\Gamma(\alpha + 2)(b + \Gamma(\alpha - \beta + 1))}{\Gamma(\alpha + 2)\Gamma(\alpha - \beta + 1) - \|c\|\Gamma(\alpha - \beta + 1) - \|b\|\Gamma(\alpha + 2)}$$

$$= \frac{6(b + 0.886227)}{5.31736 - 6b - 0.886227c}$$

$$0.2(1 - r)(1 + t) - \frac{6(b + 0.886227)}{5.31736 - 6b - 0.886227c} \cdot 0.2(1 - r)(2 - c) \tag{159}$$

$$= 0.2(1 - r) \left( (1 + t) - \frac{6(b + 0.886227)(2 - c)}{5.31736 - 6b - 0.886227c} \right)$$

$$\geq 0.2(1 - r) \left( 1 - \frac{6(b + 0.886227)(2 - c)}{5.31736 - 6b - 0.886227c} \right).$$

So, we can see that the inequality

$$1 - \frac{6(b + 0.886227)(2 - c)}{5.31736 - 6b - 0.886227c} \geq 0 \tag{160}$$

is a sufficient condition for Assumption 4. Finally, we have

$$5.31736 + 18b - 4.431135c - 6bc \leq 0. \tag{161}$$

A condition for Assumption 5: first, we can have

$$f_1(t, r_2) - f_1(t, r_1) = (1 + t)(7.9 + 0.1r_2) - (1 + t)(7.9 + 0.1r_1)$$

$$= 0.1(r_2 - r_1)(1 + t),$$

$$y_{0,1}(r_2) - y_{0,1}(r_1) = 0.1r_2 - 0.1 - (0.1r_1 - 0.1) \tag{162}$$

$$= 0.1(r_2 - r_1),$$

$$y_{1,1}(r_2) - y_{1,1}(r_1) = 0.1(r_2 - r_1),$$

$$\Delta g_1 = 0.1(r_2 - r_1)(1 + t)(1 - c).$$

We can easily obtain  $1 - 3b/\sqrt{\pi} - 1.5c \geq 0$  for  $(I - A)\Delta g_1 \geq 0$ .

As the same way, we can have the same inequality for  $(I - A)\Delta g_2 \leq 0$ .

The following inequalities are conditions for the FBVP to have a solution.

- ①  $1.12838b + c < 1$
- ②  $1 - c \geq 0$
- ③  $0.5(-2.54858 + c)(-0.78475 + c) + b(-1.88063 + 1.12838c) \geq 0$

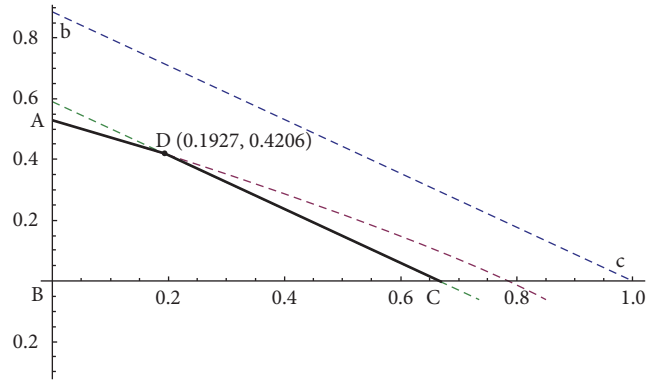


FIGURE 1: Area of  $b$  and  $c$  satisfying ①-⑤ for Example 1.

$$\textcircled{4} \quad 5.31736 + 18b - 4.431135c - 6bc \leq 0$$

$$\textcircled{5} \quad 1 - 3b/\sqrt{\pi} - 1.5c \geq 0$$

We show the area of  $b, c$  that satisfies the above conditions on Figure 1.

*Example 2.* Consider the following fuzzy boundary value problem:

$$D_{1,1}^{(2)}y(t) \oplus b(t) \otimes ({}^c D_{1,1}^{1.5}y)(t) \oplus c(t) \otimes y(t)$$

$$= (1 + t) \otimes (7.9, 8, 8.1), \quad t \in (0, 1), \tag{163}$$

$$y(0) = (-0.1, 0, 0.1),$$

$$y(1) = (-0.1, 0, 0.1),$$

where  $b, c \in C(I)$ ,  $y \in C([0, 1], \mathbf{R}_F)$ ,  $b(t), c(t) \geq 0$  and  $(-0.1, 0, 0.1)$  and  $(7.9, 8, 8.1)$  are triangular fuzzy numbers.

*Assumption 1.*  $q := (\|b\|/\Gamma(\alpha - \beta + 1) + 2\|c\|/\Gamma(\alpha + 1)) = 1.12838\|b\| + \|c\| < 1$ .

