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Research Article

The Existence and Structure of Rotational Systems in the Circle

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By a rotational system, we mean a closed subset X of the circle, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, together with a continuous transformation $f : X \to X$ with the requirements that the dynamical system (X, f) be minimal and that f respect the standard orientation of $\mathbb{T}$. We show that infinite rotational systems (X, f), with the property that map f has finite preimages, are extensions of irrational rotations of the circle. Such systems have been studied when they arise as invariant subsets of certain specific mappings, $F : \mathbb{T} \to \mathbb{T}$. Because our main result makes no explicit mention of a global transformation on $\mathbb{T}$, we show that such a structure theorem holds for rotational systems that arise as invariant sets of any continuous transformation $F : \mathbb{T} \to \mathbb{T}$ with finite preimages. In particular, there are no explicit conditions on the degree of F. We then give a development of known results in the case where $F(\theta) = d \cdot \theta \bmod 1$ for an integer $d > 1$. The paper concludes with a construction of infinite rotational sets for mappings of the unit circle of degree larger than one whose lift to the universal cover is monotonic.

1. Introduction

In what follows, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ denotes the unit circle with the standard orientation.

Suppose X is a compact metric space and $f : X \to X$ is a continuous transformation. The dynamical system (X, f) is *minimal* if, for every $\theta \in X$, the orbit

$$\mathcal{O}_\theta = \{f^n(\theta) : n \in \mathbb{N}\} \tag{1}$$

is dense in X.

Definition 1. Let $X \subseteq \mathbb{T}$. A continuous transformation $f : X \to X$ *preserves cyclic order* if, for any $A, B, C \in X$ with distinct images, the arcs ABC and $f(A)f(B)f(C)$ have the same orientation.

Definition 2. A *rotational system* is a subset $X \subseteq \mathbb{T}$ and a continuous transformation $f : X \to X$, with the properties that

(i) the dynamical system (X, f) is minimal,
(ii) the transformation $f : X \to X$ preserves cyclic order.

In this situation, we will simply say that (X, f) is rotational.

We need to recall one more definition, before stating the main theorem.

Definition 3. Let $f_i : X_i \to X_i$ be continuous transformations of compact metric spaces for $i = 1, 2$. The dynamical system (X_1, f_1) is an *extension* of (X_2, f_2) if there is a continuous, surjective function $\phi : X_1 \to X_2$ such that $\phi \circ f_1 = f_2 \circ \phi$.

Our main result is as follows.

Theorem 4. *Let (X, f) be a rotational system such that X is an infinite, proper subset of $\mathbb{T}$. In addition, suppose that the continuous mapping $f : X \to X$ has finite preimages, that is, $|T^{-1}x| < \infty$ for each $x \in X$. Then,*

(i) *the dynamical system (X, f) is an extension of an irrational rotation of the circle via a map $\phi : X \to \mathbb{T}$ that is compatible with the cyclic ordering on both X and T;*
(ii) *the function ϕ has preimages of cardinality one except at countably many points of $\mathbb{T}$. The preimages of these exceptional points have cardinality two and are the endpoints of gaps of the set X in $\mathbb{T}$;*

(iii) (X, f) *has a unique ergodic measure* μ *and* $\phi_*\mu$ *is the standard Lebesgue measure on* $\mathbb{T}$.

The angle of the rotation of $\mathbb{T}$ in the preceding theorem is called the *rotation number* of (X, f).

Such systems are of particular interest when they arise as invariant subsets of a continuous mapping on the whole circle, $F : \mathbb{T} \to \mathbb{T}$. Recall that a closed subset X of $\mathbb{T}$ is *invariant* with respect to F if $F(X) \subseteq X$. In this situation, we may set $f = F|_X$ and consider the dynamical system (X, f). Such a system, (X, f), is a *subsystem* of $(\mathbb{T}, F)$.

Our main theorem has the following obvious corollary.

Corollary 5. *Let* $F : \mathbb{T} \to \mathbb{T}$ *be a continuous mapping with finite preimages. Suppose, moreover, that* X *is a closed, infinite, proper subset of* $\mathbb{T}$ *that is invariant with respect to* F. *If* (X, f) *is rotational, then all three conclusions of Theorem 4 hold.*

In the case where $F : \mathbb{T} \to \mathbb{T}$ is given by $F(\theta) = d \cdot \theta \mod 1$ for an integer $d > 1$, Corollary 5 has an extensive history. Ideas related to the $d = 2$ case were studied by Morse and Hedlund [1] in their work on Sturmian trajectories. The problem was taken up later by several authors, including Gambaudo et al. [2], Veerman [3, 4], Goldberg [5], Goldberg and Milnor [6], and Bullett and Sentenac [7]. The $d > 2$ case was studied by Goldberg and Tresser [8], Blokh et al. [9], and Bowman et al. [10]. In sum, these works provide a complete characterization of the rotational subsystems, with rational and irrational rotation number, for the uniform cover of $\mathbb{T}$ with positive degree.

In this paper, we point out that parts of the analysis of rotational systems with irrational rotation number can be done without explicit reference to an ambient transformation on the unit circle. This leads to a structure result for rotational subsystems of a wide class of continuous transformations $F : \mathbb{T} \to \mathbb{T}$, those with finite preimages.

The proof of Theorem 4 will be accomplished over the next two sections. An important step is to solve the functional equation found in Proposition 16. This equation is mentioned in the appendix in [8] where an analytical approach to the uniform cover case is sketched. (In particular, for the direction we are interested in, a solution to the functional equation is claimed but not given.) We provide a solution to this problem in our more general setting using the existence of an invariant measure together with the Mean Ergodic Theorem. Sections 4 and 5 then revisit the known d-fold cover case (see [3, 4, 8, 9]) from our point of view.

For a given continuous $F : \mathbb{T} \to \mathbb{T}$, Theorem 4 and its corollary shed no light on how to determine which irrational numbers can be realized by rotational subsystems of $(\mathbb{T}, F)$. When $\deg F = 1$, it is well known that there can be at most one such rotation number. In the uniform cover case with $d > 1$, every irrational number in $[0, 1)$ can be achieved. In the last section, we consider degree $d > 1$ mappings of $\mathbb{T}$ that are monotonic with respect to the usual orientation. Mappings of this type can have intervals of constancy or an arbitrary number of fixed points (many of which may be attractive). In particular, they are not conjugate to the uniform cover case via a homeomorphism. We show that, with at most countable exceptions, every irrational rotation number can be realized when the degree is two. When the degree is larger than 2, there are examples for every irrational rotation number.

2. Structure of Rotational Subsets

For this section and the next, we will work under the assumptions of Theorem 4:

(i) (X, f) is a rotational system.

(ii) X is an infinite, proper subset of $\mathbb{T}$.

(iii) $f : X \to X$ has finite preimages.

Proposition 6. *The set* X *is a Cantor set; that is, it is a compact set that is perfect and has empty interior.*

Proof. Suppose $\theta_0 \in X$ is an isolated point of such a dynamical system. Minimality implies that the orbit, $\mathcal{O}_{\theta_0} = \{f^n(\theta_0) : n > 0\}$, is dense in X. As a consequence, we have that $f^n(\theta_0) = \theta_0$ for some positive integer n. The set $\mathcal{O}_{\theta_0}$ must then be finite as well as dense. Therefore, $X = \{f^n(\theta_0) : n > 0\}$. This contradiction implies that X cannot have any isolated points. Thus, X is a perfect subset of $\mathbb{T}$; that is, it is closed and has no isolated points.

