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Research Article

Existence and Attractivity Results for Coupled Systems of Nonlinear Volterra–Stieltjes Multidelay Fractional Partial Integral Equations

Saïd Abbas[1], **Mouffak Benchohra**[2], **Naima Hamidi**[2], **and Gaston N'Guérékata**[3]

[1]*Laboratory of Mathematics, Geometry, Analysis, Control and Applications, Tahar Moulay University of Saïda, P.O. Box 138, EN-Nasr, 20000 Saïda, Algeria*
[2]*Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbès, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria*
[3]*Department of Mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, MD 21252, USA*

Correspondence should be addressed to Gaston N'Guérékata; nguerekata@aol.com

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We are concerned with some existence and attractivity results of a coupled fractional Riemann–Liouville–Volterra–Stieltjes multidelay partial integral system. We prove the existence of solutions using Schauder's fixed point theorem; then we show that the solutions are uniformly globally attractive.

1. Introduction

Fractional integral and fractional differential equations are among the most fast growing field in mathematics. They are used to describe many phenomena, especially the ones with long memory. Examples include but are not limited to viscoelasticity, viscoplasticity, biochemistry, control theory, mathematical psychology, mechanics, modeling in complex media (porous, etc.), and electromagnetism [1–4]. In recent years, there has been a significant development in ordinary and partial fractional integral equations; see, for instance, the monographs of Abbas *et al.* [5–7], Agarwal *et al.* [8], Kilbas *et al.* [9], Miller and Ross [10], Podlubny [11], Samko *et al.* [12], and the papers [13–18] and the references therein.

In this paper we study the existence and attractivity of solutions to the following coupled system of nonlinear fractional Riemann–Liouville–Volterra–Stieltjes quadratic multidelay partial integral equations:

$$\begin{aligned} u_1(t,x) &= \mu_1(\alpha(t),x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t) - s)^{r_1-1}(x-y)^{r_2-1} \times f_1(t,x,s,y, \\ &\quad u_1(\gamma(s)-\tau_1, y-\xi_1), u_2(\gamma(s)-\tau_1, y-\xi_1), \dots, \\ &\quad u_1(\gamma(s)-\tau_m, y-\xi_m), \\ &\quad u_2(\gamma(s)-\tau_m, y-\xi_m))\,dy\,d_s g(t,s) \\ u_2(t,x) &= \mu_2(\alpha(t),x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t) - s)^{r_1-1}(x-y)^{r_2-1} \times f_2(t,x,s,y, \\ &\quad u_1(\gamma(s)-\tau_1, y-\xi_1), u_2(\gamma(s)-\tau_1, y-\xi_1), \dots, \\ &\quad u_1(\gamma(s)-\tau_m, y-\xi_m), \\ &\quad u_2(\gamma(s)-\tau_m, y-\xi_m))\,dy\,d_s g(t,s); \\ &\qquad (t,x) \in J, \end{aligned} \tag{1}$$

$$\begin{aligned} u_1(t,x) &= \Phi_1(t,x) \\ u_2(t,x) &= \Phi_2(t,x); \\ &\quad (t,x) \in \widetilde{J} := [-T,\infty)\times[-\xi,b] \setminus (0,\infty)\times(0,b], \end{aligned} \tag{2}$$

where $J := \mathbb{R}_+ \times [0,b]$, $b > 0$, $r_1, r_2 \in (0,\infty)$, $\mathbb{R}_+ = [0,\infty)$, $\tau_i, \xi_i \geq 0$, $i = 1,\ldots,m$, $T = \max_{i=1,\ldots,m}\{\tau_i\}$, $\xi = \max_{i=1,\ldots,m}\{\xi_i\}$, α, $\beta, \gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$, $\mu_j : J \longrightarrow \mathbb{R}$, $f_j : J' \times \mathbb{R}^{2m} \longrightarrow \mathbb{R}$, $j = 1,2$, are given continuous functions, $\lim_{t\longrightarrow\infty}\alpha(t) = \infty$, μ_j, $j = 1,2$, are bounded, $J' = \{(t,x,s,y) \in J^2 : s \leq t,\ y \leq x\}$, $\Phi_j : \tilde{J} \longrightarrow \mathbb{R}$, $j = 1,2$, are continuous and bounded functions with $\lim_{t\longrightarrow\infty}\Phi_j(t,x) = 0$, $x \in [-\xi,b]$, $\mu_j(\alpha(t),0) = \Phi_j(t,0)$ for each $t \in \mathbb{R}_+$ and $\mu_j(\alpha(0),x) = \Phi_j(0,x)$, for each $x \in [0,b]$, and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\zeta) = \int_0^\infty t^{\zeta-1}e^{-t}dt; \quad \zeta > 0. \tag{3}$$

2. Preliminaries

In this section, we recall some notations, definitions, and preliminary facts which will be used in this paper. $L^1([0,p]\times[0,q])$, $p,q > 0$, will denote the space of all Lebesgue-integrable functions $u : [0,p]\times[0,q] \longrightarrow \mathbb{R}$ equipped with the norm

$$\|u\|_1 = \int_0^p \int_0^q |u(t,x)|\,dx\,dt. \tag{4}$$

$BC := BC([-T,\infty)\times[-\xi,b])$ will denote the usual Banach space of all bounded and continuous functions from $[-T,\infty)\times[-\xi,b]$ into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC} = \sup_{(t,x)\in[-T,\infty)\times[-\xi,b]} |u(t,x)|. \tag{5}$$

It is clear that the product space $\mathscr{BC} := BC \times BC$ turns out to be a Banach space if equipped with the norm

$$\|(u_1,u_2)\|_{\mathscr{BC}} = \|u_1\|_{BC} + \|u_2\|_{BC}. \tag{6}$$

Definition 1 (see [19]). Let $r = (r_1,r_2) \in (0,\infty)\times(0,\infty)$, $\theta = (0,0)$ and $u \in L^1([0,p]\times[0,q])$. The left-sided mixed Riemann–Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \cdot \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1}u(\tau,s)\,ds\,d\tau, \tag{7}$$

provided the integral exists.

