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Research Article

Generalized Fractional Integral Operators Involving Mittag-Leffler Function

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The aim of this paper is to study various properties of Mittag-Leffler (M-L) function. Here we establish two theorems which give the image of this M-L function under the generalized fractional integral operators involving Fox's H-function as kernel. Corresponding assertions in terms of Euler, Mellin, Laplace, Whittaker, and K-transforms are also presented. On account of general nature of M-L function a number of results involving special functions can be obtained merely by giving particular values for the parameters.

1. Introduction and Preliminaries

M-L Function. In 1903, Mittag-Leffler [1] introduced the function $E_\lambda(z)$, defined by

$$E_\lambda(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\lambda n + 1)} z^n \quad (\lambda \in \mathbb{C});\ \Re(\lambda) > 0. \tag{1}$$

A further, two-index generalization of this function was given by Wiman [2] as

$$E_{\lambda,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\lambda n + \beta)} z^n \quad (\lambda, \beta \in \mathbb{C}), \tag{2}$$

where $\Re(\lambda) > 0$ and $\Re(\beta) > 0$.

By means of the series representation a generalization of M-L function (2) is introduced by Prabhakar [3] as

$$E^{\gamma}_{\lambda,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\lambda n + \beta)\, n!} z^n, \tag{3}$$

where $\lambda, \beta, \gamma \in \mathbb{C}$ $(\Re(\lambda) > 0)$. Further, it is an entire function of order $[\Re(\lambda)]^{-1}$.

Generalized Fractional Integral Operator. Now, we recall the definition of generalized fractional integral operators involving Fox's H-function as kernel, defined by Saxena and Kumbhat [4] means of the following equations:

$$R^{\mu,\alpha}_{x,r}[f(x)] = r x^{-\mu-r\alpha-1} \int_0^x t^{\mu} (x^r - t^r)^{\alpha} \cdot H^{m,n}_{p,q}\left[kU \mid \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix}\right] f(t)\, dt, \tag{4}$$

$$K^{\varepsilon,\alpha}_{x,r}[f(x)] = r x^{\varepsilon} \int_x^{\infty} t^{-\varepsilon-r\alpha-1} (t^r - x^r)^{\alpha} \cdot H^{m,n}_{p,q}\left[kV \mid \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix}\right] f(t)\, dt, \tag{5}$$

where U and V represent the expressions

$$\left(\frac{t^r}{x^r}\right)^{\tau} \left(1 - \frac{t^r}{x^r}\right)^{\upsilon}, \qquad \left(\frac{x^r}{t^r}\right)^{\tau} \left(1 - \frac{x^r}{t^r}\right)^{\upsilon}, \tag{6}$$

respectively, with $\tau, \upsilon > 0$. The sufficient conditions of operators are given below:

(i) $1 \le p, q < \infty$, $p^{-1} + q^{-1} = 1$;

(ii) $\Re(\mu + r\tau(b_j/B_j)) > -q^{-1}$; $\Re(\alpha + rv(b_j/B_j)) > -q^{-1}$;

$\Re(\varepsilon + \alpha + r\tau(b_j/B_j)) > -p^{-1}$, $(j = 1, \ldots, m)$;

(iii) $f(x) \in L_P(0, \infty)$;

(iv) $|\arg k| < \lambda\pi/2$, $\lambda > 0$,

where $\lambda = \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j + \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j > 0$.

An interest in the study of the fractional calculus associated with the Mittag-Leffler function and H-function, its application in the form of differential, and integral equations of, in particular, fractional orders (see [5–10]).

H-Function. Symbol $H_{p,q}^{m,n}(x)$ stands for well known Fox H-function [11], in operator (4) and (5) defined in terms of Mellin-Barnes type contour integral as follows:

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z \mid \begin{matrix} (a_p, A_p) \\ (b_q, B_q) \end{matrix} \right] = \frac{1}{2\pi i} \int_L \chi(s) z^s ds, \quad (7)$$

where

$$\chi(s) = \frac{\prod_{j=1}^{m} \Gamma\left(b_j + B_j s\right) \prod_{i=1}^{n} \Gamma\left(1 - a_i - A_i s\right)}{\prod_{i=n+1}^{p} \Gamma\left(a_i + A_i s\right) \prod_{j=m+1}^{q} \Gamma\left(1 - b_j - B_j s\right)}, \quad (8)$$

$m, n, p, q \in \mathbb{N}_0$ with $1 \le m \le q$, $0 \le n \le p$, $A_i, B_j \in \mathbb{R}_+$, $a_i, b_j \in \mathbb{R}$, or $\mathbb{C}$, $i = 1, 2, \ldots, p$; $j = 1, 2, \ldots, q$ such that $A_i(b_j + k) \neq B_j(a_i - l - 1)$ $(k, l \in N_0; i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$.

