Research Article

On the Rate of Convergence of P-Iteration, SP-Iteration, and D-Iteration Methods for Continuous Nondecreasing Functions on Closed Intervals

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We introduce a new iterative method called D-iteration to approximate a fixed point of continuous nondecreasing functions on arbitrary closed intervals. The purpose is to improve the rate of convergence compared to previous work. Specifically, our main result shows that D-iteration converges faster than P-iteration and SP-iteration to the fixed point. Consequently, we have that D-iteration converges faster than the others under the same computational cost. Moreover, the analogue of their convergence theorem holds for D-iteration.

1. Introduction

Let $E \subseteq \mathbb{R}$ be a closed interval. Define $f : E \longrightarrow E$ to be a continuous mapping. A point $p \in E$ is said to be a fixed point of f if $f(p) = p$. The set of all fixed points of f is denoted by $F(f)$. It is a well-known fact that f has a fixed point if the interval E is bounded. A popular way of finding a fixed point of f is an iterative method.

In 1953, Mann [1] proposed an iteration, Mann iteration, defined by $u_1 \in E$ and

$$u_{n+1} = (1 - \alpha_n) u_n + \alpha_n f(u_n) \qquad (1)$$

where $\alpha_n \in [0, 1]$ and $n \in \mathbb{N}$. Then, a two-step iteration, Ishikawa iteration [2], was introduced in 1974 and defined by $p_1 \in E$ and

$$q_n = (1 - \beta_n) p_n + \beta_n f(p_n)$$
$$p_{n+1} = (1 - \alpha_n) p_n + \alpha_n f(q_n) \qquad (2)$$

$\alpha_n, \beta_n \in [0, 1]$ and $n \in \mathbb{N}$. Two years later, Rhoades [3] showed that Mann and Ishikawa iterations converge for the class of continuous nondecreasing functions on a unit closed interval. Next, Borwein and Borwein [4] proved that Mann iteration converges for the class of continuous mappings on a bounded closed interval in 1991. In 2000, Noor [5] introduced a new three-step iterative method, Noor iteration, defined by $v_1 \in E$ and

$$r_n = (1 - \gamma_n) v_n + \gamma_n f(v_n)$$
$$o_n = (1 - \beta_n) v_n + \beta_n f(r_n) \qquad (3)$$
$$v_{n+1} = (1 - \alpha_n) v_n + \alpha_n f(o_n)$$

where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ and $n \in \mathbb{N}$. Then, Qing and Qihou [6] extended the results of Rhoades [3] and Borwein and Borwein [4] to the class of continuous functions on an arbitrary interval in 2006. On top of that, a necessary and sufficient condition for the convergence of Ishikawa iteration on an arbitrary interval was provided.

In 2011, Phuengrattana and Suantai [7] introduced an iteration, called SP-iteration, defined by $h_1 \in E$ and

$$b_n = (1 - \gamma_n) h_n + \gamma_n f(h_n)$$
$$l_n = (1 - \beta_n) b_n + \beta_n f(b_n) \quad (4)$$
$$h_{n+1} = (1 - \alpha_n) l_n + \alpha_n f(l_n)$$

where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ and $n \in \mathbb{N}$. In addition, the convergence of this three-step iteration holds for continuous functions on an arbitrary interval. Moreover, they showed that SP-iteration converges faster than Mann, Ishikawa, and Noor iterations for the class of continuous nondecreasing functions.

Two years later, Kosol [8] studied the convergence of S-iteration [9] for the class of continuous nondecreasing functions on a closed interval. S-iteration was first introduced by Agarwal et al. [9] and defined by $d_1 \in E$ and

$$e_n = (1 - \beta_n) d_n + \beta_n f(d_n)$$
$$d_{n+1} = (1 - \alpha_n) f(d_n) + \alpha_n f(e_n) \quad (5)$$

where $\alpha_n, \beta_n \in [0, 1]$ and $n \in \mathbb{N}$. In 2015, Sainuan [10] constructed a new iteration, called P-iteration, and showed that this iteration converges faster than S-iteration for the class of continuous nondecreasing functions. P-iteration is defined by $w_1 \in E$ and

$$t_n = (1 - \gamma_n) w_n + \gamma_n f(w_n)$$
$$s_n = (1 - \beta_n) t_n + \beta_n f(t_n) \quad (6)$$
$$w_{n+1} = (1 - \alpha_n) f(t_n) + \alpha_n f(s_n)$$

where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ and $n \in \mathbb{N}$.