Example 2. Let $\lambda,\omega \in (0,\infty)$ and $r = (r_1,r_2) \in (0,\infty)\times(0,\infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1}x^{\omega+r_2}, \quad \text{for almost all } (t,x) \in [0,p]\times[0,q]. \tag{8}$$

If u is a real-valued function defined on the interval $[a,b]$, then we will use the symbol $\bigvee_a^b u$ to denote the variation of u on $[a,b]$. We say that u is of bounded variation on the interval $[a,b]$ whenever $\bigvee_a^b u$ is finite. If $w : [a,b]\times[c,d] \longrightarrow \mathbb{R}$, then the symbol $\bigvee_{t=p}^q w(t,s)$ indicates the variation of the function $t \longrightarrow w(t,s)$ on the interval $[p,q] \subset [a,b]$, where s is arbitrarily fixed in the interval $[c,d]$. Analogously we define $\bigvee_{s=p}^q w(t,s)$. For more details on the properties of functions of bounded variation we refer the reader to [20].

If u and φ are two real-valued functions defined on the interval $[a,b]$, then under some appropriate conditions (see [20]) we can define the Stieltjes integral (in the Riemann–Stieltjes sense)

$$\int_a^b u(t)\,d\varphi(t) \tag{9}$$

of the function u with respect to φ. In this case we say that u is Stieltjes integrable on $[a,b]$ with respect to φ. Several conditions are known to ensure Stieltjes integrability [20]. One of the most frequently used requires that u is continuous and φ is of bounded variation on $[a,b]$.

Now we recall a few properties of the Stieltjes integral included in the lemmas below.

Lemma 3 (see [20, 21]). *If u is Stieltjes integrable on the interval $[a,b]$ with respect to a function φ of bounded variation, then*

$$\left|\int_a^b u(t)\,d\varphi(t)\right| \leq \int_a^b |u(t)|\,d\left(\bigvee_a^t \varphi\right). \tag{10}$$

Lemma 4 (see [20, 21]). *Let u and v be Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function φ such that $u(t) \leq v(t)$ for $t \in [a,b]$. Then*

$$\int_a^b u(t)\,d\varphi(t) \leq \int_a^b v(t)\,d\varphi(t). \tag{11}$$

From now on, we will also consider Stieltjes integrals of the form

$$\int_a^b u(t)\,d_s g(t,s) \tag{12}$$

and Riemann–Liouville–Stieltjes integrals of fractional order of the form

$$\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}u(s)\,d_s g(t,s), \tag{13}$$

where $g : \mathbb{R}_+\times\mathbb{R}_+ \longrightarrow \mathbb{R}$, $r \in (0,\infty)$ and the symbol d_s indicates the integration with respect to s.

Let $\emptyset \neq \Omega \subset BC$, and let $G : \Omega \longrightarrow \Omega$, and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{14}$$

In light of the definition of the attractivity of solutions of integral equations (for instance, [15]), we will introduce the following concept of attractivity of solutions for (14).

Definition 5. A solutions of (14) is said to be locally attractive if there exists a ball $B(u_0,\eta)$ in the space BC such that, for arbitrary solutions $v = v(t,x)$ and $w = w(t,x)$ of (14) belonging to $B(u_0,\eta)\cap\Omega$, we have that, for each $x\in[0,b]$,

$$\lim_{t\longrightarrow\infty}(v(t,x)-w(t,x))=0. \quad (15)$$

When the limit (15) is uniform with respect to $B(u_0,\eta)\cap\Omega$, solutions of (14) are said to be uniformly locally attractive (or equivalently that solutions of (14) are locally asymptotically stable).

Definition 6 (see [15]). The solution $v = v(t,x)$ of (14) is said to be globally attractive if (15) holds for each solution $w = w(t,x)$ of (14). If condition (15) is satisfied uniformly with respect to the set Ω, solutions of (14) are said to be globally asymptotically stable (or uniformly globally attractive).

Lemma 7 (see [22], p. 62). *Let $D\subset BC$. Then D is relatively compact in BC if the following conditions hold:*

(a) *D is uniformly bounded in BC.*

(b) *The functions belonging to D are almost equicontinuous on $[-T,\infty)\times[-\xi,b]$, i.e., equicontinuous on every compact subset of $[-T,\infty)\times[-\xi,b]$.*

(c) *The functions from D are equiconvergent; that is, given $\epsilon>0$, $x\in[-\xi,b]$, there corresponds $\lambda(\epsilon,x)>0$ such that $|u(t,x)-\lim_{t\longrightarrow\infty}u(t,x)|<\epsilon$ for any $t\geq\lambda(\epsilon,x)$ and $u\in D$.*

3. Existence and Attractivity Results

Definition 8. By a solution to problem (1)-(2), we mean every coupled functions $(u,v)\in\mathscr{BC}$ such that (u,v) satisfies (1) on J and (2) on $\tilde{J}$.

We will use the following assumptions in the sequel:

(H_1) There exist positive functions $p_j\in BC$; $j=1,2$ such that

$$(1+\alpha(t))\left|\mu_j(\alpha(t),x)\right|\leq p_j(t,x);\quad (t,x)\in J. \quad (16)$$

(H_2) For all $t_1,t_2\in\mathbb{R}_+$ such that $t_1<t_2$, the function $s\longmapsto g(t_2,s)-g(t_1,s)$ is nondecreasing on $\mathbb{R}_+$.

(H_3) The function $s\longmapsto g(0,s)$ is nondecreasing on $\mathbb{R}_+$.

(H_4) The functions $s\longmapsto g(t,s)$ and $t\longmapsto g(t,s)$ are continuous on $\mathbb{R}_+$ for each fixed $t\in\mathbb{R}_+$ or $s\in\mathbb{R}_+$, respectively.

(H_5) There exist continuous functions $q_{ji}:J'\longrightarrow\mathbb{R}_+$; $i=1,\ldots,m$, $j=1,2$ such that

$$\left(1+\sum_{i=1}^{m}\left(|u_{1i}|+|u_{2i}|\right)\right)\cdot\left|f_j(t,x,s,y,u_{11},u_{21},\ldots,u_{1m},u_{2m})\right|\leq\sum_{i=1}^{m}\left(q_{1i}(t,x,s,y)|u_{1i}|+q_{2i}(t,x,s,y)|u_{2i}|\right); \quad (17)$$

for $(t,x,s,y)\in J'$, $u_{1i},u_{2i}\in\mathbb{R}$; $i=1,\ldots,m$. Moreover, assume that

$$\lim_{t\longrightarrow\infty}\int_0^{\beta(t)}(\beta(t)-s)^{r_1-1}q_{ji}(t,x,s,y)\,d_sg(t,s)=0;\quad i=1,\ldots,m,\ j=1,2. \quad (18)$$

Remark 9. Set $\Phi_j^*:=\sup_{(t,x)\in\tilde{J}}\Phi_j(t,x)$, $p_j^*:=\sup_{(t,x)\in J}p_j(t,x)$,

$$q_{ji}^*:=\sup_{(t,x)\in J}\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^{x}(\beta(t)-s)^{r_1-1}\cdot(x-y)^{r_2-1}q_{ji}(t,x,s,y)\,dy\,d_s\left(\bigvee_{k=0}^{s}g(t,k)\right); \quad (19)$$

for $i=1,\ldots,m$ and $j=1,2$. From the above assumptions, we infer that Φ_j^*, p_j^*, q_{ji}^* are finite.

