Research Article

On Computability and Applicability of Mann-Reich-Sabach-Type Algorithms for Approximating the Solutions of Equilibrium Problems in Hilbert Spaces

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We establish the existence of a strong convergent selection of a modified Mann-Reich-Sabach iteration scheme for approximating the common elements of the set of fixed points $F(T)$ of a multivalued (or single-valued) $k-$strictly pseudocontractive-type mapping T and the set of solutions $EP(F)$ of an equilibrium problem for a bifunction F in a real Hilbert space H. This work is a continuation of the study on the computability and applicability of algorithms for approximating the solutions of equilibrium problems for bifunctions involving the construction of a sequence $\{K_n\}_{n=1}^{\infty}$ of closed convex subsets of H from an arbitrary $x_0 \in H$ and a sequence $\{x_n\}_{n=1}^{\infty}$ of the metric projections of x_0 into K_n. The obtained result is a partial resolution of the controversy over the computability of such algorithms in the contemporary literature.

1. Introduction

Let H be a real Hilbert space with an inner product $\langle ., . \rangle$ and a norm $\|.\|$, respectively and let K be a nonempty closed convex subset of H. Let $A : H \longrightarrow H$ be an operator on H and $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction on K, where $\mathbb{R}$ is the set of real numbers. The variational inequality problem of A in K denoted by $VIP(A, K)$ is to find an $x^* \in K$ such that

$$\langle x - x^*, A(x^*) \rangle \geq 0, \quad \forall x \in K, \tag{1}$$

while the equilibrium problem for F is to find $x^* \in K$ such that

$$F(x^*, x) \geq 0, \quad \forall x \in K. \tag{2}$$

The set of solutions of (2) is denoted by $EP(F)$. Suppose $F(x, y) = \langle y - x, Ax \rangle$ for all $x, y \in K$, then $w \in EP(F)$ if and only if w is a solution of (1). Many problems in optimization, economics, and physics reduce to finding a solution of (1) (see, for example, [1–3]) and the references therein. The following conditions are assumed for solving the equilibrium problems for a bifunction $F : K \times K \longrightarrow \mathbb{R}$,

(A1) $F(x, x) = 0$ for all $x \in K$.

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$, for all $x, y \in K$.

(A3) For each $x, y, z \in K, \lim_{t \downarrow 0} F(tz+(1-t)x, y) \leq F(x, y)$.

(A4) For each $x \in K$, $y \longmapsto F(x, y)$ is convex and lower semicontinuous.

Several algorithms have been introduced by authors for approximating the solutions of an equilibrium problem for a bifunction (or the common elements of the sets of solutions of equilibrium problems for a finite family of bifunctions). Many authors have also approximated the common elements of the set of fixed points $F(T)$ of a multivalued (or single-valued) mapping T and the set of solutions $EP(F)$ of an equilibrium problem for a bifunction F (or the common elements of the sets of fixed points of a finite family of multivalued (or single-valued) mappings and the sets of solutions of equilibrium problems for a finite family of bifunctions) (see, for example, [4–10] and references therein). In a real Hilbert space, many authors have studied the algorithms involving

the construction of the sequences of sets $\{K_n\}_{n=1}^{\infty}$ and the metric projections $\{x_n\}_{n=1}^{\infty}$, from an arbitrary $x_0 \in H$, where $K_{n+1} = \{z \in K_n : \|z - u_n\|^2 \leq \|z - x_n\|^2\}$, $x_{n+1} = P_{K_{n+1}} x_0$, while P_{K_n} is the projection map and $\{u_n\}_{n=1}^{\infty}$ is the sequence of the resolvent of the bifunctions, (see, for example, [4, 9] and references therein).

Among the iteration schemes studied are the modified Reich-Sabach-type Algorithm 1 and Mann-Reich-Sabach-type Algorithm 2 below defined for the approximation of (i) the solutions of an equilibrium problem for a bifunction; (ii) the common elements of the set of fixed points $F(T)$ of a multivalued (or single-valued) mapping T, and the set of solutions $EP(F)$ of an equilibrium problem for a bifunction F respectively.

(i) Let H be a real Hilbert space and K a closed and convex subset of H. Let $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction and $r \in [a, \infty)$ for some $a > 0$. Then from an arbitrary $x_0 \in H$ the algorithm is generated as follows.

Algorithm 1.

$$y_n = x_n,$$

$$u_n \in K \text{ such that } F(u_n, y) + \frac{1}{r} \langle y - u_n, u_n - y_n \rangle$$

$$\geq 0, \quad \forall y \in K, \tag{3}$$

$$K_{n+1} = \left\{ z \in K_n : \|z - u_n\|^2 \leq \|z - x_n\|^2 \right\}$$

$$x_{n+1} = P_{K_{n+1}} x_0.$$

(ii) Let H be a real Hilbert space, K a closed and convex subset of H, $F : K \times K \longrightarrow \mathbb{R}$ a bifunction, and $T : K \longrightarrow P(K)$ multivalued $k-$strictly pseudocontractive-type mapping. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1]$ and $r \in [a, \infty)$ for some $a > 0$. Then from an arbitrary $x_0 \in H$ the algorithm is generated as follows.

Algorithm 2.

$$y_n = \alpha_n x_n + (1 - \alpha_n) v_n,$$

$$u_n \in K \text{ such that } F(u_n, y) + \frac{1}{r} \langle y - u_n, u_n - y_n \rangle$$

$$\geq 0, \quad \forall y \in K, \tag{4}$$

$$K_{n+1} = \left\{ z \in K_n : \|z - u_n\|^2 \leq \|z - x_n\|^2 \right\}$$

$$x_{n+1} = P_{K_{n+1}} x_0,$$

where $v_n \in Tx_n$ for multivalued mapping T.