Motivated by the above results, we define D-iteration by $x_1 \in E$ and

$$z_n = (1 - \gamma_n) x_n + \gamma_n f(x_n)$$
$$y_n = (1 - \beta_n) f(x_n) + \beta_n f(z_n) \quad (7)$$
$$x_{n+1} = (1 - \alpha_n) f(z_n) + \alpha_n f(y_n)$$

where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ and $n \in \mathbb{N}$.

In this work, we give a necessary and sufficient condition for the convergence of D-iteration. Then, we show that D-iteration converges faster than other iterations for the class of continuous nondecreasing functions. Also, numerical examples are provided to support our result.

2. Convergence Theorem

In this section, we provide the convergence theorem of D-iteration for the class of continuous nondecreasing functions on an arbitrary closed interval. First, we begin with the following lemma.

Lemma 1. Let $f : E \longrightarrow E$ be a continuous nondecreasing function, and let (x_n) be a sequence defined by (7).

(i) If $f(x_1) < x_1$, then $f(x_n) \leq x_n$ for all $n \in \mathbb{N}$ and (x_n) is nonincreasing.

(ii) If $f(x_1) > x_1$, then $f(x_n) \geq x_n$ for all $n \in \mathbb{N}$ and (x_n) is nondecreasing.

Proof. (i) Assume that $f(x_1) < x_1$. We will show that $f(x_n) \leq x_n$ for all $n \in \mathbb{N}$ by induction on n. Clearly, this is true for $n = 1$. Assume that $f(x_k) \leq x_k$ for some $k \geq 1$. From (7), we have that $f(x_k) \leq z_k \leq x_k$. Since f is nondecreasing, $f(z_k) \leq f(x_k) \leq z_k$. By the definition of y_k, $f(z_k) \leq y_k \leq z_k$. Then, $f(y_k) \leq f(z_k) \leq y_k$ since f is nondecreasing. Similarly, $f(y_k) \leq x_{k+1} \leq y_k$, and finally, we obtain that $f(x_{k+1}) \leq f(y_k) \leq x_{k+1}$. Thus, $f(x_n) \leq x_n$ for all $n \in \mathbb{N}$. Moreover, by the proof above, we have that $x_{k+1} \leq x_k$ for all $k \in \mathbb{N}$. Therefore, (x_n) is nonincreasing.

(ii) The proof can be done similarly as in (i). □

Theorem 2. Let $f : E \longrightarrow E$ be a continuous nondecreasing function, and let (x_n) be a sequence defined by (7), where $\lim_{n \to \infty} \alpha_n = 0$ and $\lim_{n \to \infty} \gamma_n = 0$. Then, (x_n) is bounded if and only if it converges to a fixed point of f.

Proof. Assume that (x_n) is bounded. First, we will show that it is convergent. If $f(x_1) = x_1$, by (7), we obtain that $x_n = x_1$ for all $n \in \mathbb{N}$. Therefore, (x_n) is convergent. Suppose that $f(x_1) \neq x_1$. From Lemma 1, we have that (x_n) is either nonincreasing or nondecreasing. Since (x_n) is bounded, it follows that (x_n) is convergent. Assume that (x_n) converges to p for some $p \in E$. Next, we will show that p is a fixed point of f. Since f is continuous and (x_n) is bounded, we have that $(f(x_n))$ is bounded and so are $(z_n), (f(z_n)), (y_n)$ and $(f(y_n))$. Note that $z_n = x_n + \gamma_n(f(x_n) - x_n) \longrightarrow p$ since $\gamma_n \longrightarrow 0$.