Theorem 10. *Assume that hypotheses $(H_1)-(H_5)$ hold. Then problem (1)-(2) has at least one solution in the space $\mathscr{BC}$. Moreover, solutions to problem (1)-(2) are uniformly globally attractive.*

Proof. Define the operators $N_j: BC\longrightarrow BC$; $j=1,2$ by

$$\begin{aligned}
\left(N_ju_j\right)(t,x)&=\Phi_j(t,x);\quad (t,x)\in\tilde{J},\\
\left(N_ju_j\right)(t,x)&=\mu_j(\alpha(t),x)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\cdot\int_0^{\beta(t)}\int_0^{x}(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\times f_j(t,x,s,y,\\
&\quad u_1(\gamma(s)-\tau_1,y-\xi_1),u_2(\gamma(s)-\tau_1,y-\xi_1),\ldots,\\
&\quad u_1(\gamma(s)-\tau_m,y-\xi_m),\\
&\quad u_2(\gamma(s)-\tau_m,y-\xi_m))\,dy\,d_sg(t,s);\quad (t,x)\in J,
\end{aligned} \quad (20)$$

and consider the operator $N:\mathscr{BC}\longrightarrow\mathscr{BC}$ such that, for any $(u_1,u_2)\in\mathscr{BC}$,

$$(N(u_1,u_2))(t,x)=((N_1u_1)(t,x),(N_2u_2)(t,x)). \quad (21)$$

From the hypotheses above, we deduce that $N(u)$ is continuous on $[-T,\infty)\times[-\xi,b]$. Now let us prove that $N(u_1,u_2)\in$

$\mathscr{BC}$ for any $u_j \in BC$; $j = 1, 2$. For arbitrarily fixed $(t, x) \in J$, we have

$$\begin{aligned}
&\left|\left(N_j u_j\right)(t,x)\right| = \left|\mu(\alpha(t),x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\right. \\
&\cdot \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times f_j(t,x,s,y,u_1(\gamma(s)-\tau_1,y-\xi_1),u_2(\gamma(s)-\tau_1,y \\
&-\xi_1),\dots,u_1(\gamma(s)-\tau_m,y-\xi_m),u_2(\gamma(s)-\tau_m,y \\
&\left.-\xi_m))\,dy\,d_s g(t,s)\right| \le \frac{p_j(t,x)}{1+\alpha(t)} \\
&+\left|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\right. \\
&\times \sum_{i=1}^m \left(q_{1i}(t,x,s,y)\left|u_1(\gamma(s)-\tau_i,y-\xi_i)\right|\right. \\
&\left.+q_{2i}(t,x,s,y)\left|u_2(\gamma(s)-\tau_i,y-\xi_i)\right|\right)dy\,d_s \\
&\left.\cdot g(t,s)\right| \le p_j(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t) \\
&-s)^{r_1-1}(x-y)^{r_2-1}\times\sum_{i=1}^m (q_{1i}(t,x,s,y) \\
&+q_{2i}(t,x,s,y))\,dy\,d_s\left(\bigvee_{k=0}^s g(t,k)\right) \\
&\le p_j^* + \sum_{i=1}^m q_{1i}^* + q_{2i}^*,
\end{aligned} \tag{22}$$

and for all $(t, x) \in \tilde{J}$ and each $u_j \in BC$, $j = 1, 2$, we have

$$\left|\left(N_j u_j\right)(t,x)\right| = \left|\Phi_j(t,x)\right| \le \Phi_j^*. \tag{23}$$

Thus,

$$\left\|N_j(u_j)\right\|_{BC} \le \max\left\{\Phi_j^*, p_j^* + \sum_{i=1}^m q_{ji}^*\right\} := \eta_j; \quad j = 1,2. \tag{24}$$

Hence

$$\left\|N(u_1,u_2)\right\|_{\mathscr{BC}} \le \eta_1 + \eta_2 := \eta. \tag{25}$$

Therefore $N(u) \in BC$. The problem of finding the solutions of the coupled system (1)-(2) is reduced to finding the solutions of the operator equation $N(u_1, u_2) = (u_1, u_2)$. From (25), we infer that N transforms the ball $B_\eta := \{(u_1, u_2) \in \mathscr{BC} : \|(u_1, u_2)\|_{\mathscr{BC}} \le \eta\}$ into itself. Now we will show that $N : B_\eta \longrightarrow B_\eta$ satisfies the Schauder's fixed point theorem [23]. The proof will be presented in several steps and cases.

Step 1 (N is continuous). Let $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \longrightarrow u$ and $v_n \longrightarrow v$ in B_η. Then, for each $(t, x) \in [-T, \infty) \times [-\xi, b]$, we have

$$\begin{aligned}
&\left|\left(N_1 u_n\right)(t,x) - \left(N_1 u\right)(t,x)\right| \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
&\cdot\int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\times\left|f_1(t,x,s,y,\right. \\
&u_n(\gamma(s)-\tau_1,y-\xi_1),v_n(\gamma(s)-\tau_1,y-\xi_1),\dots, \\
&u_n(\gamma(s)-\tau_m,y-\xi_m),v_n(\gamma(s)-\tau_m,y-\xi_m)) \\
&-f_1(t,x,s,y,u(\gamma(s)-\tau_1,y-\xi_1), \\
&v(\gamma(s)-\tau_1,y-\xi_1),\dots,u(\gamma(s)-\tau_m,y-\xi_m), \\
&\left.v(\gamma(s)-\tau_m,y-\xi_m))\right|dy\,d_s\left(\bigvee_{k=0}^s g(t,k)\right).
\end{aligned} \tag{26}$$

Case 1. Assume that $(t, x) \in \tilde{J} \cup ([0, a] \times [0, b])$; $a > 0$, then, since $(u_n, v_n) \longrightarrow (u, v)$ as $n \longrightarrow \infty$ and f_1, g, γ are continuous, (26) implies

$$\left\|N(u_n) - N(u)\right\|_{\mathscr{BC}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{27}$$