However, despite the fact that most of these algorithms yield strong convergence theoretically, the difficulty encountered by computers with the construction of the sequence of the metric projection $\{x_n\}_{n=1}^{\infty}$ and the sequence of sets $\{K_n\}_{n=1}^{\infty}$ has made such algorithms almost impossible for real life applications. This noncomputability and nonapplicability of such algorithms has led to the introduction of other algorithms which do not involve the construction of these

two sequences but require stronger conditions and many parameters in the hypothesis of their convergence theorems.

One of these important algorithms is the algorithm of Zhaoli Ma et al. [10].

The purpose of this research is to develop a computable version of Algorithms 1 and 2. In particular, it is established that given the modified Reich-Sabach-types Algorithm 1 for approximating the solutions of an equilibrium problem EP(F) for a bifunction $F : K \times K \longrightarrow \mathbb{R}$ in a real Hilbert space H involving the construction of the sequences $\{K_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ from an arbitrary $x_0 \in H$, where $K = K_0$ is a closed and convex subset of H, $K_{n+1} = \{z \in K_n : \|z - u_n\|^2 \leq \|z - x_n\|^2\}$, and $x_{n+1} = P_{K_{n+1}} x_0$, while P_{K_n} is the metric projection of H into K_n and $\{u_n\}_{n=1}^{\infty}$ is the sequence of the resolvents of the bifunction; there exists a selection $\{k_n\}_{n=1}^{\infty}$ of $\{K_n\}_{n=1}^{\infty}$ which converges strongly to a solution of the equilibrium problem. Furthermore, if the norm on H is order inclusion transitive on the closed convex subsets $CC(H)$ of H, then $x_{n+1} = P_{K_{n+1}} x_0$ and the selection converges strongly to $P_{EP(F)} x_0$. Where a norm $\|.\|$ on a Hilbert space H is order inclusion transitive on $CC(H)$ if given any $A, B \in CC(H)$ with $A \subseteq B$ and arbitrary $x \in H$, then $d(x, B) = \inf_{\overline{b} \in B} \|\overline{b} - x\| = \|b - x\|$ and $d(b, A) = \inf_{\overline{a} \in A} \|\overline{a} - b\| = \|a - b\|$ imply that $d(x, A) = \inf_{\overline{a} \in A} \|\overline{a} - x\| = \|a - x\|$ and $EP(F)$ is the set of the solutions of the equilibrium problem for the bifunction. Also if we set $y_n = \alpha_n x_n + (1 - \alpha_n) v_n$ in Algorithm 1, where $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1]$ satisfying some conditions and $T : K \longrightarrow P(K)$ is a multivalued $k-$strictly pseudocontractive-type mapping a similar selection existing as well which is a selection of Algorithm 2, the numerical example of the computation is presented for the selection of Algorithm 2 which is the generalization of the selections of Algorithm 1. The results of this research are great contributions towards the resolution of the controversy over the computability and applicability of algorithms for approximating the solutions of equilibrium problems for bifunctions involving the construction of the sequences $\{K_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ above.

2. Preliminaries

Let X be a nonempty set and let $T : X \longrightarrow X$ be a map. A point $x \in X$ is called a fixed point of T if $x = Tx$. If $T : X \longrightarrow 2^X$ is a multivalued map from X into the family of nonempty subsets of X, then x is a fixed point of T if $x \in Tx$. If $Tx = \{x\}$, x is called a strict fixed point of T. The set $F(T) = \{x \in D(T) : x \in Tx\}$ (respectively, $F(T) = \{x \in D(T) : x = Tx\}$) is called the fixed point set of multivalued (respectively, single-valued) map T while the set $F_s(T) = \{x \in D(T) : Tx = \{x\}\}$ is called the strict fixed point set of T.

Let X be a normed space. A subset K of X is called proximinal if for each $x \in X$ there exists $k \in K$ such that

$$\|x - k\| = \inf \{\|x - y\| : y \in K\} = d(x, K). \tag{5}$$

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote the family of all nonempty closed and bounded subsets of X by $CB(X)$, the family of all nonempty subsets of X by 2^X, the family of all nonempty closed and convex subsets of X by

$CC(X)$, and the family of all proximinal subsets of X by $P(X)$, for a nonempty set X.

Let H denote the Hausdorff metric induced by the metric d on X; that is, for every $A, B \in CB(X)$,

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}. \quad (6)$$

Let X be a normed space. Let $T : D(T) \subseteq X \longrightarrow 2^X$ be a multivalued mapping on X. A multivalued mapping $T : D(T) \subseteq X \longrightarrow 2^X$ is called $L - Lipschitzian$ if there exists $L \geq 0$ such that for all $x, y \in D(T)$

$$H(Tx, Ty) \leq L \|x - y\|. \quad (7)$$

In (7), if $L \in [0, 1)$ T is said to be a contraction while T is nonexpansive if $L = 1$.