Now, consider the possibility that X contains a closed interval of positive length. Since X is closed and a proper subset of $\mathbb{T}$, we may choose a closed interval $I \subseteq X$ of maximal length. The set I cannot be left invariant by any power of f. If n were such a power, then $f^n|_I$ would have a fixed point. The orbit of this fixed point would be a finite invariant subset of X, which is impossible. Next, fix $\theta_0 \in I$. Since the orbit of this point is dense and I has a nonempty interior, there must be a positive integer n such that $f^n(\theta_0) \in I$. On the other hand, there must a point $\theta_1 \in I$ with $f^n(\theta_1) \notin I$. Let θ_t, $0 \le t \le 1$, be the linear path in I that connects θ_0 to θ_1. Then $f^n(\theta_t)$ connects $f^n(\theta_0) \in I$ with $f^n(\theta_1) \notin I$. But the range of f^n is contained in X. An argument using the intermediate value theorem shows that this contradicts the maximality of I. We have shown that X has empty interior. □

Remark 7. The properties of the dynamical system (X, f) used in the above result are that (X, f) is minimal and that X is an infinite, closed, proper subset of $\mathbb{T}$.

Since X is a proper subset of $\mathbb{T}$, we may select a point $\theta_0 \in \mathbb{T}$ with $\theta_0 \notin X$. Parameterize $\mathbb{T}$ by the map $\chi : [0, 1[\to \mathbb{T}$ defined by

$$\chi(t) = \theta_0 + t \mod 1. \tag{2}$$

This continuous bijection is orientation preserving, provided we orient $[0, 1[$ in the standard way. χ is also a homeomorphism from $]0, 1[$ to the open set $\mathbb{T} \backslash \{\theta_0\}$. Set $\Xi = \chi^{-1}(X)$ and $g = \chi^{-1} \circ f \circ \chi$ (see Figure 1). Thus, Ξ is homeomorphic to X and (X, f) and (Ξ, g) are isomorphic dynamical systems. In particular, Ξ is an infinite, perfect compact subset of the open interval $]0, 1[$ and (Ξ, g) is a minimal dynamical system.

There is also a useful reformulation of the condition that f preserves cyclic order. Let A, B and C be three distinct points

$$\begin{tikzcd}
\Xi \arrow[r, "\chi \mid \Xi"] \arrow[d, "g"] & X \arrow[r, hook] \arrow[d, "f"] & \mathbb{T} \\
\Xi \arrow[r, "\chi \mid \Xi"] & X \arrow[r, hook] & \mathbb{T}
\end{tikzcd}$$

FIGURE 1: The dynamical systems (X, f) and (Ξ, g) are equivalent.

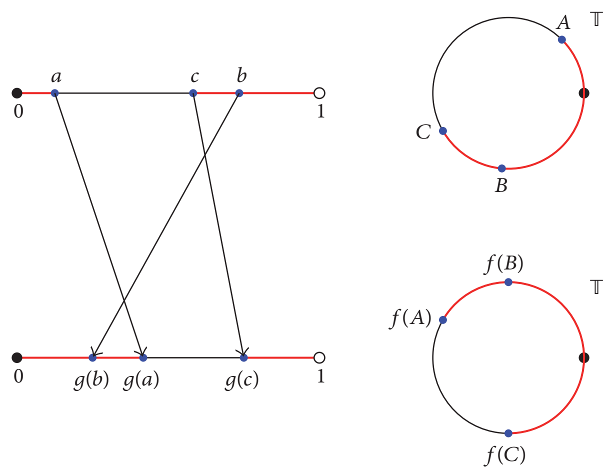

FIGURE 2: f preserving cyclic order is equivalent to all arrow diagrams for g having 0 or 2 crossings.

in $X \subset \mathbb{T}$ with distinct images under f. Then, the points $a = \chi^{-1}(A)$, $b = \chi^{-1}(B)$, and $c = \chi^{-1}(C)$ have distinct images under g. The action of g on this triple can be conveniently encoded by an arrow diagram as in Figure 2. That the arcs ABC and $f(A)f(B)f(C)$ have the same orientation in $\mathbb{T}$ is equivalent to saying that the corresponding arrow diagram has an even number (zero or two) of crossings. This elementary fact is easy to check and will be convenient in the proof of Proposition 9.

Lemma 8. *Suppose $x, y \in \Xi$ and $g(x) = y$. Then, there are sequences a_n and b_n in Ξ such that*

(i) *a_n and b_n are strictly monotone sequences,*

(ii) *$g(a_n) = b_n$ for all $n \in \mathbb{N}$,*

(iii) *$\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} b_n = y$.*

Proof. The proof uses the following elementary fact: suppose one is given a convergent sequence of real numbers whose terms are distinct from the limit. Then, there is a subsequence which is strictly monotonic.

Now, since Ξ is perfect, one can construct a sequence $a_n \in \Xi$ such that $a_n \neq x$ for all $n \in \mathbb{N}$ and $a_n \to x$ as $n \to \infty$. By passing to a subsequence, one can further arrange a_n to be strictly monotonic. Moreover, by the continuity of g, $g(a_n) \to g(x) = y$. Set $b_n = g(a_n)$. Since g has finite preimages, $g(a_n) \neq y$ for all but finitely many n. By once again passing to a subsequence, if necessary, we may arrange that b_n is strictly monotone. □

Proposition 9. *The preimages of $g : \Xi \to \Xi$ have cardinality at most two.*

Proof. Suppose, to the contrary, that $x_0, x_1, x_2 \in \Xi$ are three distinct points with the same image under g. So $g(x_i) = y$ for $i = 0, 1$ and 2. We may assume that x_0, x_1, and x_2 are arranged in increasing order in $[0, 1[$. Let $a_n^{(i)}$, $i = 0, 1, 2$ and $b_n^{(i)}$, $i = 0, 1, 2$ be the sequences in Ξ obtained by applying Lemma 8 to x_0, x_1, and x_2, respectively.

The proof proceeds according to how $b_n^{(0)}$ and $b_n^{(2)}$ approach y.

Case $b_n^{(0)}$ and $b_n^{(2)}$ Approach y from the Same Side. We give the argument when both sequences approach y from below; the other possibility is dealt with similarly. In this case, we can use the properties of sequence $b_n^{(i)}$ to find indices k and l so that

$$b_k^{(0)} < b_l^{(2)} < y \tag{3}$$

and $a_k^{(0)}$, x_1 and $a_l^{(2)}$ are in increasing order. This would mean that the arrow diagram for the triple $a_k^{(0)}$, x_1, $a_l^{(2)}$ has an odd number (one) of crossings, which is not possible.

Case $b_n^{(0)} \searrow y$ and $b_n^{(2)} \nearrow y$. Choose n sufficiently large so that

$$a_n^{(0)} < x_1 < a_n^{(2)}. \tag{4}$$

But $g(a_n^{(2)}) = b_n^{(2)} < y < b_n^{(0)} = g(a_n^{(0)})$. Hence, the arrow diagram for $a_n^{(0)}$, x, $a_n^{(2)}$ has an odd number (three) of crossings.

Case $b_n^{(0)} \nearrow y$ and $b_n^{(2)} \searrow y$. By invoking Lemma 8 again, we construct an x' that is sufficiently close to x_1 so that $x_0 < x' < x_2$ and $g(x') \neq y$. We treat the case where

$g(x') < y = g(x)$—the other case is handled similarly. In this situation, there is a sufficiently large n with the property that $x' < a_n^{(2)}$. So,

$$x_0 < x' < a_n^{(2)},$$
$$g(x') < y = g(x_0) < b_n^{(2)} = g(a_n^{(2)}). \quad (5)$$

Thus, the arrow diagram for x_0, x', $a_n^{(2)}$ has an odd number (one) of crossings.

Hence, in all three cases, we have a contradiction. □

Set $\alpha = \inf \Xi$ and $\beta = \sup \Xi$. Since Ξ is a compact, subset of the open interval $]0,1[$ we have

$$0 < \alpha < \beta < 1. \quad (6)$$

Since (Ξ, g) is minimal, g is surjective. Consequently, we may put $\alpha' = \sup\{x \in \Xi : g(x) = \beta\}$ and $\beta' = \inf\{x \in \Xi : g(x) = \alpha\}$.