From (7), we obtain that

$$x_{n+1} - x_n = (f(z_n) - z_n) + \gamma_n(f(x_n) - x_n) + \alpha_n(f(y_n) - f(z_n)) \quad (8)$$

Since $\alpha_n \longrightarrow 0, \gamma_n \longrightarrow 0$, and f is continuous, we obtain that

$$0 = p - p = \lim_{n \to \infty}(x_{n+1} - x_n) = \lim_{n \to \infty} f(z_n) - z_n$$
$$= f(p) - p. \quad (9)$$

Therefore, $f(p) = p$.

Conversely, if (x_n) is convergent, then it is obvious that (x_n) is bounded. □

Consequently, one can see from Theorem 2 that D-iteration always converges to a fixed point of f, where f is a continuous nondecreasing function defined on a bounded closed interval.

Corollary 3. Let $f : [a, b] \longrightarrow [a, b]$ be a continuous nondecreasing function, and let (x_n) be a sequence defined by (7), where $\lim_{n \to \infty} \alpha_n = 0$ and $\lim_{n \to \infty} \gamma_n = 0$. Then, (x_n) converges to a fixed point of f.

3. Rate of Convergence

To prove our main theorem, we first define how to compare the rate of convergence between two iteration methods and then give some useful lemmas to accomplish our result.

Definition 4. Let $f : E \longrightarrow E$ be a continuous function, and let (x_n) and (w_n) be two iterations which converge to the same point $p \in E$. Then (x_n) is said to converge faster than (w_n) if $|x_n - p| \le |w_n - p|$ for all $n \in \mathbb{N}$.

Lemma 5. Let $f : E \longrightarrow E$ be a continuous nondecreasing function, and let (x_n) be a sequence defined by (7). Assume that there exists a point $p \in F(f)$.

  (i) If $x_1 > p$, then $x_n \ge p$ for all $n \in \mathbb{N}$.
  (ii) If $x_1 < p$, then $x_n \le p$ for all $n \in \mathbb{N}$.

Proof. (i) Let $x_1 > p$. We will show by induction that $x_n \ge p$ for all $n \in \mathbb{N}$. It is clear that this is true for the case $n = 1$. Assume that $x_k \ge p$ for some $k \ge 1$. Since f is nondecreasing, $f(x_k) \ge p$. By the definition of z_k, we have that

$$z_k = (1 - \gamma_k) x_k + \gamma_k f(x_k) \ge p. \tag{10}$$

Thus, $f(z_k) \ge p$. Similarly,

$$y_k = (1 - \beta_k) f(x_k) + \beta_k f(z_k) \ge p. \tag{11}$$

Therefore, $f(y_k) \ge p$. From (7), we obtain that

$$x_{k+1} = (1 - \alpha_k) f(z_k) + \alpha_k f(y_k) \ge p. \tag{12}$$

Hence, $x_n \ge p$ for all $n \in \mathbb{N}$.
  (ii) By using the same proof as in (i), we are done. □

Lemma 6. Let $f : E \longrightarrow E$ be a continuous nondecreasing function, and let (h_n), (w_n), and (x_n) be sequences defined by (4), (6), and (7), respectively, where $h_1 = w_1 = x_1 \in E$.

  (i) If $f(x_1) < x_1$, then $x_n \le w_n \le h_n$ for all $n \in \mathbb{N}$.
  (ii) If $f(x_1) > x_1$, then $x_n \ge w_n \ge h_n$ for all $n \in \mathbb{N}$.