Definitions 3 (see [11]). T is said to be k-strictly pseudocontractive-type of Isiogugu [11] if there exists $k \in (0, 1)$ such that given any pair $x, y \in D(T)$ and $u \in Tx$, there exists $v \in Ty$ satisfying $\|u - v\| \leq H(Tx, Ty)$ and

$$H^2(Tx, Ty) \leq \|x - y\|^2 + k \|x - u - (y - v)\|^2. \quad (8)$$

Definitions 4. A multivalued map $T : D(T) \subseteq X \longrightarrow 2^X$ is said to be of type-one (see for example [12, 13]) if given any pair $x, y \in D(T)$, then

$$\|u - v\| \leq H(Tx, Ty), \quad \text{for all } u \in P_T x, \ v \in P_T y. \quad (9)$$

Lemma 5 (see [14]). *Let H be a real Hilbert space and let $T : D(T) \subseteq H \longrightarrow 2^H$ be a k-strictly pseudocontractive-type mapping. Then T is an L-Lipschitzian.*

Lemma 6 (see [7]). *Let K be a nonempty subset of a real Hilbert space H and let $T : K \longrightarrow P(K)$ be a k-strictly pseudocontractive-type mapping such that $F_s(T)$ is nonempty. Then $F_s(T)$ is closed and convex.*

Lemma 7. *Let H be a real Hilbert space and let K be a nonempty closed convex subset of H. Let P_K be the convex projection onto K. Then, convex projection is characterized by the following relations:*

(i) $x^* = P_K(x) \Longleftrightarrow \langle x - x^*, y - x^* \rangle \leq 0$, for all $y \in K$.

(ii) $\|x - P_K x\|^2 \leq \|x - y\|^2 - \|y - P_K x\|^2$.

(iii) $\|x - P_K y\|^2 \leq \|x - y\|^2 - \|P_K y - y\|^2$.

Lemma 8 (see [1]). *Let K be a nonempty closed convex subset of a real Hilbert space H and $F : K \times K \longrightarrow \mathbb{R}$ a bifunction satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in K$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in K. \quad (10)$$

Lemma 9 (see [2]). *Let K be a nonempty closed convex subset of a real Hilbert space H. Assume that $F : K \times K \longrightarrow \mathbb{R}$ satisfies (A1)-(A4). Let $r > 0$ and $x \in H$; define $T_r : H \longrightarrow 2^K$ by*

$$T_r(x) = \left\{ z \in K : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \right\}, \quad (11)$$
$$\forall y \in K.$$

Then the following conditions hold:

(1) T_r is single-valued.

(2) T_r is firmly nonexpansive, that is for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$.

(3) $F(T_r) = EP(F)$.

(4) $EP(F)$ is closed and convex.

Lemma 10 (see [15]). *Let K be a nonempty closed convex subset of a real Hilbert space H and $F : K \times K \longrightarrow \mathbb{R}$ a bifunction satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then for all $x \in H$ and $p \in F(T_r)$*

$$\|p - T_r x\|^2 + \|T_r x - x\|^2 \leq \|p - x\|^2. \quad (12)$$

Definition 11 (see [16, 17]). Let E be a Banach space. Let $T : D(T) \subseteq E \longrightarrow 2^E$ be a multivalued mapping. $I - T$ is said to be *weakly demiclosed* at *zero* if for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq D(T)$ such that $\{x_n\}$ converges weakly to p and a sequence $\{y_n\}$ with $y_n \in Tx_n$ for all $n \in \mathbb{N}$ such that $\{x_n - y_n\}$ converges strongly to *zero*. Then $p \in Tp$ (i.e., $0 \in (I - T)p$).

3. Main Results

Definition 12. Let $\{K_n\}_{n=1}^{\infty}$ be sequence of sets. Then a sequence $\{z_n\}_{n=1}^{\infty}$ is called a selection of $\{K_n\}_{n=1}^{\infty}$ if $z_n \in K_n$ for each n.

Definition 13. A norm $\|.\|$ on a Hilbert space H is order inclusion transitive on $CC(H)$ if given any $A, B \in CC(H)$ with $A \subseteq B$ and arbitrary $x \in H$, then $d(x, B) = \inf_{\bar{b} \in B} \|\bar{b} - x\| = \|b - x\|$ and $d(b, A) = \inf_{\bar{a} \in A} \|\bar{a} - b\| = \|a - b\|$ imply that $d(x, A) = \inf_{\bar{a} \in A} \|\bar{a} - x\| = \|a - x\|$ (i.e., if b is the point in B closet to x and a is the point in A closest to b then a is the point in A closest to x).

Definition 14. A Hilbert H is said to have order inclusion transitive property on $CC(H)$ if its norm is order inclusion transitive on $CC(H)$.

It is easy to see that the set of real numbers with the usual norm has order inclusion transitive property.

Proposition 15. *In the definition of the set $K_{n+1} := \{z \in K_n : \|z - u_n\|^2 \leq \|z - x_n\|^2\}$, if we define $x_{n+1} = (1/2)(u_n + x_n)$, then the following conditions are true:*

(C_1) $\{x_n\}_{n=1}^{\infty}$ is a selection of $\{K_n\}_{n=1}^{\infty}$.

(C_2) $x_{n+1} = P_{K_{n+1}} x_n$.

(C_3) If H has order inclusion transitive property on $CC(H)$ then, $x_{n+1} = P_{K_{n+1}} x_0$

Proof. (C_1)

$$\|x_{n+1} - u_n\|^2 = \left\| \frac{1}{2}(u_n + x_n) - u_n \right\|^2 = \frac{1}{4} \|u_n - x_n\|^2 \quad (13)$$

and

$$\|x_{n+1} - x_n\|^2 = \left\| \frac{1}{2}(u_n + x_n) - x_n \right\|^2 = \frac{1}{4} \|u_n - x_n\|^2. \quad (14)$$

Therefore, $x_{n+1} \in K_{n+1}$; thus, $\{x_n\}_{n=1}^{\infty}$ is a selection of $\{K_n\}_{n=1}^{\infty}$.