Proposition 10. *The inequality*

$$\alpha < \alpha' < \beta' < \beta \quad (7)$$

holds. Moreover, $[\alpha', \beta']$ *is a gap for* Ξ*, that is,* $X \cap]\alpha', \beta'[= \emptyset$.

Proof. Suppose that $\alpha' > \beta'$. Minimality ensures that $\beta' > \alpha$. By using Lemma 8, we can find x near α such that $\alpha < g(x) < \beta$. However, this would lead to an arrow diagram for the triple x, β', α' with an odd number of crossings.

Now, consider $x \in \Xi$ with $\alpha' < x < \beta'$. By the definition of α' and β', $g(x)$ must be distinct from α or β. Hence $g(x)$ must be strictly between α and β. This is a contradiction due to an impossible arrow diagram—this time for the triple α', x, β'.

Finally, since Ξ has no isolated points, $\alpha' \neq \alpha$ and $\beta' \neq \beta$. □

Lemma 11. *Let* $x, y \in \Xi$*. If* $x < y$ *and* $g(x) < g(y)$*, then, for any* $z \in \Xi$ *with* $x < z < y$*, we must have* $g(x) \leq g(z) \leq g(y)$.

Proof. Failure of the conclusion clearly leads to an impossible arrow diagram for the triple x, z, y. □

Proposition 12. $\alpha < g(\beta) \leq g(\alpha) < \beta$.

Proof. Minimality rules out the possibilities that $g(\alpha) = \alpha$ and $g(\beta) = \beta$.

If $g(\beta) = \alpha$, then by Lemma 8 there is $x \in \Xi$ near β such that $g(\alpha) > g(x) > \alpha$. This means that the arrow diagram for α, x, β is not allowed. In a similar way, $g(\alpha) = \beta$ can also be ruled out.

Only the middle inequality remains. Suppose, to the contrary, that $g(\beta) > g(\alpha)$. The previous proposition implies that $g(\alpha) \leq g(x) \leq g(\beta)$, for any $x \in \Xi$ that is strictly between α and β. Thus, Im g is a proper subset of X. This violates the minimality assumption. □

For any $x_0, x_1 \in [0,1)$, set

$$\Xi_{x_0,x_1} = \Xi \cap [x_0, x_1]. \quad (8)$$

Proposition 13. g *is monotone increasing on the sets* $\Xi_{\alpha,\alpha'}$ *and* $\Xi_{\beta',\beta}$*. Moreover,*

$$g(x) > x \quad \forall x \in \Xi_{\alpha,\alpha'},$$
$$g(x) < x \quad \forall x \in \Xi_{\beta,\beta'}. \quad (9)$$

Proof. Let $x, y \in \Xi \cap [\alpha, \alpha']$ with $x < y$. Suppose $g(y) < g(x)$. Since $y < \beta'$, the definition of β' forces $\alpha = g(\beta') < g(y)$. Thus, the arrow diagram for the triple x, y, β' is impossible. This contradiction shows that g is monotonic increasing on $[\alpha, \alpha']$. A similar argument applies to $\Xi \cap [\beta', \beta]$.

Next, let $x \in \Xi \cap [\alpha, \alpha']$ and suppose $g(x) \leq x$. Since g has no fixed points, $g(x) < x$. For the same reason, $\alpha < g(\alpha)$. Appealing to the monotonicity property proved in the previous paragraph, we have $\alpha < g(\alpha) \leq g(x) < x$. Lemma 11 then implies that $\Xi \cap [\alpha, x]$ is a nonempty, proper, closed invariant subset of Ξ—a contradiction. Thus $g(x) > x$ for $x \in \Xi \cap [\alpha, \alpha']$. The last inequality can be verified similarly. See Figure 3 for a diagram of the structure of g. □

3. Coding by Irrational Rotations of the Circle

By the Krylov-Bogoliubov theorem (see, e.g., [11, pp. 98]), there is a Borel probability measure on Ξ that is invariant under g. Fix one such, μ. Regard μ as a measure on $[0,1)$ and write $\tilde{\gamma}$ for its cumulative distribution function:

$$\tilde{\gamma}(x) = \int_0^x \mu(t). \quad (10)$$

Since every point of the infinite set Ξ has dense orbit, the invariant probability measure μ cannot include any point masses. Thus $\tilde{\gamma}$ is continuous. Hence, the restriction $\gamma = \tilde{\gamma}|_\Xi$ is also a continuous, monotone increasing map from Ξ to $[0,1]$. Finally, because

$$\tilde{\gamma}(\alpha) = 0,$$
$$\tilde{\gamma}(\beta) = 1 \quad (11)$$

and the fact that $\tilde{\gamma}$ is locally constant on the complement of Ξ, $\gamma : X \to [0,1]$ is surjective.

Proposition 14. *(i) If* $x_0, x_1 \in \Xi$ *and* $\Xi \cap (x_0, x_1) = \emptyset$*, then* $\gamma(x_0) = \gamma(x_1)$.

(ii) On the other hand, if $\Xi \cap (x_0, x_1) \neq \emptyset$*, then* $\gamma(x_0) > \gamma(x_1)$.

Proof. The first statement follows directly from the facts that spt $\mu \subseteq \Xi$ and μ contains no point masses.

To prove the second statement, first put $U = \Xi \cap (x_0, x_1)$ and

$$\nu_n(U, x) = \left|\left\{k : k = 0, \ldots, n-1, \ g^k(x) \in U\right\}\right|. \quad (12)$$

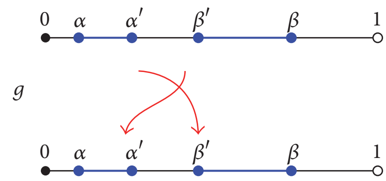

FIGURE 3: Schematic for the action of g. Ξ is a proper subset of the two intervals marked in blue.

Finally, write $\mathscr{P} : L^2(\Xi,\mu) \to L^2(\Xi,\mu)$ for the projection onto the subspace of functions left invariant by g. By the Mean Ergodic Theorem (see, e.g., [11, pp. 32]), the averages

$$A_n(x) = \frac{\nu_n(U,x)}{n} \tag{13}$$

converge in $L^2(\Xi,\mu)$ to the projection $\mathscr{P}[\mathbf{1}_U](x)$. On the other hand, since U is a nonempty open set and (Ξ, g) is a minimal dynamical system, there is an $\epsilon > 0$ such that

$$\liminf_{n\to\infty}\frac{\nu_n(U,x)}{n} > \epsilon \quad \forall x \in \Xi. \tag{14}$$

See, e.g., Proposition 4.7 in [12]. Hence, keeping Fatou's theorem in mind,

$$\begin{aligned}\gamma(x_1) - \gamma(x_0) &= \mu(U) = \|\mathbf{1}_U\|^2 \geq \|\mathscr{P}\mathbf{1}_U\|^2 \\ &= \lim_{n\to\infty}\|A_n\|^2 \\ &\geq \int \liminf_{n\to\infty}|A_n(x)|^2\,d\mu(x) \geq \epsilon^2 > 0.\end{aligned} \tag{15}$$

□

The above result can be restated as follows: for any distinct pair x, $x' \in \Xi$, $\gamma(x) = \gamma(x')$ if and only if x and x' are endpoints of a gap of $\Xi \subset\ ]0,1[$. A further useful corollary of this is that any nonempty open subset of Ξ has positive μ-measure.

Lemma 15. *One has*

$$\gamma_*\mu = \mathscr{L}, \tag{16}$$

where $\mathscr{L}$ denotes Lebesgue measure on $[0,1[$.