*Assumption 2.*

$$\text{len}(f)(t, r) = f_2(t, r) - f_1(t, r)$$

$$= 0.2(1 - r)(1 + t),$$

$$c(t)((1 - t)\text{len}(y_0) + t\text{len}(y_1)) = c(t)0.2(1 - r), \tag{164}$$

$$\text{len}(g)(t, r) = 0.2(1 - r)(1 + t) - c(t)0.2(1 - r)$$

$$= 0.2(1 - r)(1 + t - c(t)).$$

We can have  $(I - A)(1 + t - c(t)) \geq 0$  as a condition for Assumption 2.

*Assumption 3.*  $(I - A)(tc(t)) \geq 0$ .

*Assumption 4.* Denote  $\|c\| := \min_{t \in [0,1]} |c(t)|$ ; then we get

$$\begin{aligned}
 & 0.2(1-r)(1+t) - \frac{6(b(t) + 0.886227)}{5.31736 - 6\|b\| - 0.886227\|c\|} \\
 & \cdot 0.2(1-r)(2 - \|c\|) = 0.2(1-r) \\
 & \cdot \left( (1+t) - \frac{6(b(t) + 0.886227)(2 - \|c\|)}{5.31736 - 6\|b\| - 0.886227\|c\|} \right) \quad (165) \\
 & \geq 0.2(1-r) \\
 & \cdot \left( 1 - \frac{6(b(t) + 0.886227)(2 - \|c\|)}{5.31736 - 6\|b\| - 0.886227\|c\|} \right).
 \end{aligned}$$

So we obtain

$$1 \geq \frac{6(b(t) + 0.886227)(2 - \|c\|)}{5.31736 - 6\|b\| - 0.886227\|c\|}. \quad (166)$$

*Assumption 5.* We can obtain  $1 - \|c\| \geq 0, 1 - 3\|b\|/\sqrt{\pi} - 1.5\|c\| \geq 0$  as Example 1.

Sufficient conditions are given as follows:

- ①  $1.12838\|b\| + \|c\| < 1$
- ②  $(I - A)(1 + t - c(t)) \geq 0$
- ③  $(I - A)(tc(t)) \geq 0$
- ④  $1 \geq 6(b(t) + 0.886227)(2 - \|c\|)/(5.31736 - 6\|b\| - 0.886227\|c\|)$
- ⑤  $1 - 3\|b\|/\sqrt{\pi} - 1.5\|c\| \geq 0$

Let us consider especially for  $c(t) = 0.5t/(1 + t^2), b(t) = 0.2(1 - t)$ . We can obtain that

$$\begin{aligned}
 \|c\| &= 0.25, \\
 \|b\| &= 0.2, \\
 \|c\| &= 0.
 \end{aligned} \quad (167)$$

$$\begin{aligned}
 1.12838\|b\| + \|c\| &= 1.12838 \cdot 0.2 + 0.25 = 0.475676 \\
 &< 1,
 \end{aligned}$$

That is, condition 1 holds. Using the inequalities  $0.75 + t \leq c(t) \leq 1 + 0.75t$ , we can have

$$\begin{aligned}
 & (I - A)(1 + t - c(t)) \\
 &= (1 + t - c(t)) - A(1 + t - c(t)) \\
 &= 1 + t - \frac{0.5t}{(1 + t^2)} \\
 & \quad - 0.2(1 - t) I_{0+}^{\alpha-\beta} \left( 1 + t - \frac{0.5t}{(1 + t^2)} \right) \\
 & \quad - \frac{0.5t}{(1 + t^2)} I_{0+}^{\alpha} \left( 1 + t - \frac{0.5t}{(1 + t^2)} \right)
 \end{aligned}$$

$$\begin{aligned}
 & \geq t + 0.75 - 0.2(1 - t) I_{0+}^{0.5} (1 + 0.75t) \\
 & \quad - \frac{0.5t}{(1 + t^2)} I_{0+}^2 (1 + 0.75t) \\
 &= t + 0.75 - \frac{0.2(1 - t)(2t^{0.5} + t^{1.5})}{\Gamma(0.5)} \\
 & \quad - \frac{0.5t}{(1 + t^2)} \frac{0.5t^2 + 0.125t^3}{\Gamma(2)}, \\
 & \geq t + 0.75 - 0.1(2t^{0.5} + t^{1.5}) \\
 & \quad - (1 + 0.75t)(0.5t^2 + 0.125t^3) \\
 & \geq t + 0.55 - 0.1t^{1.5} \\
 & \quad - (1 + 0.75t)(0.5t^2 + 0.125t^3) \geq 0,
 \end{aligned} \quad (168)$$

i.e., condition 3 holds.