(C_2) Let $y \in K_{n+1}$ arbitrary. Then

$$
\begin{aligned}
\langle x_n - x_{n+1}, y - x_{n+1} \rangle &= \left\langle x_n - \frac{1}{2}(u_n + x_n), y \right. \\
&\left. - \frac{1}{2}(u_n + x_n) \right\rangle = \left\langle \frac{1}{2}(x_n - u_n), \right. \\
&\left. - \frac{1}{2}((u_n - y) + (x_n - y)) \right\rangle = \frac{1}{4} \langle x_n - u_n, \\
&- (u_n - y) - (x_n - y) \rangle = \frac{1}{4} \langle (x_n - y) \\
&+ (y - u_n), (y - u_n) + (y - x_n) \rangle \\
&= \frac{1}{4} \left[\langle x_n - y, y - u_n \rangle + \langle x_n - y, y - x_n \rangle \right. \\
&\left. + \langle y - u_n, y - u_n \rangle + \langle y - u_n, y - x_n \rangle \right] \\
&= \frac{1}{4} \left[\langle x_n - y, y - u_n \rangle - \langle y - x_n, y - x_n \rangle \right. \\
&\left. + \langle y - u_n, y - u_n \rangle + \langle y - u_n, y - x_n \rangle \right] \\
&= \frac{1}{4} \left[\langle x_n - y, y - u_n \rangle - \|y - x_n\|^2 + \|y - u_n\|^2 \right. \\
&\left. + \langle y - u_n, y - x_n \rangle \right] = \frac{1}{4} \left[- \langle y - x_n, y - u_n \rangle \right. \\
&\left. - \|y - x_n\|^2 + \|y - u_n\|^2 + \langle y - x_n, y - u_n \rangle \right] \\
&\leq \frac{1}{4} \left[- \langle y - x_n, y - u_n \rangle - \|y - x_n\|^2 + \|y - x_n\|^2 \right. \\
&\left. + \langle y - x_n, y - u_n \rangle \right] = 0.
\end{aligned}
\tag{15}
$$

Consequently, $x_{n+1} = P_{K_{n+1}} x_n$.

(C_3) Since $K_{n+1} \subseteq K_n$ and K_n is closed and convex for each n, condition C_2 and order inclusion transitive property of H on $CC(H)$ guarantee that $x_{n+1} = P_{K_{n+1}} x_0$. $\square$

We now consider the following algorithm which we shall refer to as a selection of Algorithm 1.

Let H be a real Hilbert space, K be a nonempty closed convex subset of H, and $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction. Let $r \in [a, \infty)$ for some $a > 0$. Then from an arbitrary $x_0 \in H$ we generate the sequence $\{x_n\}_{n=1}^{\infty}$ as follows.

Algorithm 16.

$$
\begin{aligned}
y_n &= x_n, \\
u_n &\in K \text{ such that } F(u_n, y) + \frac{1}{r} \langle y - u_n, u_n - y_n \rangle \\
&\geq 0, \quad \forall y \in K, \\
x_{n+1} &= \frac{1}{2}(u_n + x_n).
\end{aligned}
\tag{16}
$$

Theorem 17. *Let H, $\{x_n\}_{n=1}^{\infty}$ and F and r be as in Algorithm 16. Suppose F satisfying (A1)-(A4), $EP(F) \neq \emptyset$, then (i) $\{x_n\}$ converges strongly to $p \in EP(F)$; (ii) if H has order inclusion transitive property, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $P_{EP(F)} x_0$.*

Proof. Since $u_n = T_r y_n = T_r x_n$, given arbitrary $q \in EP(F)$, we have that

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &= \left\| \frac{1}{2}(u_n + x_n) - q \right\|^2 \\
&= \left\| \frac{1}{2}(u_n - q) + \frac{1}{2}(x_n - q) \right\|^2 \\
&= \frac{1}{2}\|u_n - q\|^2 + \frac{1}{2}\|x_n - q\|^2 \\
&\quad - \frac{1}{2}\|x_n - u_n\|^2 \\
&\leq \frac{1}{2}\|x_n - q\|^2 + \frac{1}{2}\|x_n - q\|^2 \\
&\quad - \frac{1}{4}\|x_n - u_n\|^2 \\
&= \|x_n - q\|^2 - \frac{1}{4}\|x_n - u_n\|^2 \leq \|x_n - q\|^2.
\end{aligned}
\tag{17}
$$

Therefore, $\{\|x_n - p\|\}_{n=0}^{\infty}$ is monotone, nonincreasing, and bounded; hence, $\lim_{n \to \infty} \|x_n - q\|$ exists. Also,

$$
\frac{1}{4}\|x_n - u_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.
\tag{18}
$$

Hence $\sum_{n=0}^{\infty} \|x_n - u_n\|^2 < \infty$ which implies that $\lim_{n \to \infty} \|x_n - u_n\| = \|x_n - T_r x_n\| = 0$. Also,

$$
\|x_{n+1} - x_n\| = \frac{1}{2}\|x_n - u_n\| \leq \|x_n - u_n\|,
\tag{19}
$$

so that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$. Consequently $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty} = \{x_n\}_{n=0}^{\infty}$, and $\{u_n\}_{n=0}^{\infty} = \{T_r x_n\}_{n=0}^{\infty}$ are bounded. From $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$ we have that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy in K and hence converges strongly to $p \in K$. From the Opial condition of H, the firmly nonexpansive and demiclosedness property of $(I - T_r)$ established that $p \in F(T_r) = EP(F)$. (ii) If H has order inclusion transitive property then $x_n = P_{K_n} x_0$, consequently, from Lemma 7(i)

$$
\langle x_n - y, x_0 - x_n \rangle \geq 0, \quad \forall y \in K_n.
\tag{20}
$$