Case 2. Let $(t, x) \in (a, \infty) \times [0, b]$; $a > 0$, then from (H_5) and (26) we obtain

$$\begin{aligned}
&\left|\left(N_1 u_n\right)(t,x) - \left(N_1 u\right)(t,x)\right| \le \frac{2}{\Gamma(r_1)\Gamma(r_2)} \\
&\cdot\int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times\sum_{i=1}^m (q_{1i}(t,x,s,y)+q_{2i}(t,x,s,y))\,dy\,d_s \\
&\cdot\left(\bigvee_{k=0}^s g(t,k)\right) \le \sum_{i=1}^m \frac{2}{\Gamma(r_1)\Gamma(r_2)} \\
&\cdot\int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\
&\times(q_{1i}(t,x,s,y)+q_{2i}(t,x,s,y))\,dy\,d_s \\
&\cdot\left(\bigvee_{k=0}^s g(t,k)\right).
\end{aligned} \tag{28}$$

Since $t \longrightarrow \infty$, then (28) gives

$$\left\|N_1(u_n) - N_1(u)\right\|_{BC} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{29}$$

Let us show that N_2 is continuous in the same way as continuity of N_1.

Step 2 ($N(B_\eta)$ is uniformly bounded). This fact is obvious because $N(B_\eta) \subset B_\eta$ and B_η is a bounded set.

Step 3 ($N(B_\eta)$ is equicontinuous on every compact subset $[0,a] \times [0,b]$ of J, $a > 0$). Let $(t_1, x_1), (t_2, x_2) \in [0,a] \times [0,b]$, $t_1 < t_2$, $x_1 < x_2$, and let $(u, v) \in B_\eta$. Without loss of generality, let us assume that $\beta(t_1) \le \beta(t_2)$. Then we obtain

$$\begin{aligned}
&\left|(N_1u)(t_2,x_2) - (N_1u)(t_1,x_1)\right| \le \left|\mu_1(\alpha(t_2),x_2) - \mu_1(\alpha(t_1),x_1)\right| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t_2)} \int_0^{x_2} (\beta(t_2) \\
&- s)^{r_1-1}(x_2-y)^{r_2-1} \times \left|f_1(t_2,x_2,s,y,u(\gamma(s)-\tau_1,y - \xi_1), v(\gamma(s)-\tau_1,y-\xi_1),\dots,u(\gamma(s)-\tau_m,y \right.\\
&- \xi_m), v(\gamma(s)-\tau_m,y-\xi_m)) - f_1(t_1,x_1,s,y, u(\gamma(s)-\tau_1,y-\xi_1), v(\gamma(s)-\tau_1,y-\xi_1),\dots,\\
&\left. u(\gamma(s)-\tau_m,y-\xi_m), v(\gamma(s)-\tau_m,y - \xi_m))\right| dy\, d_s\left(\bigvee_{k=0}^{s} g(t_2,k)\right) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\beta(t_1)}^{\beta(t_2)} \int_0^{x_2} (\beta(t_2)-s)^{r_1-1}(x_2 - y)^{r_2-1} \times \left|f_1(t_1,x_1,s,y,u(\gamma(s)-\tau_1,y-\xi_1)\right., \\
&v(\gamma(s)-\tau_1,y-\xi_1),\dots,u(\gamma(s)-\tau_m,y-\xi_m), \left. v(\gamma(s)-\tau_m,y-\xi_m))\right| dy\, d_s\left(\bigvee_{k=0}^{s} g(t_2,k)\right) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \cdot \int_0^{\beta(t_1)} \int_0^{x_1} \left|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} - (\beta(t_1)-s)^{r_1-1}(x_1-y)^{r_2-1}\right| \times \left|f_1(t_1,x_1,s,y,\right. \\
&u(\gamma(s)-\tau_1,y-\xi_1), v(\gamma(s)-\tau_1,y-\xi_1),\dots, u(\gamma(s)-\tau_m,y-\xi_m), v(\gamma(s)-\tau_m,y \\
&\left. - \xi_m))\right| dy\, d_s\left(\bigvee_{k=0}^{s} g(t_1,k)\right) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \cdot \int_0^{\beta(t_1)} \int_{x_1}^{x_2} \left|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}\right| \times \left|f_1(t_1,\right. \\
&x_1,s,y,u(\gamma(s)-\tau_1,y-\xi_1), v(\gamma(s)-\tau_1,y-\xi_1),\dots, u(\gamma(s)-\tau_m,y-\xi_m), v(\gamma(s)-\tau_m,y \\
&\left. - \xi_m))\right| dy\, d_s\left(\bigvee_{k=0}^{s} g(t_1,k)\right).
\end{aligned} \tag{30}$$

Thus

$$\begin{aligned}
&\left|(N_1u)(t_2,x_2) - (N_1u)(t_1,x_1)\right| \le \left|\mu_1(\alpha(t_2),x_2) - \mu_1(\alpha(t_1),x_1)\right| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t_2)} \int_0^{x_2} (\beta(t_2) \\
&- s)^{r_1-1}(x_2-y)^{r_2-1} \times \left|f_1(t_2,x_2,s,y,u(\gamma(s)-\tau_1,y - \xi_1), v(\gamma(s)-\tau_1,y-\xi_1),\dots,u(\gamma(s)-\tau_m,y\right. \\
&- \xi_m), v(\gamma(s)-\tau_m,y-\xi_m)) - f_1(t_1,x_1,s,y, u(\gamma(s)-\tau_1,y-\xi_1), v(\gamma(s)-\tau_1,y-\xi_1),\dots,\\
&u(\gamma(s)-\tau_m,y-\xi_m), v(\gamma(s)-\tau_m,y \left. - \xi_m))\right| dy\, d_s\left(\bigvee_{k=0}^{s} g(t_2,k)\right) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\beta(t_1)}^{\beta(t_2)} \int_0^{x_2} (\beta(t_2)-s)^{r_1-1}(x_2 - y)^{r_2-1} \times \sum_{i=1}^{m} (q_{1i}(t,x,s,y) + q_{2i}(t,x,s,y))\, dy\, d_s \\
&\cdot \left(\bigvee_{k=0}^{s} g(t_2,k)\right) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \cdot \int_0^{\beta(t_1)} \int_0^{x_1} \left|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} - (\beta(t_1)-s)^{r_1-1}(x_1-y)^{r_2-1}\right| \times \sum_{i=1}^{m} (q_{1i}(t,x,s,y) \\
&+ q_{2i}(t,x,s,y))\, dy\, d_s\left(\bigvee_{k=0}^{s} g(t_1,k)\right) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \cdot \int_0^{\beta(t_1)} \int_{x_1}^{x_2} \left|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}\right| \times \sum_{i=1}^{m} (q_{1i}(t,x,s,y) + q_{2i}(t,x,s,y))\, dy\, d_s \\
&\cdot \left(\bigvee_{k=0}^{s} g(t_1,k)\right)
\end{aligned} \tag{31}$$