Proof. It is enough to show that

$$\gamma_*\mu([0,t]) = \mathscr{L}([0,t]) \quad \forall t \in [0,1[\,. \tag{17}$$

The left hand side is just $\mu(\gamma^{-1}[0,t])$. Since γ is monotone increasing, continuous, and surjective, $\gamma^{-1}[0,t] = [\alpha,x]$ for some x with $\gamma(x) = t$. But this just means $\mu(\Xi_{[\alpha,x]}) = t = \mathscr{L}([0,t])$. □

Proposition 16. *There is an $\omega_0 \in (0,1)$ such that*

$$\gamma(g(x)) = \omega_0 + \gamma(x) \mod 1 \tag{18}$$

for all $x \in \Xi$.

Proof. Put $\omega_0 = \mu([\beta',\beta])$. If $x \in \Xi \cap [\alpha,\alpha']$, then the g invariance of the measure μ implies that

$$\begin{aligned}\gamma(gx) &= \mu([\alpha,gx]) = \mu([\alpha,g\beta]) + \mu([g\alpha,gx]) \\ &= \mu([g\beta',g\beta]) + \mu([g\alpha,gx]) \\ &= \omega_0 + \mu([\alpha,x]) = \omega_0 + \gamma(x).\end{aligned} \tag{19}$$

If $x \in X \cap [\beta',\beta]$, then

$$\begin{aligned}\gamma(gx) &= \mu([\alpha,gx]) = \mu([g\beta',gx]) = \mu([\beta',x]) \\ &= \gamma(x) - \mu([\alpha,\alpha']) = \gamma(x) + \mu([\beta',\beta]) - 1 \\ &= \gamma(x) + \omega_0 \mod 1.\end{aligned} \tag{20}$$

□

Remark 17. In the appendix to the article by Goldberg and Tresser [8], an analytic argument for characterizing rotational subsystems for the map $F(\theta) = d\cdot\theta \mod 1$ (where $d > 1$) is sketched. The preceding proposition is claimed, but no argument is given. In addition, the authors then start with data equivalent to what is given in Section 5 and solve this functional equation to produce rational systems with a given rotation number.

Define the maps $\Gamma : \Xi \to \mathbb{T}$ and $r : \mathbb{T} \to \mathbb{T}$ by

$$\begin{aligned}\Gamma(x) &= \gamma(x) \mod 1, \\ r(\theta) &= \theta + \omega_0 \mod 1.\end{aligned} \tag{21}$$

Proposition 16 proved that

$$\Gamma \circ g = r \circ \Gamma. \tag{22}$$

In other words, Γ is a surjective, continuous mapping of the dynamical system (Ξ, g) to the dynamical system $(\mathbb{T}, r)$. In view of Proposition 14, if $\Gamma(x) = \Gamma(x')$ for distinct x and x', then either x and x' are endpoints of a gap for Ξ or $x = \alpha$ and $x' = \beta$. Finally, from Lemma 15, it is clear that $\Gamma_*\mu$ is Lebesgue measure on $\mathbb{T}$.

Proposition 18. *ω_0 is irrational.*

Proof. If ω_0 were rational, the set

$$\{r^n(\omega_0) : n \in \mathbb{N}\} = \{n\omega_0 \mod 1 : n \in \mathbb{N}\} \tag{23}$$

would be finite. Recalling that Γ has preimages of cardinality no greater that 2, this means that

$$\{x \in \Xi : \Gamma(x) = n\omega_0 \mod 1 \text{ for some } n \in \mathbb{N}\} \tag{24}$$

is finite. It is easy to check that it is also invariant under g. This contradicts the minimality of (Ξ, g). □

Lemma 19. *Let Z_0 be those points $x \in \Xi$ with the property that for every $\delta > 0$, the intervals $(x, x+\delta)$ and $(x, x-\delta)$ contain infinitely many points of Ξ. Then Z_0 is a dense, uncountable subset of Ξ.*

Proof. A point is in the complement of Z_0 in Ξ precisely when it is the endpoint of a gap, that is, a maximal open interval contained in $[\alpha, \beta]\backslash\Xi$. Since there are at most countably many such open intervals, we conclude that $\Xi\backslash Z_0$ is countable. Since the perfect set Ξ is uncountable, Z_0 is uncountable as well.

The density follows from a small modification of the standard argument that a perfect subset of a complete metric space is uncountable. Suppose $U \subseteq \Xi$ is a nonempty open set with $U \cap Z_0 = \emptyset$. Then, U is countable. Let $U = \{x_1, x_2, \ldots\}$. Write $U_i = \Xi\backslash\{x_i\}$ and note that U_i is dense and open in Ξ for each i. Baire's theorem yields that $\bigcap_{i=1}^{\infty} U_i$ is dense in Ξ. But, by construction, $U \cap (\bigcap_{i=1}^{\infty} U_i) = \emptyset$—a contradiction. So, Z_0 is dense. □

Theorem 20. *The dynamical system (Ξ, g) is uniquely ergodic.*

Proof. Let $x_0 \in Z_0$. By Proposition 14 and Lemma 19,

(i) $x \in \Xi$ and $x < x_0 \Rightarrow \gamma(x) < \gamma(x_0)$,

(ii) $x \in \Xi$ and $x > x_0 \Rightarrow \gamma(x) > \gamma(x_0)$.

Therefore, for any $y \in \Xi$ and $n \geq 0$,

$$\begin{gathered} g^n y \leq x_0 \Longleftrightarrow \\ \{\gamma(y) + n\theta_0\} \leq \gamma(x_0). \end{gathered} \tag{25}$$

(Here the braces denote fractional part.) Consequently,

$$\begin{aligned} &\lim_{n\to\infty} \frac{\left|\{k : g^k y \leq x_0,\ 0 \leq k < n\}\right|}{n} \\ &= \lim_{n\to\infty} \frac{\left|\{k : \{\gamma(y) + n\theta_0\} \leq \gamma(x_0),\ 0 \leq k < n\}\right|}{n} \\ &= \gamma(x_0) = \mu(\Xi \cap [\alpha, x_0]). \end{aligned} \tag{26}$$

The second limit has been evaluated by invoking Weyl's equidistribution theorem (see [13, 14]). Since such x_0 are dense in Ξ, the cumulative distribution of the measure μ is uniquely determined by (Ξ, g). In other words, (Ξ, g) is uniquely ergodic. □

With this last result, we have completed the proof of Theorem 4. Indeed, the analysis of this section transfers over to the dynamical system (X, f) by means of the isomorphism, χ^{-1}, of dynamical systems. In particular, one sets $\phi = \Gamma \circ \chi^{-1}$.

4. The Uniform d-Fold Cover

In this section and the next we are concerned with the map $F : \mathbb{T} \to \mathbb{T}$ defined by

$$Fx = d \cdot x \mod 1 \tag{27}$$

for some fixed integer $d \geq 2$. We will analyse the structure of rotational systems (X, f) with $f = F|_X$ using the findings of the last two sections.

Since $0 \mod 1$ is a fixed point of F, it is not in X. So, choosing this point for θ_0, we have that the parameterization $\chi : [0, 1[\ \to \mathbb{T}$ is given by

$$\chi(t) = t \mod 1 \tag{28}$$

and that $G = \chi^{-1} \circ F \circ \chi$ satisfies

$$G(x) = \{d \cdot x\}. \tag{29}$$

Define Ξ and g as before, and note that $g = G|_{\Xi}$. We will find it convenient to work with (Ξ, g), $[0, 1[$ and G, noting that findings in this setting convert readily to statements about (X, f), $\mathbb{T}$ and F.

The inverse image of 0 under G consists of the d points:

$$\xi_k = \frac{k}{d} \quad k = 0, \ldots, d-1. \tag{30}$$

In addition, G has $d-1$ fixed points:

$$\eta_k = \frac{k}{d-1} \quad k = 0, \ldots, d-2. \tag{31}$$

We also set $\xi_d = \eta_{d-1} = 1$.