Since $EP(F) \subseteq K_n$ for all $n \geq 1$, we have that

$$
\langle x_n - q, x_0 - x_n \rangle \geq 0, \quad \forall q \in EP(F).
\tag{21}
$$

Taking the limit as $n \longrightarrow \infty$ in (21) we have

$$
\langle p - q, x_0 - p \rangle \geq 0, \quad \forall q \in EP(F).
\tag{22}
$$

Thus, from Lemma 7(i) $p = P_{EP(F)} x_0$. This completes the proof. $\square$

Remark 18. It is important to note that the strong convergence of Algorithm 16 to a $p \in EP(F)$ does not depend on the order inclusion transitive property condition on H.

Motivated by Algorithm 16 we now obtain the following algorithm which is a selection of Algorithm 2.

Let H be a real Hilbert space, K be a closed and convex subset of H, $F : K \times K \longrightarrow \mathbb{R}$ be a bifunction, and $T : K \longrightarrow P(K)$ be a multivalued k–strictly pseudocontractive-type mapping. Let $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1]$ and $r_n \in [a, \infty)$ for some $a > 0$. Then from an arbitrary $x_0 \in H$ the algorithm is generated as follows.

Algorithm 19.

$$y_n = \alpha_n x_n + (1 - \alpha_n) v_n,$$

$$u_n \in K \text{ such that } F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle$$

$$\geq 0, \quad \forall y \in K, \tag{23}$$

$$x_{n+1} = \frac{1}{2}(u_n + x_n),$$

where $v_n \in Tx_n$.

Theorem 20. *Let H, K, F, T, $\{\alpha_n\}$, and $\{r_n\}$ be as in Algorithm 19. Suppose that $(I-T)$ is weakly demiclosed at zero, F satisfies (A1)-(A4), $\mathbb{F} = F_s(T) \cap EP(F) \neq \emptyset$, and $\{\alpha_n\}$ satisfies (i) $\alpha_n \longrightarrow \alpha < 1 - k$; (ii) $\alpha > 0$; (iii) $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to $q \in \mathbb{F}$. Also, if H has order inclusion transitive property, then $q = P_{\mathbb{F}} x_0$.*

Proof. Let $p \in \mathbb{F}$ be arbitrary. Then

$$\|x_{n+1} - p\|^2 = \left\| \frac{1}{2}(x_n + u_n) - p \right\|^2 = \frac{1}{2}\|x_n - p\|^2$$

$$+ \frac{1}{2}\|u_n - p\|^2 - \frac{1}{4}\|x_n - u_n\|^2 \leq \frac{1}{2}\|x_n - p\|^2$$

$$+ \frac{1}{2}\|y_n - p\|^2 - \frac{1}{4}\|x_n - u_n\|^2 = \frac{1}{2}\|x_n - p\|^2$$

$$- \frac{1}{4}\|x_n - u_n\|^2 + \frac{1}{2}\|(1 - \alpha_n)x_n + \alpha_n v_n - p\|^2$$

$$= \frac{1}{2}\|x_n - p\|^2 - \frac{1}{4}\|x_n - u_n\|^2$$

$$+ \frac{1}{2}\left[\|(1 - \alpha_n)(x_n - p) + \alpha_n(v_n - p)\|^2\right] = \frac{1}{2}\|x_n$$

$$- p\|^2 - \frac{1}{4}\|x_n - u_n\|^2 + \frac{1}{2}\left[(1 - \alpha_n)\|x_n - p\|^2\right.$$

$$+ \alpha_n\|v_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - v_n\|^2\right] \leq \frac{1}{2}\|x_n$$

$$- p\|^2 - \frac{1}{4}\|x_n - u_n\|^2 + \frac{1}{2}\left[(1 - \alpha_n)\|x_n - p\|^2\right.$$

$$+ \alpha_n H^2(Tx_n, Tp) - \alpha_n(1 - \alpha_n)\|x_n - v_n\|^2\right]$$

$$\leq \frac{1}{2}\|x_n - p\|^2 - \frac{1}{4}\|x_n - u_n\|^2$$

$$+ \frac{1}{2}\left[(1 - \alpha_n)\|x_n - p\|^2\right]$$

$$+ \alpha_n\left[\|x_n - p\|^2 + k\|x_n - v_n\|^2\right]$$

$$- \alpha_n(1 - \alpha_n)\|x_n - v_n\|^2\right] = \frac{1}{2}\|x_n - p\|^2 - \frac{1}{4}\|x_n$$

$$- u_n\|^2 + \frac{1}{2}\left[\|x_n - p\|^2\right]$$

$$- \alpha_n(1 - (\alpha_n + k))\|x_n - v_n\|^2\right] = -\frac{1}{4}\|x_n - u_n\|^2$$

$$+ \|x_n - p\|^2 - \frac{1}{2}\alpha_n(1 - (\alpha_n + k))\|x_n - v_n\|^2. \tag{24}$$

It then follows that $\lim_{n\to\infty}\|x_n - p\|$ exists; hence, $\{x_n\}$ is bounded. Also,

$$\sum_{n=1}^{\infty} \frac{1}{2}\alpha_n(1 - (\alpha_n + k))\|x_n - v_n\|^2 \leq \|x_0 - p\|^2 < \infty. \tag{25}$$

Since $\alpha > 0$ from (ii), we have that $\lim_{n\to\infty}\|x_n - v_n\| = 0$. Furthermore, $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Consequently, $\lim_{n\to\infty}\|x_{n+1} - x_n\|^2 = \lim_{n\to\infty}\|(1/2)(x_n - u_n)\|^2 = 0$ which implies that $\{x_n\}$ is a Cauchy sequence in K. Also, since K is closed and convex, $\{x_n\}$ converges strongly to some $q \in K$. From the Opial condition of H, weakly demiclosed of $(I - T)$ at *zero*, we have that $q \in Tq$.