Using continuity of the functions $\mu_1, \alpha, \beta, \gamma, f_1, g, h, q_i$, $i = 1, \dots m$, and since $t_1 \longrightarrow t_2$ and $x_1 \longrightarrow x_2$, the right-hand side of the above inequality tends to zero. The equicontinuity of N_1 for the cases $t_1 < t_2 < 0$, $x_1 < x_2 < 0$ and $t_1 \leq 0 \leq t_2$, $x_1 \leq 0 \leq x_2$ is immediate.

We can also prove that

$$\begin{aligned}
&\left|(N_2(u,v))(t_2,x_2) - (N_2(u,v))(t_1,x_1)\right| \\
&\leq \left|\mu_2(\alpha(t_2),x_2) - \mu_2(\alpha(t_1),x_1)\right| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
&\cdot \int_0^{\beta(t_2)}\int_0^{x_2} (\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} \times \left|f_2(t_2,\right. \\
&x_2,s,y,u(\gamma(s)-\tau_1,y-\xi_1),v(\gamma(s)-\tau_1,y-\xi_1),\ldots, \\
&u(\gamma(s)-\tau_m,y-\xi_m),v(\gamma(s)-\tau_m,y-\xi_m)) \\
&- f_2(t_1,x_1,s,y,u(\gamma(s)-\tau_1,y-\xi_1),v(\gamma(s)-\tau_1,y \\
&-\xi_1),\ldots,u(\gamma(s)-\tau_m,y-\xi_m),v(\gamma(s)-\tau_m,y \\
&\left.-\xi_m))\right| dy\, d_s\left(\bigvee_{k=0}^{s} g(t_2,k)\right) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2 \\
&-y)^{r_2-1}\times\sum_{i=1}^{m}(q_{1i}(t,x,s,y)+q_{2i}(t,x,s,y))\,dy\,d_s \\
&\cdot\left(\bigvee_{k=0}^{s} g(t_2,k)\right) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
&\cdot\int_0^{\beta(t_1)}\int_0^{x_1}\left|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}\right. \\
&\left.-(\beta(t_1)-s)^{r_1-1}(x_1-y)^{r_2-1}\right|\times\sum_{i=1}^{m}(q_{1i}(t,x,s,y) \\
&+q_{2i}(t,x,s,y))\,dy\,d_s\left(\bigvee_{k=0}^{s} g(t_1,k)\right) \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
&\cdot\int_0^{\beta(t_1)}\int_{x_1}^{x_2}\left|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}\right| \\
&\times\sum_{i=1}^{m}(q_{1i}(t,x,s,y)+q_{2i}(t,x,s,y))\,dy\,d_s \\
&\cdot\left(\bigvee_{k=0}^{s} g(t_1,k)\right)\longrightarrow 0, \quad as\ t_1\longrightarrow t_2,\ x_1\longrightarrow x_2.
\end{aligned} \tag{32}$$

Hence

$$\begin{aligned}
&\left|(N(u,v))(t_2,x_2)-(N(u,v))(t_1,x_1)\right| \\
&\leq\left|(N_1u)(t_2,x_2)-(N_1u)(t_1,x_1)\right| \\
&\quad+\left|(N_2v)(t_2,x_2)-(N_2v)(t_1,x_1)\right|\longrightarrow 0, \\
&\qquad as\ t_1\longrightarrow t_2,\ x_1\longrightarrow x_2.
\end{aligned} \tag{33}$$

Step 4 ($N(B_\eta)$ is equiconvergent). Let $(t,x)\in J$ and $u\in B_\eta$, then we get

$$\begin{aligned}
&\left|(N(u,v))(t,x)\right| \leq \left|\mu_1(\alpha(t),x)\right| + \left|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\right. \\
&\cdot\int_0^{\beta(t)}\int_0^{x}(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\times f_1(t,x,s,y, \\
&u(\gamma(s)-\tau_1,y-\xi_1),v(\gamma(s)-\tau_1,y-\xi_1),\ldots, \\
&u(\gamma(s)-\tau_m,y-\xi_m), \\
&\left.v(\gamma(s)-\tau_m,y-\xi_m))\,dy\,d_s g(t,s)\right| \\
&+\left|\mu_2(\alpha(t),x)\right|+\left|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\right. \\
&\cdot\int_0^{\beta(t)}\int_0^{x}(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\times f_2(t,x,s,y, \\
&u(\gamma(s)-\tau_1,y-\xi_1),v(\gamma(s)-\tau_1,y-\xi_1),\ldots, \\
&u(\gamma(s)-\tau_m,y-\xi_m), \\
&\left.v(\gamma(s)-\tau_m,y-\xi_m))\,dy\,d_s g(t,s)\right| \\
&\leq\frac{p_1(t,x)}{1+\alpha(t)}+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^{x}(\beta(t)-s)^{r_1-1} \\
&\cdot(x-y)^{r_2-1}\times\sum_{i=1}^{m}(q_{1i}(t,x,s,y) \\
&+q_{2i}(t,x,s,y))\,dy\,d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&+\frac{p_2(t,x)}{1+\alpha(t)}+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^{x}(\beta(t)-s)^{r_1-1} \\
&\cdot(x-y)^{r_2-1}\times\sum_{i=1}^{m}(q_{1i}(t,x,s,y) \\
&+q_{2i}(t,x,s,y))\,dy\,d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&\leq\frac{p_1^*}{1+\alpha(t)}+\frac{p_2^*}{1+\alpha(t)}+\sum_{i=1}^{m}\frac{2}{\Gamma(r_1)\Gamma(r_2)} \\
&\cdot\int_0^{\beta(t)}\int_0^{x}(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\times(q_{1i}(t,x,s, \\
&y)+q_{2i}(t,x,s,y))\,dy\,d_s\left(\bigvee_{k=0}^{s} g(t,k)\right).
\end{aligned} \tag{34}$$

Now, since $\alpha(t) \longrightarrow \infty$ as $t \longrightarrow \infty$, we conclude that, for each $x \in [0, b]$, we obtain

$$|(N(u,v))(t,x)| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{35}$$

Also, for each $x \in [-\xi, 0]$, we get

$$|(N(u,v))(t,x)| \leq |\Phi_1(t,x)| + |\Phi_2(t,x)| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{36}$$

Then, for each $x \in [-\xi, b]$, we get

$$|(N(u,v))(t,x)| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{37}$$

Hence,

$$|(N(u,v))(t,x) - (N(u,v))(+\infty,x)| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{38}$$

In view of Steps 1 to 4, along with the Lemma 7, we deduce that $N : B_\eta \longrightarrow B_\eta$ is continuous and compact. From an application of Schauder's theorem [23], we conclude that N has a fixed point (u, v) which is a solution of the coupled system (1)-(2).