For the conditions of analytically continuations together with the convergence conditions of H-function, one can see [12, 13]. Throughout the present paper, we assume that these conditions are satisfied by the function.

2. Images of M-L Function Involving the Generalized Fractional Integral Operators

In this section, we consider two generalized fractional integral operators involving the Fox's H-function as the kernels and derived the following theorems.

Theorem 1. *Let $\lambda, \beta, \vartheta, \gamma \in \mathbb{C}$, $x > 0$, $\Re(\lambda) > 0$, $\Re(\vartheta) > 0$, $f(x) \in L_P(0, \infty)$, $1 \le p \le 2$, $|\arg k| < \lambda\pi/2$, $\lambda > 0$, $a \in \mathbb{C}$; then the fractional integration $R_{x,r}^{\mu,\alpha}$ of the product of M-L function exists, under the condition*

$$p^{-1} + q^{-1} = 1;$$
$$\Re\left(\mu + r\tau\left(\frac{b_j}{B_j}\right)\right) > -q^{-1}; \quad (9)$$
$$\Re\left(\alpha + rv\left(\frac{b_j}{B_j}\right)\right) > -q^{-1};$$

then there holds the following formula:

$$R_{x,r}^{\mu,\alpha}\left(t^{\vartheta-1} E_{\lambda,\beta}^{\gamma}\left(at^{\nu}\right)\right)(x) = x^{\vartheta-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\lambda n + \beta) n!} \left(ax^{\nu}\right)^n \times H_{p+2,q+1}^{m,n+2}\left[k \mid \begin{matrix} (a_p, A_p), \left(1 - \frac{(\mu + \vartheta + 1 + \nu n)}{r}, \tau\right), (-\alpha, v) \\ \left(-\frac{(\mu + \vartheta + 1 + \nu n)}{r} - \alpha, \tau + v\right), (b_q, B_q) \end{matrix} \right]. \quad (10)$$

Proof. Let ℓ be the left-hand side of (10); using (3) and (4), we have

$$\ell = rx^{-\mu - r\alpha - 1} \int_0^x t^{\mu + \vartheta - 1} \left(x^r - t^r\right)^{\alpha} \cdot \frac{1}{2\pi i} \int_L \chi(s) (kU)^s ds \frac{(\gamma)_n}{\Gamma(\lambda n + \beta) n!} \left(ax^{\nu}\right)^n dt. \quad (11)$$

Changing the order of the integration valid under the condition given with the theorem, we obtain

$$\ell = rx^{-\mu - r\alpha - 1} \sum_{n=0}^{\infty} \frac{(\gamma)_n a^n}{\Gamma(\lambda n + \beta) n!} \times \frac{1}{2\pi i} \int_L \chi(s) \cdot k^s x^{r\alpha - r\tau s} \left\{ \int_0^x t^{\mu + \vartheta + \nu n + r\tau s - 1} \left(1 - \frac{t^r}{x^r}\right)^{\alpha + vs} dt \right\} ds. \quad (12)$$

Let the substitution $t^r / x^r = w$; then $t = xw^{(1/r)}$ in the above term; we get

$$= x^{\vartheta - 1} \sum_{n=0}^{\infty} \frac{(\gamma)_n a^n}{\Gamma(\lambda n + \beta) n!} \frac{x^{\nu n}}{2\pi i} \int_L \chi(s) k^s x^{\nu s} \times \left\{ \int_0^1 w^{(1/r)(\mu + \vartheta + \nu n + r\tau s) - 1} (1 - w)^{\alpha + vs} dw \right\} ds. \quad (13)$$

Using beta function for (13), the inner integral reduces to

$$= x^{\vartheta - 1} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\lambda n + \beta) n!} \left(ax^{\nu}\right)^n \frac{1}{2\pi i} \int_L \chi(s) k^s \times \frac{\Gamma\left(\left((\mu + \vartheta + \nu n)/r\right) + \tau s\right) \Gamma(\alpha + 1 + vs)}{\Gamma\left(\left((\mu + \vartheta + \nu n)/r\right) + \alpha + 1 + (\tau + v) s\right)} ds. \quad (14)$$

Interpreting the right-hand side of (14), in view of the definition (7), we arrive at the result (10). □

Theorem 2. *Let $\lambda, \beta, \vartheta, \gamma \in \mathbb{C}$, $x > 0$, $\Re(\lambda) > 0$, $\Re(\vartheta) < 1$, $f(x) \in L_P(0, \infty)$, $1 \le p \le 2$, $|\arg k| < \lambda\pi/2$, $\lambda > 0$, and*