It remains to show that q is in $EP(F)$ since

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{26}$$

It follows from $\lim_{n\to\infty}\|x_n - q\| = 0$ and (26) that

$$\lim_{n\to\infty}\|u_n - q\| = 0. \tag{27}$$

Also,

$$\|y_n - q\|^2 \leq \|x_n - q\|^2 - \alpha_n(1 - (\alpha_n + k))\|x_n - v_n\|^2. \tag{28}$$

Observe that

$$\|q - x_n\|^2 - \|q - u_n\|^2$$

$$= \|x_n\|^2 - \|u_n\|^2 - 2\langle q, x_n - u_n \rangle \tag{29}$$

$$\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|q\|\|x_n - u_n\|.$$

It follows from (26) that

$$\lim_{n\to\infty}\|q - x_n\| - \|q - u_n\| = 0. \tag{30}$$

Now from (28),

$$\|q - y_n\| \leq \|q - x_n\|. \tag{31}$$

Also, using $u_n = T_{r_n} y_n$, Lemma 10, and (31) we have

$$
\begin{aligned}
\left\| u_n - y_n \right\|^2 &= \left\| T_{r_n} y_n - y_n \right\|^2 \\
&\leq \left\| q - y_n \right\|^2 - \left\| q - T_{r_n} y_n \right\|^2 \\
&\leq \left\| q - x_n \right\|^2 - \left\| q - T_{r_n} y_n \right\|^2 \\
&= \left\| q - x_n \right\|^2 - \left\| q - u_n \right\|^2 .
\end{aligned}
\tag{32}
$$

Therefore, from (30) and (32),

$$
\lim_{n \to \infty} \left\| u_n - y_n \right\| = 0.
\tag{33}
$$

Consequently, from (27) and (33),

$$
\lim_{n \to \infty} \left\| y_n - q \right\| = 0.
\tag{34}
$$

From the assumption that $r_n \geq a > 0$,

$$
\lim_{n \to \infty} \frac{\left\| u_n - y_n \right\|}{r_n} = 0.
\tag{35}
$$

Since $u_n = T_{r_n} y_n$, we have

$$
F\left(u_n, y \right) + \frac{1}{r_n} \left\langle y - u_n, u_n - y_n \right\rangle \geq 0.
\tag{36}
$$

We then deduce from (A2) that

$$
\begin{aligned}
\frac{\left\| u_n - y_n \right\|^2}{r_n} &\geq \frac{1}{r_n} \left\langle y - u_n, u_n - y_n \right\rangle \geq -F\left(u_n, y \right) \\
&\geq F\left(y, u_n \right), \quad \forall y \in K.
\end{aligned}
\tag{37}
$$

By taking limit as $n \to \infty$ of the above inequality and from (A4), (27), and (34), $F(y, q) \leq 0$, for all $y \in K$. Let $t \in (0, 1)$ and for all $y \in K$, since $q \in K$, $y_t = ty + (1 - t)q \in K$. Hence $F(y_t, q) \leq 0$. Therefore, from (A1),

$$
\begin{aligned}
0 = F\left(y_t, y_t \right) &\leq t F\left(y_t, y \right) + (1 - t) F\left(y_t, q \right) \\
&\leq t F\left(y_t, y \right);
\end{aligned}
\tag{38}
$$

that is, $F(y_t, y) \geq 0$. Letting $t \downarrow 0$, from (A3) we obtain $F(q, y) \geq 0$ for all $y \in K$ so that $q \in EP(F)$ for all $i = 1, 2, \ldots, N$. Hence $q \in \mathbb{F}$.

Finally, if H has order inclusion transitive property, $x_n = P_{K_n} x_0$ consequently, from Lemma 7(i)

$$
\left\langle x_n - y, x_0 - x_n \right\rangle \geq 0, \quad \forall y \in K_n.
\tag{39}
$$

Since $EP(F) \subseteq K_n$ for all $n \geq 1$, we have that

$$
\left\langle x_n - p, x_0 - x_n \right\rangle \geq 0, \quad \forall p \in EP(F).
\tag{40}
$$

Taking the limit as $n \to \infty$ in (40) we have

$$
\left\langle q - p, x_0 - q \right\rangle \geq 0, \quad \forall p \in EP(F).
\tag{41}
$$

Thus, from Lemma 7(i) $q = P_{EP(F)} x_0$. This completes the proof. □

Remark 21. The above proof shows that the strong convergence of Algorithm 19 to a common solution $q \in \mathbb{F}$ does not depend on order inclusion transitive property condition on H. However, order inclusion transitive property is only required on H if we want to have that $p = P_{EP(F)} x_0$.

Remark 22. It is also of a great interest to us to get the same results in normed spaces which enjoy the order inclusion transitive property.

4. Numerical Examples of the Computations

We shall use Algorithm 19 to recompute the example presented by Isiogugu et al. [13], when $i = 1$ is defined as follows.