Step 5 (the uniform global attractivity of solutions). Now let us study the stability of solutions of the coupled system (1)-(2). Let (u_1, u_2) and (v_1, v_2) be two solutions of (1)-(2). Then, for each $(t, x) \in [-T, \infty) \times [-\xi, b]$, we obtain

$$\begin{aligned}
&|(u_1,u_2)(t,x) - (v_1,v_2)(t,x)| = |(N(u_1,u_2))(t,x) \\
&\quad - (N(v_1,v_2))(t,x)| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
&\quad \cdot \int_0^{\beta(t)} \int_0^x (\beta(t) - s)^{r_1 - 1} (x - y)^{r_2 - 1} \times |f_1(t,x,s,y, \\
&\quad u_1(\gamma(s) - \tau_1, y - \xi_1), u_2(\gamma(s) - \tau_1, y - \xi_1), \ldots, \\
&\quad u_1(\gamma(s) - \tau_m, y - \xi_m), u_2(\gamma(s) - \tau_m, y - \xi_m)) \\
&\quad - f_1(t,x,s,y, v_1(\gamma(s) - \tau_1, y - \xi_1), \\
&\quad v_2(\gamma(s) - \tau_1, y - \xi_1), \ldots, v_1(\gamma(s) - \tau_m, y - \xi_m), \\
&\quad v_2(\gamma(s) - \tau_m, y - \xi_m))| \, dy \, d_s g(t,s) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t) - s)^{r_1 - 1} (x - y)^{r_2 - 1} \\
&\quad \times |f_2(t,x,s,y, u_1(\gamma(s) - \tau_1, y - \xi_1), \\
&\quad u_2(\gamma(s) - \tau_1, y - \xi_1), \ldots, u_1(\gamma(s) - \tau_m, y - \xi_m), \\
&\quad u_2(\gamma(s) - \tau_m, y - \xi_m)) - f_2(t,x,s,y, \\
&\quad v_1(\gamma(s) - \tau_1, y - \xi_1), v_2(\gamma(s) - \tau_1, y - \xi_1), \ldots, \\
&\quad v_1(\gamma(s) - \tau_m, y - \xi_m), \\
&\quad v_2(\gamma(s) - \tau_m, y - \xi_m))| \, dy \, d_s g(t,s).
\end{aligned} \tag{39}$$

Thus

$$\begin{aligned}
&|(u_1,u_2)(t,x) - (v_1,v_2)(t,x)| = |(N(u_1,u_2))(t,x) \\
&\quad - (N(v_1,v_2))(t,x)| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \\
&\quad \cdot \int_0^{\beta(t)} \int_0^x (\beta(t) - s)^{r_1 - 1} (x - y)^{r_2 - 1} \times |f_1(t,x,s,y, \\
&\quad u_1\gamma(s) - \tau_1, y - \xi_1), u_2\gamma(s) - \tau_1, y - \xi_1), \ldots, \\
&\quad u_1(\gamma(s) - \tau_m, y - \xi_m), u_2(\gamma(s) - \tau_m, y - \xi_m)) \\
&\quad - f_1(t,x,s,y, v_1(\gamma(s) - \tau_1, y - \xi_1), \\
&\quad v_2(\gamma(s) - \tau_1, y - \xi_1), \ldots, v_1(\gamma(s) - \tau_m, y - \xi_m), \\
&\quad v_2(\gamma(s) - \tau_m, y - \xi_m))| \, dy \, d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t) - s)^{r_1 - 1} (x - y)^{r_2 - 1} \\
&\quad \times |f_2(t,x,s,y, u_1\gamma(s) - \tau_1, y - \xi_1), u_2\gamma(s) - \tau_1, y \\
&\quad - \xi_1), \ldots, u_1(\gamma(s) - \tau_m, y - \xi_m), u_2(\gamma(s) - \tau_m, y \\
&\quad - \xi_m)) - f_2(t,x,s,y, v_1(\gamma(s) - \tau_1, y - \xi_1), \\
&\quad v_2(\gamma(s) - \tau_1, y - \xi_1), \ldots, v_1(\gamma(s) - \tau_m, y - \xi_m), \\
&\quad v_2(\gamma(s) - \tau_m, y - \xi_m))| \, dy \, d_s\left(\bigvee_{k=0}^{s} g(t,k)\right).
\end{aligned} \tag{40}$$

Hence

$$\begin{aligned}
&|(u_1,u_2)(t,x) - (v_1,v_2)(t,x)| = |(N(u_1,u_2))(t,x) \\
&\quad - (N(v_1,v_2))(t,x)| \leq \frac{2}{\Gamma(r_1)\Gamma(r_2)} \\
&\quad \cdot \int_0^{\beta(t)} \int_0^x (\beta(t) - s)^{r_1 - 1} (x - y)^{r_2 - 1} \\
&\quad \times \sum_{i=1}^{m} (q_{1i}(t,x,s,y) + q_{2i}(t,x,s,y)) \, dy \, d_s \\
&\quad \cdot \left(\bigvee_{k=0}^{s} g(t,k)\right) \leq \frac{2}{\Gamma(r_1)\Gamma(r_2)} \\
&\quad \cdot \sum_{i=1}^{m} \int_0^{\beta(t)} \int_0^x (\beta(t) - s)^{r_1 - 1} (x - y)^{r_2 - 1} \\
&\quad \times (q_{1i}(t,x,s,y) + q_{2i}(t,x,s,y)) \, dy \, d_s \\
&\quad \cdot \left(\bigvee_{k=0}^{s} g(t,k)\right).
\end{aligned} \tag{41}$$

By using (41) and (H_5), we obtain

$$\lim_{t \longrightarrow \infty} |(u_1,u_2)(t,x) - (v_1,v_2)(t,x)| = 0. \tag{42}$$

Therefore, all solutions of the coupled system (1)-(2) are uniformly globally attractive.