Proof. (i) Let $f(x_1) < x_1$. First, we show that $x_n \le w_n$ for all $n \in \mathbb{N}$ by induction. It is obvious that this inequality holds for the case $n = 1$. Assume that $x_k \le w_k$ for some $k \ge 1$. Since f is nondecreasing, $f(x_k) \le f(w_k)$. Since $f(x_1) < x_1$, by Lemma 1(i), $f(x_k) \le x_k$. It follows that $f(x_k) \le z_k$ by the definition of z_k. From iterations (6) and (7), we have that

$$z_k - t_k = (1 - \gamma_k)(x_k - w_k) + \gamma_k (f(x_k) - f(w_k))$$
$$\le 0. \tag{13}$$

Thus, $z_k \le t_k$. Therefore, $f(z_k) \le f(t_k)$. Then,

$$y_k - s_k = (1 - \beta_k)(f(x_k) - t_k)$$
$$+ \beta_k (f(z_k) - f(t_k)) \le 0. \tag{14}$$

That is, $y_k \le s_k$ which implies $f(y_k) \le f(s_k)$. Consider

$$x_{k+1} - w_{k+1} = (1 - \alpha_k)(f(z_k) - f(t_k))$$
$$+ \alpha_k (f(y_k) - f(s_k)) \le 0. \tag{15}$$

We obtain that $x_{k+1} \le w_{k+1}$. By induction, we can conclude that $x_n \le w_n$ for all $n \in \mathbb{N}$. Next, we show that $w_n \le h_n$ for all $n \in \mathbb{N}$ by induction. It is clear that this is true for the case $n = 1$. Assume that $w_k \le h_k$ for some $k \ge 1$. Then $f(w_k) \le f(h_k)$. Since $f(h_1) < h_1$, we have that $f(h_k) \le h_k$ (see [7] Lemma 3.2 (vii)). From (4), $f(h_k) \le b_k \le h_k$. Since f is nondecreasing, $f(b_k) \le f(h_k) \le b_k$. By the definition of l_k, $f(b_k) \le l_k$. By (4) and (6), we have that

$$t_k - b_k = (1 - \gamma_k)(w_k - h_k) + \gamma_k (f(w_k) - f(h_k))$$
$$\le 0. \tag{16}$$

Thus, $t_k \le b_k$. Since f is nondecreasing, $f(t_k) \le f(b_k) \le l_k$. Then,

$$s_k - l_k = (1 - \beta_k)(t_k - b_k) + \beta_k (f(t_k) - f(b_k)) \le 0. \tag{17}$$

That is, $s_k \le l_k$. Therefore, $f(s_k) \le f(l_k)$. Consider

$$w_{k+1} - h_{k+1} = (1 - \alpha_k)(f(t_k) - l_k)$$
$$+ \alpha_k (f(s_k) - f(l_k)) \le 0. \tag{18}$$

We have that $w_{k+1} \le h_{k+1}$. By induction, we can conclude that $w_n \le h_n$ for all $n \in \mathbb{N}$.
  (ii) By using similar arguments as in (i) together with Lemma 1(ii) and Lemma 3.2 (viii) in [7], we are done. □

Proposition 7. Let $f : E \longrightarrow E$ be a continuous nondecreasing function such that $F(f)$ is nonempty and bounded. If $x_1 > \sup F(f)$ and $f(x_1) > x_1$, then (x_n) defined by (7) does not converge to a fixed point of f.

Proof. Assume that $x_1 > \sup F(f)$ and $f(x_1) > x_1$. Then, by Lemma 1(ii), (x_n) is nondecreasing. Since $x_1 > \sup F(f)$, it follows that (x_n) does not converge to a fixed point of f. □

Proposition 8. Let $f : E \longrightarrow E$ be a continuous nondecreasing function such that $F(f)$ is nonempty and bounded. If $x_1 < \inf F(f)$ and $f(x_1) < x_1$, then (x_n) defined by (7) does not converge to a fixed point of f.

Proof. Assume that $x_1 < \inf F(f)$ and $f(x_1) < x_1$. Then, by Lemma 1(i), (x_n) is nonincreasing. Since $x_1 < \inf F(f)$, (x_n) does not converge to a fixed point of f. □

In 2011, Phuengrattana and Suantai [7] compared the rate of convergence of Mann, Ishikawa, and Noor iterations with SP-iteration. Four years later, Sainuan [10] studied the rate of convergence between P-iteration and S-iteration. Their results are concluded as the following.