Let $H = \mathbb{R}$ (the reals with the usual metric and inner product) and $K = [-4, 10]$; we define
(i) $T : [-4, 10] \longrightarrow P([-4, 10])$ by

$$
Tx =
\begin{cases}
\left[-2x, -\dfrac{9x}{4} \right], & x \in [-4, 0] \\
\left\{ -\dfrac{3x}{9} \right\}, & x \in (0, 10].
\end{cases}
\tag{42}
$$

We have that, for any $x, y \in [-4, 0]$,

$$
\begin{aligned}
H^2\left(Tx, Ty \right) &= \left\| \frac{9}{4} \left(x - y \right) \right\|^2 = \left(\frac{9}{4} \right)^2 \left\| x - y \right\|^2 \\
&= \left\| x - y \right\|^2 + \left(\left(\frac{9}{4} \right)^2 - 1 \right) \left\| x - y \right\|^2 .
\end{aligned}
\tag{43}
$$

Also, given any $u \in Tx$, $u = -\alpha x$, $2 \leq \alpha \leq 9/4$, and we can choose $v = -\alpha y \in Ty$ so that

$$
\left\| x - u - \left(y - v \right) \right\|^2 = (1 + \alpha)^2 \left\| x - y \right\|^2 .
\tag{44}
$$

It then follows that

$$
\begin{aligned}
H^2\left(Tx, Ty \right) &= \left\| x - y \right\|^2 \\
&\quad + \frac{(9)^2 - 16}{16(1 + \alpha)^2} \left\| x - u - \left(y - v \right) \right\|^2 \\
&\leq \left\| x - y \right\|^2 \\
&\quad + \frac{(9)^2 - 16}{16(1 + 2)^2} \left\| x - u - \left(y - v \right) \right\|^2 .
\end{aligned}
\tag{45}
$$

Similarly, for any $x \in [-4, 0]$, $y \in (0, 10]$,

$$
\begin{aligned}
H^2\left(Tx, Ty \right) &= \left\| \frac{9}{4} x - \frac{3y}{9} \right\|^2 \leq \left\| \frac{9}{4} x - \frac{9}{4} y \right\|^2 \\
&\leq \left\| x - y \right\|^2 \\
&\quad + \frac{(9)^2 - 16}{16(1 + 2)^2} \left\| x - u - \left(y - v \right) \right\|^2 .
\end{aligned}
\tag{46}
$$

TABLE 1: Sequences generated by Algorithms 2 and 19.

	Algorithm 2	Algorithm 2	Algorithm 4	Algorithm 2	Algorithm 2	Algorithm 4
n	K_n	x_n	x_n	K_n	x_n	x_n
0	[−4, 10]	2	2	[−4, 10]	−3	−3
1	[−4, 1.126855044]	1.126855044	1.126855044	[−1.532563756, 10]	−1.532563756	−1.532563756
2	[−4, 0.652769535]	0.652769535	0.652769535	[−0.787076007, 10]	−0.787076007	−0.787076007
3	[−4, 0.38284414]	0.38284414	0.38284414	[−0.405188015, 10]	−0.405188015	−0.405188015
4	[−4, 0.226111845]	0.226111845	0.226111845	[−0.208877029, 10]	−0.208877029	−0.208877029
5	[−4, 0.134146663]	0.134146663	0.134146663	[−0.107772707, 10]	−0.107772707	−0.107772707
6	[−4, 0.079836206]	0.079836206	0.079836206	[−0.055641074, 10]	−0.055641074	−0.055641074
7	[−4, 0.047623897]	0.047623897	0.047623897	[−0.028739594, 10]	−0.028739594	−0.028739594
8	[−4, 0.028459121]	0.028459121	0.028459121	[−0.014849721, 10]	−0.014849721	−0.014849721
9	[−4, 0.01703056]	0.01703056	0.01703056	[−0.007674974, 10]	−0.007674974	−0.007674974
10	[−4, 0.010203099]	0.010203099	0.010203099	[−0.003967653, 10]	−0.003967653	−0.003967653
11	[−4, 0.006118509]	0.006118509	0.006118509	[−0.002051501, 10]	−0.002051501	−0.002051501
12	[−4, 0.003672014]	0.003672014	0.003672014	[−0.001060909, 10]	−0.001060909	−0.001060909
13	[−4, 0.002205248]	0.002205248	0.002205248	[−0.00054871, 10]	−0.00054871	−0.00054871
14	[−4, 0.001325149]	0.001325149	0.001325149	[−0.00028383, 10]	−0.00028383	−0.00028383
15	[−4, 0.000796697]	0.000796697	0.000796697	[−0.000146831, 10]	−0.000146831	−0.000146831
16	[−4, 0.0004792]	0.0004792	0.0004792	[−0.000075965, 10]	−0.000075965	−0.000075965
17	[−4, 0.000288345]	0.000288345	0.000288345	[−0.000039304, 10]	−0.000039304	−0.000039304
18	[−4, 0.000173565]	0.000173565	0.000173565	[−0.000020337, 10]	−0.000020337	−0.000020337
19	[−4, 0.000104508]	0.000104508	0.000104508	[−0.000010523, 10]	−0.000010523	−0.000010523
20	[−4, 0.000062945]	0.000062945	0.000062945	[−0.000005445, 10]	−0.000005445	−0.000005445
21	[−4, 0.000037921]	0.000037921	0.000037921	[−0.000002817, 10]	−0.000002817	−0.000002817
22	[−4, 0.000022851]	0.000022851	0.000022851	[−0.000001457, 10]	−0.000001457	−0.000001457
23	[−4, 0.000013772]	0.000013772	0.000013772	[−0.000000754, 10]	−0.000000754	−0.000000754
24	[−4, 0.000008302]	0.000008302	0.000008302	[−0.00000039, 10]	−0.00000039	−0.00000039
25	[−4, 0.000005005]	0.000005005	0.000005005	[−0.000000201, 10]	−0.000000201	−0.000000201
26	[−4, 0.000003018]	0.000003018	0.000003018	[−0.000000103, 10]	−0.000000103	−0.000000103
27	[−4, 0.00000182]	0.00000182	0.00000182	[−0.000000053, 10]	−0.000000053	−0.000000053
28	[−4, 0.000001097]	0.000001097	0.000001097	[−0.000000027, 10]	−0.000000027	−0.000000027
29	[−4, 0.000000661]	0.000000661	0.000000661	[−0.000000013, 10]	−0.000000013	−0.000000013
30	[−4, 0.000000398]	0.000000398	0.000000398	[−0.000000006, 10]	−0.000000006	−0.000000006
31	[−4, 0.00000024]	0.00000024	0.00000024	[−0.000000003, 10]	−0.000000003	−0.000000003
32	[−4, 0.000000144]	0.000000144	0.000000144	[−0.000000001, 10]	−0.000000001	−0.000000001
33	[−4, 0.000000086]	0.000000086	0.000000086	{0}	0	0
34	[−4, 0.000000051]	0.000000051	0.000000051			
35	[−4, 0.00000003]	0.00000003	0.00000003			
36	[−4, 0.000000018]	0.000000018	0.000000018			
37	[−4, 0.00000001]	0.00000001	0.00000001			
38	[−4, 0.000000005]	0.000000005	0.000000005			
39	[−4, 0.000000002]	0.000000002	0.000000002			
40	[−4, 0.000000001]	0.000000001	0.000000001			
41	{0}	0	0			