Let $\overline{BC} := BC^n$ (product space) be the Banach space equipped with the following norm:

$$\left\|(u_1, u_2, \ldots, u_n)\right\|_{\overline{BC}} = \sum_{k=1}^{n} \left\|u_k\right\|_{BC}. \tag{43}$$

□

From the above theorem, we deduce the following consequence.

Corollary 11. *Consider the system of nonlinear fractional Riemann–Liouville–Volterra–Stieltjes quadratic multidelay partial integral equations of the form*

$$\begin{aligned}
u_1(t,x) &= \mu_1(\alpha(t),x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \cdot \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \times f_1(t,x,s,y, \\
&\quad u_1(\gamma(s)-\tau_1, y-\xi_1), u_2(\gamma(s)-\tau_1, y-\xi_1), \ldots, \\
&\quad u_n(\gamma(s)-\tau_1, y-\xi_1), \ldots, u_1(\gamma(s)-\tau_m, y-\xi_m), \\
&\quad u_2(\gamma(s)-\tau_m, y-\xi_m), \ldots, \\
&\quad u_n(\gamma(s)-\tau_m, y-\xi_m))\, dy\, d_s g(t,s) \\
u_2(t,x) &= \mu_2(\alpha(t),x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \cdot \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \times f_2(t,x,s,y, \\
&\quad u_1(\gamma(s)-\tau_1, y-\xi_1), u_2(\gamma(s)-\tau_1, y-\xi_1), \ldots, \\
&\quad u_n(\gamma(s)-\tau_1, y-\xi_1), \ldots, u_1(\gamma(s)-\tau_m, y-\xi_m), \\
&\quad u_2(\gamma(s)-\tau_m, y-\xi_m), \ldots, \\
&\quad u_n(\gamma(s)-\tau_m, y-\xi_m))\, dy\, d_s g(t,s) \\
&\quad \vdots \\
u_n(t,x) &= \mu_n(\alpha(t),x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \cdot \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \times f_n(t,x,s,y, \\
&\quad u_1(\gamma(s)-\tau_1, y-\xi_1), u_2(\gamma(s)-\tau_1, y-\xi_1), \ldots, \\
&\quad u_n(\gamma(s)-\tau_1, y-\xi_1), \ldots, u_1(\gamma(s)-\tau_m, y-\xi_m), \\
&\quad u_2(\gamma(s)-\tau_m, y-\xi_m), \ldots, \\
&\quad u_n(\gamma(s)-\tau_m, y-\xi_m))\, dy\, d_s g(t,s); \quad (t,x) \in J,
\end{aligned} \tag{44}$$

$$\begin{aligned}
u_1(t,x) &= \Phi_1(t,x) \\
u_2(t,x) &= \Phi_2(t,x) \\
&\vdots \\
u_n(t,x) &= \Phi_n(t,x); \\
&(t,x) \in \widetilde{J} := [-T,\infty)\times[-\xi,b] \setminus (0,\infty)\times(0,b],
\end{aligned} \tag{45}$$

where $r_1, r_2 \in (0,\infty)$, $\tau_i, \xi_i \geq 0$; $i = 1,\ldots,m$, $T = \max_{i=1,\ldots,m}\{\tau_i\}$, $\xi = \max_{i=1,\ldots,m}\{\xi_i\}$, α, β, $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $g : \mathbb{R}_+\times\mathbb{R}_+ \longrightarrow \mathbb{R}$, $\mu_j : J \longrightarrow \mathbb{R}$, $f_j : J'\times\mathbb{R}^{2m} \longrightarrow \mathbb{R}$, $j = 1,\ldots,n$ *are given continuous functions,* $\lim_{t\longrightarrow\infty}\alpha(t) = \infty$, μ_j; $j = 1,\ldots,n$ *are bounded,* $J' = \{(t,x,s,y) \in J^2 : s \leq t,\ y \leq x\}$, $\Phi_j : \widetilde{J} \longrightarrow \mathbb{R}$; $j = 1,2$ *are continuous and bounded functions with* $\lim_{t\longrightarrow\infty}\Phi_j(t,x) = 0$; $x \in [-\xi,b]$, $\mu_j(\alpha(t),0) = \Phi_j(t,0)$ *for each* $t \in \mathbb{R}_+$ *and* $\mu_j(\alpha(0),x) = \Phi_j(0,x)$; *for each* $x \in [0,b]$.

Suppose that (H_2)–(H_4) *and the following assumptions are verified:*

(H_1') *There exist positive functions* $p_j \in BC$, $j = 1,\ldots,n$, *such that*

$$(1+\alpha(t))\left|\mu_j(\alpha(t),x)\right| \leq p_j(t,x); \quad (t,x) \in J. \tag{46}$$

(H_2') *There exist continuous functions* $q_{ji} : J' \longrightarrow \mathbb{R}_+$, $i = 1,\ldots,m$, $j = 1,\ldots,n$, *such that*

$$\begin{aligned}
&\left(1+\sum_{i=1}^{m}\sum_{j=1}^{n}\left|u_{ji}\right|\right)\left|f_j(t,x,s,y,u_{11},u_{21},\ldots,u_{n1},\ldots,\right. \\
&\quad \left. u_{1m},u_{2m},\ldots,u_{nm},u_{nm})\right| \leq \sum_{i=1}^{m}\sum_{j=1}^{n} q_{ji}(t,x,s,y)\left|u_{ji}\right|;
\end{aligned} \tag{47}$$

for $(t,x,s,y) \in J'$, $u_{ji} \in \mathbb{R}$, $i = 1,\ldots,m$, $j = 1,\ldots,n$. *Moreover, assume that*

$$\begin{aligned}
&\lim_{t\longrightarrow\infty}\int_0^{\beta(t)}(\beta(t)-s)^{r_1-1} q_{ji}(t,x,s,y)\, d_s g(t,s) = 0; \\
&\qquad i = 1,\ldots,m,\ j = 1,\ldots,n.
\end{aligned} \tag{48}$$

Then problem (44)-(45) has at least one solution in the space $\overline{BC}$. *In addition, the solutions are uniformly globally attractive.*