Set $I_i = [\xi_i, \xi_{i+1}[$ for $i = 0, \ldots, d-1$ and note that the interior of I_i are precisely those points in $\mathbb{T}$ with a canonical d-adic expansion that starts with the digit k. (Recall that every real number has a canonical d-adic expansion that does not end with an infinite string of $d-1$'s.) Note also that $GI_k = [0, 1]$, since G is just the shift map on the d-adic expansion. Moreover, G is monotonic increasing on the interior of each I_i. Each closed interval I_i contains a unique fixed point η_i, for $0 \leq i < d-1$. The behavior of G at these fixed points can be readily determined. In particular, one checks that

$$\begin{aligned} Gx < x \quad &\text{for } \xi_i < x < \eta_i, \\ Gx > x \quad &\text{for } \eta_i < x < \xi_{i+1}. \end{aligned} \tag{32}$$

(See Figure 4 for the $d = 5$ case.) Under our operating assumptions, Ξ is infinite and minimal. Hence, none of ξ_i and η_i can lie in Ξ. In particular, any point in Ξ must lie in the interior of precisely one of the intervals $I_0, \ldots, I_{d-1}$.

Write $\mathscr{A} = \{0, 1, \ldots, d-1\}$ and $\mathscr{A}^{\mathbb{N}_0}$ for the set of functions from the nonnegative integers to $\mathscr{A}$. We use the usual product topology on $\mathscr{A}^{\mathbb{N}_0}$ and let S denote the usual continuous shift on $\mathscr{A}^{\mathbb{N}_0}$. Define a map $\mathscr{E} : \Xi \to \mathscr{A}^{\mathbb{N}_0}$ by

$$\mathscr{E}(x) = a_0 a_1 a_2 \ldots, \tag{33}$$

where a_i are defined by the requirement that $g^i x \in [\xi_{a_i}, \xi_{a_i+1})$. $\mathscr{E}(x)$ is just the canonical d-adic expansion of x. Since each $x \in [0, 1)$ has a unique expansion of this type, $\mathscr{E}$ is injective. It is also obvious that $S \circ \mathscr{E} = \mathscr{E} \circ G$.

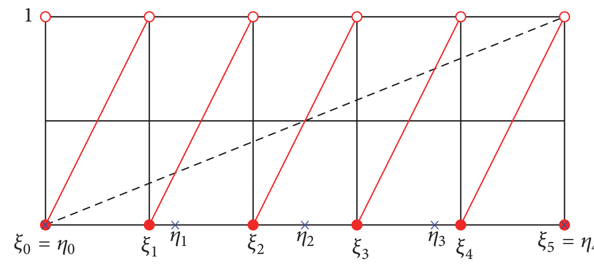

FIGURE 4: Plot of $G(x)$ when $d = 5$.

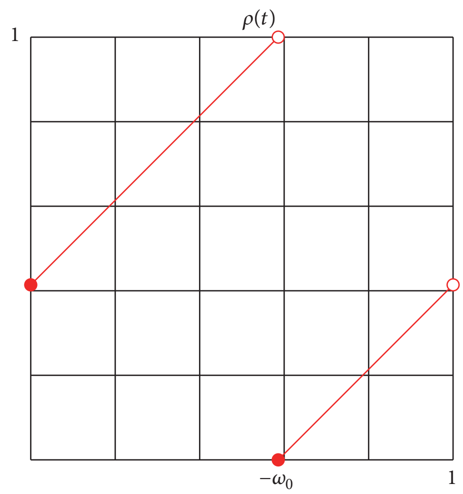

FIGURE 5: Plot of ρ.

Proposition 21. *The map $\mathscr{E}$ is a homeomorphism from Ξ to* $\mathrm{Im}(\mathscr{E})$.

Proof. Since Ξ is compact, it is enough to show that $\mathscr{E}$ is continuous on Ξ. Suppose $x_i, x \in \Xi$ and $x_i \to x$ as $i \to \infty$. For any n, we must have that $g^n x$ lies in an open interval of the form $\mathring{I}_j = (\xi_j, \xi_{j+1})$. Since g is continuous on $[0,1)\backslash\{\xi_0, \xi_1, \ldots, \xi_{d-1}\}$,

$$g^n x_i \longrightarrow g^n x \quad \text{as } i \longrightarrow \infty \tag{34}$$

for each $n \in \mathbb{N}_0$. This implies that the nth digit of x_i coincides with that of x for all sufficiently large i. Hence $\mathscr{E}$ is continuous. □

Let $\Xi_1, \ldots, \Xi_\ell$ be the nonempty sets in the list

$$\Xi \cap I_0, \Xi \cap I_1, \ldots, \Xi \cap I_{d-1}. \tag{35}$$

The indexing can be arranged so $\Xi_i \subset I_{k_i}$ for some $0 \le k_i < d$ with

$$k_1 < k_2 < \cdots < k_\ell. \tag{36}$$

Set

$$\alpha_i = \inf \Xi_i, \quad \beta_i = \sup \Xi_i. \tag{37}$$

Since Ξ is perfect, $\alpha_i < \beta_i$ for $i = 1, \ldots, \ell$ and $\Xi \cap (\alpha_i, \beta_i)$ is nonempty. Consequently, $\gamma(\Xi_i)$ is the closed interval $[\gamma(\alpha_i), \gamma(\beta_i)]$ and has positive length (see Proposition 14). Because μ has no mass on the open interval (β_i, α_{i+1}), $\gamma(\beta_i) = \gamma(\alpha_{i+1})$. Set $t_0 = 0$ and $t_i = \gamma(\beta_i)$ for $i = 1, \ldots, \ell$. Then,

$$0 = t_0 < t_1 < \cdots < t_\ell = 1, \quad \gamma(\Xi_i) = [t_{i-1}, t_i] \quad \text{for } i = 1, \ldots, \ell. \tag{38}$$

Recall a consequence of our previous analysis: every point $x \in \Xi$ with $g(x) > x$ must lie to the left of every point $y \in \Xi$ with $g(y) < y$. Thus, each Ξ_i must lie completely to one side of the sole fixed point within I_{k_i}. Moreover, if $\sup \Xi_i < \eta_{k_i}$ and $\inf \Xi_j > \eta_{k_j}$ then $i > j$ and $k_i > k_j$. Let m be the last index, j between 1 and ℓ with $\inf \Xi_j > \eta_{k_j}$. Clearly, $m < \ell$ and $\sup \Xi_m = \alpha'$ and $\inf \Xi_{m+1} = \beta'$. Hence, $t_m = \gamma(\alpha_{m+1}) = \gamma(\beta_m) = 1 - \omega_0$.

We next seek to understand the preimages of γ. Define $\rho : [0,1[\ \to [0,1[$ by

$$\rho(t) = \{t + \omega_0\}. \tag{39}$$

(See Figure 5.) Set

$$\mathfrak{D}_0 = \{t \in [0,1[: \rho^n(t) = \{t + n\omega_0\} \neq t_i \text{ for any } n \ge 0,\ i = 1, \ldots, \ell\}. \tag{40}$$

(Except for a countable subset, every $t \in [0,1[$ is in $\mathfrak{D}_0$.) For any such t, each member of the sequence of points $\rho^n(t) = \{t + n\omega_0\}$ with $n \geq 0$ lies in exactly one of the intervals of the form $[t_i, t_{i+1}[$. Write i_n for the index with this property:

$$\rho^n(t) \in \left[t_{i_n}, t_{i_n+1}\right[. \tag{41}$$

Then, for any x in $\gamma^{-1}(t)$,

$$g^n(x) \in \Xi \cap \left[\xi_{k_{i_n}}, \xi_{k_{i_{n+1}}}\right[\quad \forall n \geq 0. \tag{42}$$

This just says the string $k_{i_0}k_{i_1}k_{i_2}\cdots$ is the canonical d-adic expansion of x. Thus, there is a unique point in the preimage $\gamma^{-1}(t)$.