We can obtain

$$\begin{aligned}
 (I - A)(tc(t)) &= tc(t) - A(tc(t)) \\
 &= \frac{0.5t^2}{(1 + t^2)} - A\left(\frac{0.5t^2}{(1 + t^2)}\right) \\
 &= \frac{0.5t^2}{(1 + t^2)} + 0.01t I_{0+}^{\alpha-\beta} \left( \frac{t^2}{(1 + t^2)} \right) \\
 & \quad - 0.01 I_{0+}^{\alpha-\beta} \left( \frac{t^2}{(1 + t^2)} \right) \\
 & \quad - \frac{0.25t}{(1 + t^2) I_{0+}^{\alpha} (t^2/(1 + t^2))} \\
 & \geq \frac{0.5t^2}{(1 + t^2)} + 0.005t I_{0+}^{\alpha-\beta} t^2 \\
 & \quad - 0.01 I_{0+}^{\alpha-\beta} t^2 - \frac{0.25t}{(1 + t^2) I_{0+}^{\alpha} t^2} \\
 &= \frac{0.5t^2}{(1 + t^2)} + 0.005 \cdot 0.601802t^{3.5} \\
 & \quad - 0.01 \cdot 0.601802t^{2.5} - \frac{0.25t^5}{12(1 + t^2)} \\
 &= -0.00601802t^{2.5} + 0.00300901t^{3.5} \\
 & \quad + \frac{0.5t^2}{1 + t^2} - \frac{0.0208333t^5}{1 + t^2} \geq 0,
 \end{aligned} \quad (169)$$

$$t \in [0, 1]$$

and since  $1 - 3\|b\|/\sqrt{\pi} - 1.5\|c\| = 1 - 0.338514 - 0.375 = 0.286486 > 0$ , condition 5 holds.

Through the investigations, we can see that the problem of Example 2 has a solution when  $c(t) = 0.5t/(1 + t^2), b(t) =$



TABLE 1: In case of  $r = 1$  for Example 2.

$t$	$y_1(t, 1)$	$y_2(t, 1)$	$y_1''(t, 1)$	$y_2''(t, 1)$
0.015025	-0.104378	-0.104378	8.04222	8.04222
0.140625	-0.867268	-0.867268	9.80614	9.80614
0.265625	-1.47628	-1.47628	12.0982	12.0982
0.390625	-1.89576	-1.89576	14.7463	14.7463
0.515625	-2.08441	-2.08441	17.7042	17.7042
0.640625	-1.99603	-1.99603	20.9508	20.9508
0.765625	-1.5799	-1.5799	24.4836	24.4836
0.890625	-0.780805	-0.780805	28.3149	28.3149
0.984375	0.105687	0.105687	31.3979	31.3979

TABLE 2: In case of  $r = 0.5$  for Example 2.

$t$	$y_1(t, 1)$	$y_2(t, 1)$	$y_1''(t, 1)$	$y_2''(t, 1)$
0.015625	-0.153774	-0.054983	7.99234	8.0921
0.140625	-0.912267	-0.822269	9.74817	9.86412
0.265625	-1.51779	-1.43476	12.0285	12.1679
0.390625	-1.93488	-1.85664	14.6623	14.8303
0.515625	-2.12245	-2.04637	17.6034	17.805
0.640625	-2.03457	-1.95749	20.8309	21.0706
0.765625	-1.62082	-1.53898	24.3424	24.6248
0.890625	-0.826309	-0.7353	28.1502	28.4796
0.984375	0.0550747	0.1563	31.2141	31.5816

$0.2(1 - t)$  and we can find out the solution approximately by using scheme of successive approximation.

We have presented the numerical result of Example 2 by using proposed method in Tables 1 and 2. Table 1 is the numerical results for Example 2 when the level value is equal to 1 and Table 2 is the numerical results when the level value is equal to 0.5. The first rows of Tables 1 and 2 are the numeral values of the dependent variables and the second rows are the values of the lower functions and the third rows are the values of upper functions and the fourth and fifth rows are the values of lower and upper functions for  $(1, 1)$ -derivatives of  $y(t)$  for Example 2. For Tables 1 and 2, we can know that  $y_1(t, r) \leq y_2(t, r)$  and  $y_1''(t, r) \leq y_2''(t, r)$  have been satisfied. Also we can see that  $y_1(t, r) \leq y_2(t, r)$  and  $y_1''(t, r) \leq y_2''(t, r)$  have been satisfied for all  $r \in (0, 1]$  and  $y_1(t, r), y_2(t, r)$  constitute intervals. We compute by Mathematica6.0.

### 6. Conclusion

We have studied one condition for a two-point boundary value problem to have a solution and investigated one method to calculate the solution. In conclusion, we have to consider the cut-equations and problem (43) with the inequality condition to get the fuzzy solution. The conditions were given by the coefficients, boundary conditions, and nonhomogeneous terms of fuzzy differential equations. The next step in the research is to extend the results of this paper to generalized cases.

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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