4. An Example

To illustrate our results, we consider the following coupled system of nonlinear fractional order Riemann–Liouville–Volterra–Stieltjes quadratic multidelay partial integral equation

$$\begin{aligned}
u_1(t,x) &= \frac{x^2 e^{-t}}{1+t} + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t) \\
&\quad - s)^{r_1-1}(x-y)^{r_2-1} \times f_1\Big(t,x,s,y, \\
&\quad u_1\Big(\gamma(s)-\frac{1}{2}, y-1\Big), u_1\Big(\gamma(s)-\frac{1}{4}, y-2\Big), \\
&\quad u_2\Big(\gamma(s)-\frac{1}{2}, y-1\Big), \\
&\quad u_2\Big(\gamma(s)-\frac{1}{4}, y-2\Big)\Big)\, dy\, d_s g(t,s)
\end{aligned}$$

$$u_2(t,x) = \frac{x^2 e^{-t}}{1+t} + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t) - s)^{r_1-1}(x-y)^{r_2-1} \times f_2\Big(t,x,s,y, u_1\Big(\gamma(s)-\frac{1}{2}, y-1\Big), u_1\Big(\gamma(s)-\frac{1}{4}, y-2\Big), u_2\Big(\gamma(s)-\frac{1}{2}, y-1\Big), u_2\Big(\gamma(s)-\frac{1}{4}, y-2\Big)\Big)\,dy\,d_s g(t,s); \quad (t,x)\in J, \tag{49}$$

$$u_1(t,x) = x^2 e^{-t}$$
$$u_2(t,x) = \frac{x^2}{1+t^2}; \tag{50}$$
$$(t,x)\in \tilde{J} := \Big[-\frac{1}{2},\infty\Big)\times[-2,1] \setminus (0,\infty)\times(0,1],$$

where $J := \mathbb{R}_+ \times [0,1]$, $r_1 = 1/4$, $r_2 = 1/2$, $\alpha(t) = \beta(t) = \gamma(t) = t$; $t \in \mathbb{R}_+$,

$$\mu(\alpha(t),x) = \frac{x^2 e^{-t}}{1+t}; \quad (t,x)\in\mathbb{R}_+\times[0,1],$$

$$f_1(t,x,s,y,u_1,u_2,v_1,v_2) = \frac{xs^{-3/4}(|u_1|+|u_2|+|v_1|+|v_2|)\sin\sqrt{t}\sin s}{(1+y^2+t^2)(1+|u_1|+|u_2|+|v_1|+|v_2|)};$$
$$\text{if } (t,x,s,y)\in J',\ s\neq 0,\ y\in[0,1],\ u_1,u_2,v_1,v_2\in\mathbb{R}, \tag{51}$$
$$f_1(t,x,0,y,u_1,u_2,v_1,v_2) = 0; \quad \text{if } (t,x)\in J,\ y\in[0,1],\ u_1,u_2,v_1,v_2\in\mathbb{R},$$
$$f_2(t,x,s,y,u_1,u_2,v_1,v_2) = \frac{xs^{-3/4}(|u_1|+|u_2|+|v_1|+|v_2|)\sin\sqrt{t}e^{-s}}{1+|u_1|+|u_2|+|v_1|+|v_2|};$$

$$\text{if } (t,x,s,y)\in J',\ y\in[0,1] \text{ and } u_1,u_2,v_1,v_2\in\mathbb{R},$$
$$J' = \{(t,x,s,y)\in J^2 : s\le t,\ y\le x\},$$
$$g(t,s) = s, \quad (t,s)\in\mathbb{R}_+^2. \tag{52}$$

First, we can see that $\lim_{t\to\infty}\alpha(t) = \infty$ and $\lim_{t\to\infty}\Phi_j(t,x) = 0$; $j = 1,2$. Next, the assumption (H_1) is satisfied with $p_j(t,x) = x^2e^{-t}$ and consequently $p_j^* = 1$. Also, it is clear that the function g satisfies assumptions $(H_2)-(H_4)$.

Finally, the functions f_j, $j = 1,2$, satisfy the assumption (H_5). Indeed, f_j are continuous and satisfy the inequality

$$|f_j(t,x,s,y,u_1,u_2,v_1,v_2)| \le \frac{q_1(t,x,s,y)(|u_1|+|u_2|)+q_2(t,x,s,y)(|v_1|+|v_2|)}{1+|u_1|+|u_2|+|v_1|+|v_2|}; \tag{53}$$

$(t,x,s,y)\in J'$, $u_1,u_2,v_1,v_2\in\mathbb{R}$. Also, we have

$$q_1(t,x,s,y) = \frac{xs^{-3/4}\sin\sqrt{t}\sin s}{1+y^2+t^2}; \quad (t,x,s,y)\in J',\ y\in[0,1],\ s\neq 0, \tag{54}$$
$$q_1(t,x,0,y) = 0; \quad (t,x)\in J,\ y\in[0,1].$$

and

$$q_2(t,x,s,y) = xs^{-3/4}\sin\sqrt{t}e^{-s}; \quad (t,x,s,y)\in J',\ y\in[0,1]. \tag{55}$$

For $i = 1,2$, we have also

$$\Big|\int_0^t (t-s)^{r_1-1} q_i(t,x,s,y)\,d_s g(t,s)\Big| \le \int_0^t (t-s)^{-3/4} xs^{-3/4}|\sin\sqrt{t}\sin s|\,d_s \cdot \Big(\bigvee_{k=0}^{s} g(t,k)\Big) \le x|\sin\sqrt{t}|\int_0^t (t-s)^{-3/4}s^{-3/4}ds \le \frac{x\Gamma^2(1/4)}{\sqrt{\pi}}\Big|\frac{\sin\sqrt{t}}{\sqrt{t}}\Big| \le \frac{x\Gamma^2(1/4)}{\sqrt{\pi t}} \longrightarrow 0 \quad \text{as } t\longrightarrow\infty, \tag{56}$$

and

$$q_i^* := \sup_{(t,x)\in J}\frac{1}{\Gamma(r_1)\Gamma(r_2)} \cdot \int_0^t\int_0^x (t-s)^{r_1-1}(x-y)^{r_2-1} q_i(t,x,s,y)\,dy\,d_s \cdot \Big(\bigvee_{k=0}^{s} g(t,k)\Big) \le \sup_{(t,x)\in J}\frac{x\Gamma(1/4)}{\pi}\Big|\frac{\sin\sqrt{t}}{\sqrt{t}}\Big| = \frac{\Gamma(1/4)}{\pi}. \tag{57}$$

Consequently, Theorem 10 implies that the coupled system (49)-(50) has a solution defined on $[-1/2,\infty)\times[-2,1]$; moreover solutions of this system are uniformly globally attractive.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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