In summary, this discussion shows how any rotational subsystem (X, g) of the uniform d-fold cover of the unit circle must arise from the symbolic flow of an irrational rotation of $\mathbb{T}$ relative to a suitable partition. The next section shows that this process can be reversed.

5. The Inverse Process

In this section, we begin with an irrational number $\omega_0 \in [0,1[$ and a partition of $[0,1]$ into $\ell \leq d$ subintervals with the requirement that one of the interior nodes is $1-\omega_0$:

$$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} < \cdots < t_\ell = 1 \tag{43}$$

and $t_m = 1 - \omega_0$. Set $J_k = \,]t_k, t_{k+1}[$ for $k = 0, \ldots, \ell-1$. Next, select a coding that maps $\{0, \ldots, \ell-1\}$ to the set of digits $\{0, \ldots, d-1\}$. More precisely, choose integers $k_1, \ldots, k_\ell$ that satisfy

$$0 \leq k_1 < k_2 < \cdots < k_\ell < d. \tag{44}$$

We will show that this data determines a rotational subset of $\mathbb{T}$ that inverts the process described in previous section.

Let $\rho : [0,1[\;\to [0,1[$ be defined by $\rho(t) = \{t+\omega_0\}$. Write $r : \mathbb{T} \to \mathbb{T}$ for the corresponding rotation of $\mathbb{T}$ and note that $\chi \circ \rho = r \circ \chi$:

$$\begin{array}{ccc} [0,1[& \xrightarrow{\chi} & \mathbb{T} \\ \Big\downarrow \rho & & \Big\downarrow r \\ [0,1[& \xrightarrow{\chi} & \mathbb{T} \end{array} \tag{45}$$

Remark 22. It is well known that $(\mathbb{T}, r)$ is a minimal dynamical system. Hence, for any non-empty open $V \subseteq \mathbb{T}$ there is an $N > 0$ with the following property: for every $\theta \in \mathbb{T}$,

$$r^k(\theta) \in V \tag{46}$$

for at least one k with $0 \leq k \leq N$. (See, e.g., Proposition 4.7 in [12].)

It follows easily that the analogous statement holds for $([0,1[, \rho)$ and any non-empty open $U \subset \;]0,1[$. (Simply apply the previous observation to $V = \chi(U)$.)

As before, let $\mathfrak{D}_0$ be the set of $t \in [0,1[$ satisfying $\rho^n(t) = \{t + n\omega_0\} \neq t_i$ for all $n \in \mathbb{N}_0$ and $i = 0, \ldots, \ell$. In other words, $\mathfrak{D}_0$ consists of those points of $[0,1[\backslash\{t_0, t_1, t_2, \ldots, t_{\ell-1}\}$ whose forward orbit does not contain any of the nodes t_i. $\mathfrak{D}_0$ is invariant under the map ρ.

Proposition 23. *(i) The complement of $\mathfrak{D}_0$ in $[0,1[$ is countable.*

(ii) For every $t \in [0,1[$, there is an integer $n \geq 0$ with the property that $\rho^n(t) \in \mathfrak{D}_0$.

Proof. The map ρ is invertible. The complement of $\mathfrak{D}_0$ is just the countable set:

$$\left\{\rho^{-k} t_i : i = 1, \ldots, \ell,\; k \geq 0\right\}. \tag{47}$$

For the proof of the second claim, fix $t \in [0,1[$. If the orbit of t hits the $\{t_i : i = 0, \ldots, \ell\}$ infinitely often, then there must be an index i such that

$$\left\{t + n\omega_0\right\} = t_i \tag{48}$$

for infinitely many $n \in \mathbb{N}$. This means that there are two distinct, positive integers n_0, n_1 with the property that

$$\left(n_1 - n_0\right)\omega_0 = 0 \mod 1. \tag{49}$$

But this contradicts the condition that ω_0 is irrational. This argument proves that the forward orbit of such a t must eventually lie completely in $\mathfrak{D}_0$. □

The trajectory of any point $t \in \mathfrak{D}_0$ can be encoded by an infinite string:

$$a_0 a_1 a_2 \ldots, \tag{50}$$

where $a_n = k_i$ precisely when $\{\omega + n\omega_0\}$ is in J_i. The Kronecker approximation theorem implies that each of the digits k_i, $i = 0, \ldots, \ell-1$ occurs infinitely often. Thus,

$$E(\omega) = 0.a_0 a_1 a_2 \ldots \tag{51}$$

may be interpreted as the canonical d-adic expansion of a real number in the open unit interval $]0,1[$. So, $E : \mathfrak{D}_0 \to [0,1)$. Note that $E(\{\omega + \omega_0\})$ is the shift $0.a_1 a_2 \ldots$.

Proposition 24. *The map $E : \mathfrak{D}_0 \to [0,1[$ is a continuous, injective, strictly monotonic increasing and*

$$E\left(\rho(t)\right) = G\left(E(t)\right). \tag{52}$$

Proof. Kronecker's theorem also implies injectivity of E: Let t and t' be distinct points in $\mathfrak{D}_0$ with $t' > t$. Set $\Delta = t' - t$ and note that there must be an s with the property that s and $s' = s + \Delta$ lie in the interior of different J_i's. Note that if $|\rho^n t - s|$ is sufficiently small, then $|\rho^n t - s| = |\rho^n t' - s'|$. In particular, by choosing n so that $|\rho^n t - s|$ is sufficiently small, we may insure that $\rho^n t$ and $\rho^n t'$ will be in different J_i's. Hence the d-adic encodings of t and t' are different. Since neither of these encodings can end with an infinite string of $d-1$'s, $E(\mathrm{t}) \neq E(t')$.

Equation (52) follows directly from (51) and the ensuing discussion.

Let $t \in \mathfrak{D}_0$ and t_n be a sequence in $\mathfrak{D}_0$ that converges to t. Then, since $\rho^k t$ is in the interior of one of J_i, $\rho^k t_n$ must eventually have this property as well. Thus, E must be continuous.

It only remains to prove that E is monotone increasing. Let $t, t' \in \mathfrak{D}_0$ with $t < t'$. Write $E(t) = a_0 a_1 a_2 \ldots$ and $E(t') = a'_0 a'_1 a'_2 \ldots$. Because E is injective, there is a first index $i \geq 0$ for which $a_i \neq a'_i$. If $i = 0$, $t < t'$ implies $a_0 < a'_0$. This in turn yields $E(t) < E(t')$. If $i > 0$, then $a_{i-1} = a'_{i-1}$ entails that both $\rho^{i-1} t$ and $\rho^{i-1} t'$ are in interior of the same J_k. Since ρ is increasing on the interior of any J_n, we must have $a_i < a'_i$. As a consequence, $E(t) < E(t')$ in this case as well. □

Write Ξ_0 for the image of $\mathfrak{D}_0$ under the map E and let Ξ be its closure in $[0, 1[$. It is straightforward to check that Ξ_0 is invariant under G. Establishing this for its closure is complicated by the possibility that G might not be continuous on Ξ. Most of the effort in the proof of the next result is to rule this out.

Proposition 25. *Ξ is a compact subset of $]0, 1[$ and is invariant under G.*

Proof. By Remark 22, there is an N with the property that for each $t \in [0, 1[$, the set

$$\left\{\rho^i (t) : 0 \leq i \leq N\right\} \tag{53}$$

has nonempty intersection with the interiors of each of the intervals $J_0, \ldots, J_{\ell-1}$. Now consider the corresponding set of words. In particular, let $\mathscr{W}$ denote the set of all words in the alphabet $\{k_1, \ldots, k_\ell\}$ of length N for which each k_i occurs at least once. Let w and w' be the smallest and largest word, respectively, in the finite set $\mathscr{W}$. Let k be the first digit of w' that is not $d - 1$. Write v for the word obtained by replacing k with $k + 1$ in w'. The words w and v are lexicographically smaller and larger, respectively, than every element of Ξ_0. Write m and M for the numbers in $[0, 1[$ corresponding to w and v, noting that $0 < m < M < 1$. It is clear that $\Xi_0 \subseteq [m, M]$. Hence Ξ is compact in the open unit interval.