Theorem 9 (see [7, 10]). Let $f : E \longrightarrow E$ be a continuous nondecreasing function such that $F(f)$ is nonempty and bounded. For the same initial point and $p \in F(f)$, the following are satisfied.

  (i) Ishikawa iteration converges to p if and only if Mann iteration converges to p. Moreover, Ishikawa iteration converges faster than Mann iteration.

(ii) *Noor iteration converges to p if and only if Ishikawa iteration converges to p. Moreover, Noor iteration converges faster than Ishikawa iteration.*

(iii) *SP-iteration converges to p if and only if Noor iteration converges to p. Moreover, SP-iteration converges faster than Noor iteration.*

(iv) *If S-iteration converges to p, then P-iteration converges to p. Moreover, P-iteration converges faster than S-iteration.*

Remark 10. From Theorem 9, one can conclude that SP-iteration is better than Noor, Ishikawa, and Mann iterations. However, one can come to the different conclusion if we take the computational cost into consideration. As mentioned in [11] Remark 3.3, SP-iteration is exactly three-step Mann iteration. Thus, Mann iteration converges faster than Noor iteration and also Ishikawa iteration under the same computational cost because Ishikawa iteration is a special case of Noor iteration.

Next, we compare the rate of convergence of D-iteration with SP-iteration and P-iteration.

Theorem 11. *Let $f : E \longrightarrow E$ be a continuous nondecreasing function such that $F(f)$ is nonempty and bounded, and let $p \in F(f)$. Let (h_n), (w_n), and (x_n) be sequences defined by (4), (6), and (7), respectively, where $h_1 = w_1 = x_1 \in E$. Then, the following are satisfied.*

(i) *If P-iteration (w_n) converges to p, then D-iteration (x_n) converges to p. Moreover, D-iteration converges faster than P-iteration.*

(ii) *If SP-iteration (h_n) converges to p, then P-iteration (w_n) converges to p. Moreover, P-iteration converges faster than SP-iteration.*

Proof. (i) Assume that P-iteration (w_n) converges to p. Note that if $w_1 = p$, then we are done. Assume that $w_1 \neq p$. Consider the following two cases.

Case 1 ($w_1 > p$). If $f(x_1) > x_1$, then, by the proof of Proposition 3.6 in [10], it follows that (w_n) does not converge to p which leads to a contradiction. Thus, $f(x_1) < x_1$. Using Lemma 5(i) and Lemma 6(i), we obtain that $p \leq x_n \leq w_n$ for all $n \in \mathbb{N}$. This implies $|x_n - p| \leq |w_n - p|$ for all $n \in \mathbb{N}$. By the assumption, we have that (x_n) converges to p. Furthermore, we also have that D-iteration (x_n) converges faster than P-iteration (w_n).

Case 2 ($w_1 < p$). Similarly, $f(x_1) > x_1$ since if $f(x_1) < x_1$, then (w_n) does not converge to p by the proof of Proposition 3.5 in [10]. Then, by Lemmas 5(ii) and 5(ii), we obtain that $w_n \leq x_n \leq p$ for all $n \in \mathbb{N}$. This implies that $|x_n - p| \leq |w_n - p|$ for all $n \in \mathbb{N}$. Therefore, D-iteration (x_n) converges faster than P-iteration (w_n) to p.