Furthermore, for any $x, y \in (0, 10]$,

$$
\begin{aligned}
H^2(Tx, Ty) &= \frac{3}{9} \|x - y\|^2 \\
&\leq \|x - y\|^2 \\
&\quad + \frac{(9)^2 - 16}{16(1 + 2)^2} \|x - u - (y - v)\|^2.
\end{aligned}
\tag{47}
$$

Thus, T is $((9)^2 - 16)/16(1 + 2)^2$–strictly pseudocontractive-type. It is easy to see that, given any pair $x, y \in [-4, 0]$, we have that $H(Tx, Ty) = (9/4)\|x - y\|$. Therefore T is not nonexpansive and $F_s(T) = \{0\} \neq \emptyset$. We then set

(ii)

$$
v_n = \begin{cases} -\dfrac{9x_n}{4}, & x_n \in [-4, 0] \\ -\dfrac{3x_n}{9}, & x_n \in (0, 10]. \end{cases}
\tag{48}
$$

(iii)

$\{\alpha_n\}_{n=1}^\infty$

$$
= \frac{\left((9)^2 - 16 + 16(1 + 2)^2\right)\left(n + 32(1 + 2)^2\right)^2 + 32(1 + 2)^2}{32(1 + 2)^2 \left(n + 32(1 + 2)^2\right)^2}.
\tag{49}
$$

We also let $F : R \times R \longrightarrow R$, $\{r_n\}_{n=1}^\infty$, and $\{u_n\}_{n=1}^\infty$ to be as defined in [9] for $i = 1$. That is,

(iv) $F(x, y) = -x^2 + y^2$.

Observe that $EP(F) = \{0\}$, $F_s(T) \cap EP(F) = \{0\}$, and

$$
\begin{aligned}
&F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \\
&\Longrightarrow y^2 - z^2 + \frac{1}{r}(y - z)(z - x) \geq 0, \\
&\Longrightarrow y^2 - z^2 + \frac{1}{r}\left[yz - xy - z^2 + xz\right] \geq 0, \\
&\Longrightarrow ry^2 - rz^2 + yz - xy - z^2 + xz \geq 0, \\
&\Longrightarrow ry^2 + (z - x)y - rz^2 - z^2 + xz \geq 0.
\end{aligned}
\tag{50}
$$

Now $F(y) = ry^2 + (z - x)y - rz^2 - z^2 + xz$ is a quadratic function of y with coefficients $a = r$, $b = z - x$, and $c = -rz^2 - z^2 + xz$. Therefore, we can compute the discriminant Δ of F as follows:

$$
\begin{aligned}
\Delta &= (z - x)^2 + 4r\left(rz^2 + z^2 - xz\right) \\
&= z^2 + x^2 - 2xz + 4r^2z^2 + 4rz^2 - 4rxz \\
&= \left(1 + 4r^2 + 4r\right)z^2 - 2(2r + 1)xz + x^2 \\
&= (1 + 2r)^2 z^2 - 2(1 + 2r)xz + x^2 \\
&= [(1 + 2r)z - x]^2.
\end{aligned}
\tag{51}
$$

Obviously, $F(y) \geq 0$ for all $y \in \mathbb{R}$ if it has at most one solution in $\mathbb{R}$. Thus $\Delta \leq 0$ and hence $z = T_{r_n}(x) = x/(1 + 2r)$. Consequently,

(v) $\{u_n\}_{n=1}^\infty = \{T_{r_n}(y_n)\}_{n=1}^\infty = \{y_n/(2r_n + 1)\}_{n=1}^\infty$,

(vi) $\{r_n\}_{n=1}^\infty = \{(n + 1)/n\}_{n=1}^\infty$.

The algorithm is computed with Microsoft word Excel 97-2003 Workbook. Table 1 shows the sequences $\{x_n\}$ and $\{K_n\}$ generated from our computation using two different values of $x_0 = 2, -3$.

Disclosure

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

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