By examining G explicitly, it is easy to check that $G^{-1}[m, M]$ is a compact subset of $[0, 1[$ that does not contain any of the points $\xi_0, \ldots, \xi_{d-1}$. Since these are precisely the points of discontinuity of G, G is continuous on $G^{-1}[m, M]$. Because Ξ_0 is invariant, $G\Xi_0 \subseteq \Xi_0 \subset [m, M]$. Consequently, $\Xi = \overline{\Xi_0} \subset G^{-1}[m, M]$ and G is continuous on Ξ.

Let $x \in \Xi$. Then $x_n \to x$ for a sequence $x_n \in \Xi_0$. By the observations in the previous paragraph, $G(x_n) \to G(x)$. Since, Ξ_0 is G invariant, this proves that $G(x) \in \overline{\Xi_0} = \Xi$. □

As before, define $g : \Xi \to \Xi$ by $g = G|_\Xi$.

Theorem 26. *The dynamical system (Ξ, g) is rotational.*

Proof. We first prove the minimality of (Ξ, g).

Consider a point $x_0 \in \Xi_0$. An $y \in \Xi$ can be approximated to within an arbitrary accuracy by an element of $y_0 \in \Xi_0$. Write t_0, s_0 for the elements of $\mathfrak{D}_0$ with $E(t_0) = x_0$ and $E(s_0) = y_0$. By Kronecker's theorem, there is a sequence n_i of indices such that $\rho^{n_i}(t_0) \to s_0$. The continuity of E together with Proposition 24 yields

$$\lim_i G^{n_i} (x_0) = y_0. \tag{54}$$

Hence, y can be approximated to arbitrary accuracy by an element of the orbit, under g, of x_0. Thus, for any x_0 in Ξ_0, the orbit of x_0 is dense in Ξ.

Now let $x \in \Xi \backslash \Xi_0$. It suffices to prove that an element in the forward orbit of x is in Ξ_0. We may write x as a limit of a sequence x_i in Ξ_0. Write t_i for the elements of $[0, 1[$ with $E(t_i) = x_i$. By passing to a subsequence, if needed, we may assume that $\chi(t_i)$ converges to an element of $\mathbb{T}$, say $\chi(t)$. By Proposition 23, there is m such that $\rho^m(t) \in \mathfrak{D}_0$. By continuity of r, we have that $\chi(\rho^m(t_i)) = r^m(\chi(t_i)) \to r^m(\chi(t)) = \chi(\rho^m(t))$. Since $\rho^m(t) \in \mathfrak{D}_0$, $\chi(\rho^m(t)) \in \mathbb{T}\backslash\{0 \mod 1\}$. Since χ^{-1} is continuous on the latter set, $\rho^m(t_i) \to \rho^m(t)$. We may apply E to both sides of this last limit to get

$$\begin{aligned} E\rho^m t &= \lim_i E\left(\rho^m t_i\right) = \lim_i G^m \left(Et_i\right) = \lim_i G^m \left(x_i\right) \\ &= G^m (x). \end{aligned} \tag{55}$$

It follows that $G^m(x) \in \Xi_0$. Since the orbit of this point is dense in Ξ, so is the orbit of x.

The only point left is to prove that g respects cyclic order on Ξ. To this end, let x_0, x_1, and x_2 be an increasing triple of points in Ξ with distinct images under g. We may approximate each of these points by elements of Ξ_0 without changing the number of crossings in the arrow diagram. Hence, we may assume that x_0, x_1, and x_2 are in Ξ_0. Let t_0, t_1, and t_2 be the preimages under E of x_0, x_1, and x_2, respectively. The t_i's are in increasing order due to Proposition 24. This monotonicity property of E together with the identity in that proposition, also force the arrow diagram for t_0, t_1, and t_2 under the map ρ to have the same number of crossings as that for x_0, x_1, and x_2 under g. But it is easy to verify that ρ respects cyclic order. Hence, so does g. □

6. Examples for a Class of Continuous Maps

As the last two sections show, every irrational $\omega_0 \in [0.1)$ arises as the rotation number of some rotational system with respect to a standard cover of degree $d > 1$. (This is true for rational $\omega_0 \in [0, 1)$ as well.) It is tempting to ask if this is true for any continuous transformation of degree $d > 1$.

In this section, we look at mappings of the oriented unit circle, $K : \mathbb{T} \to \mathbb{T}$, whose lifts to the universal cover are monotonic increasing (though not strictly). The degree of K will be assumed to be at least 2. It is well known that such maps must have at least one fixed point. Conjugating by the appropriate rotation, we may arrange that 0 mod 1 is a fixed point. As before, we parameterize by the unit interval: $H = \chi^{-1} \circ K \circ \chi$. There is a partition

$$0 = s_0 < s_1 < \cdots < s_d = 1 \tag{56}$$

with the property that, for each $i = 0, \ldots, d - 1$,

(i) H is monotone increasing function on each half-open interval $[s_k, s_{k+1})$,

(ii) $H(s_k) = 0$ and $\lim_{x\to s_{k+1}^-} H(x) = 1$.

Applying a piecewise linear change of coordinates, we may assume that $s_i = i/d$ for $i = 0, \ldots, d$.

Next, we set up a standard symbolic encoding for H in terms of infinite strings of the alphabet $\mathscr{A} = \{0, 1, \ldots, d-1\}$. In particular, for each finite (nonempty) word $w = i_1 i_2 \cdots i_n$, let

$$A_w = \Big\{x \in [0,1[:\ \forall k = 1, \ldots, n,\ H^{k-1}(x) \in \big[s_{i_k}, s_{i_k+1}\big[\Big\}. \tag{57}$$

As in Section 4, denote by $\mathscr{A}^{\mathbb{N}_0}$ the set of infinite words in the alphabet $\mathscr{A}$ with the usual product topology. The shift on $S : \mathscr{A}^{\mathbb{N}_0} \to \mathscr{A}^{\mathbb{N}_0}$ defined by

$$S(i_0 i_1 i_2 \ldots) = i_1 i_2 \ldots. \tag{58}$$

is continuous with respect to this topology.

Definition 27. A finite word v is a *prefix* of a word w if $v \star u$ where $\star$ means concatenation of words.

We collect some useful remarks that are easy to check.

Remark 28. (i) A_w are half-open intervals that are closed on the left.

(ii) Let w and v be finite words. Suppose that w is a prefix for the word v. Then $A_v \subseteq A_w$. In addition, if $v_k \neq d-1$ for some $k > j$, then $\overline{A_v} \subset A_w$.

(iii) For any fixed w and $n \geq |w|$, the collection

$$\{A_v : |v| = n \text{ and } w \text{ is a prefix for } v\} \tag{59}$$

is a partition of A_w.

(iv) If $w = i_1 i_2 \ldots i_n$ and $w' = i_2 \ldots i_n$, then

$$x \in A_w \Longrightarrow Hx \in A_{w'}. \tag{60}$$

(v) If w and v are of the same length and w precedes v in lexicographical order, then

$$x \in A_w, \quad y \in A_v \Longrightarrow x < y. \tag{61}$$

Each point $x \in [0,1)$ determines a unique infinite word $\iota(x) \in \mathscr{A}^{\mathbb{N}_0}$. Specifically, $\iota(x) = i_0 i_1 i_2 \ldots$ if

$$H^n x \in \big[s_{i_n}, s_{i_n+1}\big) \quad \forall n \in \mathbb{N}_0. \tag{62}$$

By construction,

$$\iota \circ H = S \circ \iota. \tag{63}$$

Finally, for a given finite word $w = i_0 i_1 \ldots i_n$, the half-interval A_w consists precisely of those $x \in [0,1)$ with the property that w_n is a prefix for $\iota(x)$.