(ii) Assume that SP-iteration (h_n) converges to p. By using the same proof as in (i) together with Proposition 3.5, 3.6 in [7], Lemma 5, and Lemma 6, we obtain the desired result. □

TABLE 1: SP-, P-, and D-iterations for $f(x) = (x^2 + 2\sqrt{x+5})/8$.

n	SP	P	D-iteration	
	h_n	w_n	x_n	$|f(x_n) - x_n|$
5	4.089767594	2.761477135	1.948539462	4.20216E-01
...	...	...	...	...
15	1.379380405	1.000194745	1.000004283	3.67813E-01
16	1.295812521	1.000072757	1.000001234	3.64817E-01
17	1.230394331	1.000027194	1.000000357	3.62012E-01
18	1.179372155	1.000010168	1.000000103	3.59376E-01
19	1.139668491	1.000003803	1.000000030	3.56890E-01
20	1.108810300	1.000001423	1.000000009	3.54539E-01

TABLE 2: The rate of convergence of SP-, P-, and D-iterations for $f(x) = (x^2 + 2\sqrt{x+5})/8$.

n	SP	P	D-iteration
	$|h_n - p|$	$|w_n - p|$	$|x_n - p|$
23	5.1733132295E-02	7.4597958699E-08	2.1734436473E-10
24	4.0468736811E-02	2.7929774493E-08	6.3705707376E-11
...	...	...	...
29	1.2079291278E-02	2.0582890947E-10	1.4099832413E-13
30	9.5201332284E-03	7.7109207908E-11	4.1744385726E-14
31	7.5121079051E-03	2.8889335368E-11	1.2212453271E-14
32	5.9345143329E-03	1.0824230401E-11	3.5527136788E-15

It follows from Theorems 9 and 11 that D-iteration converges faster than Mann, Ishikawa, Noor, SP-, S-, and P-iterations for the class of continuous nondecreasing functions.

Remark 12. As a result from Theorem 11, we can also conclude that D-iteration converges faster than Mann iteration and P-iteration under the same computational cost. Since S-iteration is a special case of P-iteration, Mann iteration converges faster than S-iteration under the same computational cost as well. From Remark 10, we have that D-iteration is better than other iterations despite whether computational costs being considered or not.

Next, we give numerical examples of SP-, P-, and D-iterations, where $\alpha_n = 1/(n^{0.2} + 1)$ and $\beta_n = \gamma_n = 1/(n^2 + 1)$ for all $n \in \mathbb{N}$.

Example 13. Let $f : [0, 8] \longrightarrow [0, 8]$ be defined by $f(x) = (x^2 + 2\sqrt{x} + 5)/8$. We have that f is a nondecreasing continuous function. Given the initial point $h_1 = w_1 = x_1 = 6$. Then, SP-, P-, and D-iterations are presented in Table 1, where the fixed point $p = 1$. It can be seen that D-iteration converges faster than other iterations as a result from Theorem 11. In addition, Table 2 shows the rate of convergence for each iteration. Notice that at least 24 steps of D-iteration must be computed to obtain an error less than 1×10^{-10}, at least 30 steps for P-iteration and more than 32 steps for SP-iteration. In fact, at least 119 steps are needed for SP-iteration.

TABLE 3: SP-, P-, and D-iterations for $f(x) = \cos(0.2x + 3) + x$.

	SP	P	D-iteration	
n	h_n	w_n	x_n	$\|f(x_n) - x_n\|$
87	-7.1308754582	-7.1460183352	-7.1460183659	1.92024E-11
88	-7.1317558732	-7.1460183413	-7.1460183660	1.44693E-11
89	-7.1325837416	-7.1460183463	-7.1460183660	1.09042E-11
...	...	...	...	...
113	-7.1427309049	-7.1460183659	-7.1460183660	1.24345E-14
114	-7.1429149653	-7.1460183660	-7.1460183660	9.76996E-15
115	-7.1430884985	-7.1460183660	-7.1460183660	7.10543E-15

TABLE 4: The rate of convergence of SP-, P-, and D-iterations for $f(x) = \cos(0.2x + 3) + x$.