Remark 29. The lexicographical ordering on $\mathscr{A}^{\mathbb{N}_0}$ and the usual order on $[0,1[$ are also compatible with the factor map ι. In particular, for any $x, y \in [0,1[$,

(i) $x < y$ implies $\iota(x) \leq \iota(y)$,

(ii) $\iota(x) < \iota(y)$ implies $x < y$.

Thus, for any $w \in \mathscr{A}^{\mathbb{N}_0}$, the preimage $\iota^{-1}\{w\}$ is either empty, a singleton, or an interval of positive length. In particular, the cardinality of $\iota^{-1}\{w\}$ exceeds one for only countable many words $w \in \mathscr{A}^{\mathbb{N}_0}$.

Lemma 30. *Let x_n be a sequence in $[0,1)$ with the property that $\iota(x_k)$ converges in $\mathscr{A}^{\mathbb{N}_0}$ to w. Suppose, in addition, that the infinite word w does not end with an infinite sequence of $d-1$'s. Then any convergent subsequence x_{k_i} with a limit, say x, has the property that $x \in [0,1)$ with $\iota(x) = w$.*

Proof. Let $w = i_0 i_1 i_2 \ldots$ and set $w_n = i_0 i_1 \ldots i_n$ for each $n \in \mathbb{N}$. Since $\iota(x_i) \to w$, we have that, for each m, $x_k \in A_{w_m}$ for all sufficiently large k. By Remark 28 and the hypothesis on w, we have that for any n there is an $m > n$ with the property that

$$\overline{A_{w_m}} \subset A_{w_n}. \tag{64}$$

Thus, x must be in $A_{w_n} \subset [0,1)$ for each $n \in \mathbb{N}_0$. This just means that $\iota(x) = w$. □

Remark 31. If the infinite word $w \in \mathscr{A}^{\mathbb{N}_0}$ does not terminate with an infinite sequence of $d-1$'s, there is an $x \in [0,1[$ with the property that $\iota(x) = w$. To see this, first construct a sequence x_n by the requirement that $x_n \in A_{w_n}$ for each $n \in \mathbb{N}_0$. (Here w_n is defined as in the previous proof.) By construction, $\iota(x_n) \to w$ as $n \to \infty$. Let x be any limit point of this sequence in $\mathbb{R}$. By Lemma 30, $x \in [0,1)$ and $\iota(x) = w$.

The map ι is not continuous everywhere on $[0,1)$. However, one can characterize the points of continuity.

Lemma 32. *Let $x_0 \in [0,1)$ be a point with the property that $\iota(x_0)$ does not terminate with an infinite sequence of 0's. Then x_0 is a point of continuity for the map $\iota : [0,1) \to \mathscr{A}^{\mathbb{N}_0}$.*

Proof. Write $w = i_0 i_1 i_2 \ldots$ for $\iota(x_0)$ and let $w_n = i_0 i_1 \ldots i_n$. The hypothesis entails that x_0 does not lie on the left endpoint of any of the A_{w_n}. In other words, x_0 is in the interior of all the A_{w_n}. Suppose $x_k \to x_0$ as $k \to \infty$. For any $n \in \mathbb{N}$, $x_k \in A_{w_n}$ for all sufficiently large k. For such k, $\iota(x_k)$ will have w_n as a prefix. Since $n \in \mathbb{N}_0$ is arbitrary, $\iota(x_n) \to w = \iota(x_0)$ as $n \to \infty$. □

Now choose an irrational number $\omega_0 \in [0,1)$ and a compatible partition as in (43) of the last section. Let (X, f) be the rotational system constructed from this data and write (Ξ, g) for the associated system on $[0,1)$.

Let $Z \subset [0,1)$ consist of those points z with the property that the associated word $\iota(z)$ is in $\mathscr{E}(\Xi)$. The word $w = \mathscr{E}(x)$ associated with an $x \in \Xi$ cannot end with an infinite sequence of $d-1$'s; this would contradict the minimality of the infinite system (Ξ, g), for instance. Hence, by Remark 31, for any $x \in \Xi$ there is a $z \in Z$ with $\iota(z) = \mathscr{E}(x)$.

We next claim that Z is (sequentially) compact in $[0,1)$. Indeed, recall that $\mathscr{E}(\Xi)$ is compact (see Proposition 21). Let

z_n be any sequence in Z. By passing to a subsequence, we may assume that z_n converges in $\mathbb{R}$ and $\iota(z_n)$ converges to a $w \in \mathscr{E}(\Xi)$. By Lemma 30, we have that $z = \lim z_i$ satisfies $z \in [0, 1)$ and $\iota(z) = w$.

Another feature is that no $z \in Z$ can have the property that $\iota(z)$ ends with an infinite string of 0's. (The same statement holds for any $\mathscr{E}(x)$ where $x \in \Xi$.) Consequently, by Lemma 32, ι is continuous on Z.

Consider the composition $\psi = \mathscr{E}^{-1} \circ \iota$. The function $\psi : Z \to \Xi$ assigns to a $z \in Z$ the element in Ξ with the same symbolic representation. In view of Proposition 21 and the previous paragraph, ψ is continuous. It is also clear that it is a map between the dynamical systems $(Z, H|_Z)$ and (Ξ, g), that is, $\psi \circ H = g \circ \psi$.

Let $(Y, H|_Y)$ be a compact, minimal subsystem of (Z, H) and write $h = H|_Y$. The factor map ψ restricts to Y in a natural way: $\psi \circ h = g \circ \psi$. Since (Ξ, g) is minimal as well, ψ maps Y to Ξ surjectively. In particular, Y is infinite.

The only point left open is whether (Y, h) preserves cyclic order. We do not know if this is true in general, but it is true generically as will now be shown.

Proposition 33. *In the $d = 2$ case, (Y, h) is a rotational system for all irrational $\omega_0 \in [0, 1)$ except for a countable number of possible exceptions.*

Proof. In this case, the choice of ω_0 determines the partition in (43). We will write $(\Xi_{\omega_0}, h_{\omega_0})$ for the associated rotational system. Note that, for distinct choices ω_0 and ω_0', the corresponding sets, Ξ_{ω_0} and $\Xi_{\omega_0'}$ must be disjoint. Indeed, because of minimality, the orbit of any common point would have to be dense in both sets. This would imply that $\Xi_{\omega_0} = \Xi_{\omega_0'}$, which yields $\omega_0 = \omega_0'$.

Thus, the family of sets, $Y_{\omega_0} = \iota^{-1}(\mathscr{E}\Xi_{\omega_0})$, is an uncountable collection of disjoint sets. By Remark 29, ι is injective on all but countably many of Y_{ω_0}. For such ω_0, $h = h_\omega$ is a continuous bijection between compact sets. So h is a homeomorphism. Since h respects the ordering of $[0, 1)$, (Y, h) preserves cyclic order. □

The $d > 2$ case differs in that, for each ω_0, there are uncountably many choices for the partition, $\mathscr{P}$, described in (43). Let $(\Xi_{\omega_0,\mathscr{P}}, g)$ denote the rotational system uniquely associated with ω_0 and $\mathscr{P}$, according to the method of Section 5. For distinct partitions $\mathscr{P}$ and $\mathscr{P}'$ compatible with θ_0, $\Xi_{\omega_0,\mathscr{P}}$ and $\Xi_{\omega_0,\mathscr{P}'}$ are disjoint. Thus, ι will be injective on $\iota^{-1}(\mathscr{E}\Xi_{\omega_0,\mathscr{P}})$ for all but countably many of the partitions $\mathscr{P}$ compatible with ω_0. This yields the following.

Proposition 34. *If $d > 2$ and ω_0 is an irrational number in $[0, 1)$, there are uncountable many rotational systems (Y, h) with rotation number ω_0.*

Conflicts of Interest

The author declares that he has no conflicts of interest regarding the publication of this paper.

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