	SP	P	D-iteration
n	$\|h_n - p\|$	$\|w_n - p\|$	$\|x_n - p\|$
86	1.6079317678E-02	3.8590673768E-08	1.2743317512E-10
87	1.5142907816E-02	3.0871461831E-08	9.6013863526E-11
88	1.4262492814E-02	2.4696326761E-08	7.2347461355E-11
...	...	...	...
112	3.4827043677E-03	1.1655210130E-10	8.3488771452E-14
113	3.2874611183E-03	9.3240970500E-11	6.3948846218E-14
114	3.1034007327E-03	7.4591888222E-11	4.7961634664E-14
...	...	...	...
121	2.0776963242E-03	1.5641710149E-11	7.1054273576E-15
122	1.9625188948E-03	1.2514433934E-11	6.2172489379E-15
123	1.8538565287E-03	1.0011547147E-11	6.2172489379E-15

TABLE 5: SP-, P-, and D-iterations for $f(x) = (0.5)^x$.

	SP	P	D-iteration	
n	h_n	w_n	x_n	$\|f(x_n) - x_n\|$
10	0.6412050446	0.6410282913	0.6411857269	2.54983E-08
11	0.6411940179	0.6412543436	0.6411857479	4.90076E-09
12	0.6411893615	0.6411557546	0.6411857438	9.59270E-10
13	0.6411873528	0.6411988907	0.6411857446	1.90767E-10
14	0.6411864703	0.6411799699	0.6411857445	3.84718E-11
15	0.6411860764	0.6411882852	0.6411857445	7.85594E-12
16	0.6411858981	0.6411846252	0.6411857445	1.62237E-12
17	0.6411858163	0.6411862382	0.6411857445	3.38507E-13

TABLE 6: The rate of convergence of SP-, P-, and D-iterations for $f(x) = (0.5)^x$.

	SP	P	D-iteration
n	$\|h_n - p\|$	$\|w_n - p\|$	$\|x_n - p\|$
13	1.608284337E-06	1.314622597E-05	1.320704657E-10
14	7.258204218E-07	5.774555100E-06	2.663447241E-11
15	3.319134519E-07	2.540677713E-06	5.438760553E-12
16	1.535913483E-07	1.119313535E-06	1.123212634E-12
17	7.184125961E-08	4.936533222E-07	2.343680805E-13
18	3.393466175E-08	2.179109104E-07	4.940492460E-14
19	1.617459799E-08	9.626305075E-08	1.043609643E-14
20	7.774100030E-09	4.255138120E-08	2.331468352E-15
21	3.765638468E-09	1.881918510E-08	4.440892099E-16

Example 14. Let $f : [-10, 5] \longrightarrow [-10, 5]$ be defined by $f(x) = \cos(0.2x + 3) + x$. Then, f is a nondecreasing continuous function. Given the initial point $h_1 = w_1 = x_1 = 0$. Then, Table 3 provides SP-, P-, and D-iterations, where the fixed point $p = 5\pi/2 - 15 = -7.1460183660\ldots$. Then, D-iteration converges faster than other iterations satisfying Theorem 11. Moreover, the rate of convergence of each iteration is shown in Table 4. Note that at least 87 steps of D-iteration must be computed to obtain an error less than 1×10^{-10}, at least 113 steps for P-iteration, and more than 123 steps for SP-iteration. Precisely, at least 455 steps are needed for SP-iteration.

Example 15. Let $f : [0, 5] \longrightarrow [0, 5]$ be defined by $f(x) = (0.5)^x$. Then, f is a nonincreasing continuous function. Given the initial point, $h_1 = w_1 = x_1 = 4$. Then, we have SP-, P-, and D-iterations as shown in Table 5, where the fixed point $p = 0.6411857445\ldots$ which is calculated by NSolve command in Mathematica to accuracy of 40 decimal places. One can see that D-iteration converges faster than other iterations. Further, the rate of convergence for each iteration is given in Table 6. As a result, at least 14 steps of D-iteration must be computed to obtain an error less than 1×10^{-10}, at least 28 steps for P-iteration, and at least 27 steps for SP-iteration.

Consequently, Example 15 suggests that this may be true for the class of nonincreasing continuous functions. This remains open. Furthermore, one can consider these iterations for larger classes of continuous functions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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