$a \in \mathbb{C}$; then the fractional integration $K_{x,r}^{\varepsilon,\alpha}$ of the product of M-L function exists, under the condition

$$p^{-1}+q^{-1}=1,\quad \Re\left(\alpha+rv\left(\frac{b_j}{B_j}\right)\right)>-q^{-1},\quad \Re\left(\varepsilon+\alpha+r\tau\left(\frac{b_j}{B_j}\right)\right)>-p^{-1} \tag{15}$$

and then the following formula holds:

$$K_{x,r}^{\varepsilon,\alpha}\left(t^{-\vartheta}E_{\lambda,\beta}^{\gamma}\left(at^{-\nu}\right)\right)(x) = x^{-\vartheta}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)\,n!}\left(ax^{-\nu}\right)^n \times H_{p+2,q+1}^{m,n+2}\left[k \,\middle|\, \begin{matrix}(a_p,A_p),\left(1-\frac{(\varepsilon+\vartheta+\nu n)}{r},\tau\right),(-\alpha,v)\\ \left(-\alpha-\frac{(\varepsilon+\vartheta+\nu n)}{r},\tau+v\right),(b_q,B_q)\end{matrix}\right]. \tag{16}$$

Proof. Let $\wp$ be the left-hand side of (16); using (3) and (5), we have

$$\wp = rx^{\varepsilon}\int_x^{\infty} t^{-\varepsilon-\vartheta-r\alpha-1}\left(t^r-x^r\right)^{\alpha} \times\frac{1}{2\pi i}\int_L \chi(s)(kV)^{-s}ds\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)\,n!}\left(ax^{-\nu}\right)^n dt. \tag{17}$$

Changing the order of the integration valid under the condition given with the theorem statement, we obtain

$$\wp = rx^{\varepsilon}\sum_{n=0}^{\infty}\frac{(\gamma)_n a^n}{\Gamma(\lambda n+\beta)\,n!}\frac{1}{2\pi i}\int_L \chi(s)k^{-s}x^{-r\tau s} \times\left\{\int_x^{\infty} t^{-\varepsilon-\vartheta-\nu n+r\tau s-1}\left(1-\frac{x^r}{t^r}\right)^{\alpha-\upsilon s}dt\right\}ds. \tag{18}$$

Letting the substitution $x^r/t^r = u$, then $t = x/u^{(1/r)}$ in the above term and, using beta function, we get

$$= x^{-\vartheta}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)\,n!}\left(ax^{-\nu}\right)^n\frac{1}{2\pi i}\int_L \chi(s)k^{-s} \times\frac{\Gamma\left(((\varepsilon+\vartheta+\nu n)/r)-\tau s\right)\Gamma(\alpha+1-\upsilon s)}{\Gamma\left(((\varepsilon+\vartheta+\nu n)/r)+\alpha+1-(\tau+\upsilon)s\right)}ds. \tag{19}$$

Interpreting the right-hand side of (19), in view of definition (7), we arrive at the result (16). □

3. Integral Transforms of Fractional Integral Involving M-L Function

In this section, Mellin, Laplace, Euler, Whittaker, and K-transforms of the results established in Theorems 1 and 2 have been obtained.

Euler Transform (Sneddon [14]). The Euler transform of a function $f(t)$ is defined as

$$B\{f(t);a,b\} = \int_0^1 t^{a-1}(1-t)^{b-1}f(t)\,dt,\qquad a,b\in\mathbb{C},\ \Re(a)>0,\ \Re(b)>0. \tag{20}$$

Theorem 3. *Let $\lambda,\beta,\vartheta,\gamma,c,d \in \mathbb{C}$, $\Re(c)>0$, $\Re(d)>0$, $\Re(\vartheta)>0$, $\Re(\lambda)>0$, $p^{-1}+q^{-1}=1$; $f(x)\in L_P(0,\infty)$, $1\le p\le 2$, $|\arg k|<\lambda\pi/2$, $\lambda>0$, $p^{-1}+q^{-1}=1$; $\Re(\mu+r\tau(b_j/B_j))>-q^{-1}$; $\Re(\alpha+rv(b_j/B_j))>-q^{-1}$; $(j=1,\ldots,m)$; then*

$$B\left\{R_{x,r}^{\mu,\alpha}\left(t^{\vartheta-1}E_{\lambda,\beta}^{\gamma}\left(at^{\nu}\right)\right);c,d\right\} = \Gamma(d) \cdot\sum_{n=0}^{\infty}\frac{(\gamma)_n\left(a^n\right)}{\Gamma(\lambda n+\beta)\,n!}\frac{\Gamma(c+\vartheta-1+\nu n)}{\Gamma(c+d+\vartheta-1+\nu n)} \times H_{p+2,q+1}^{m,n+2}\left[k \,\middle|\, \begin{matrix}(a_p,A_p),\left(1-\frac{(\mu+\vartheta+1+\nu n)}{r},\tau\right),(-\alpha,v)\\ \left(-\frac{(\mu+\vartheta+1+\nu n)}{r}-\alpha,\tau+v\right),(b_q,B_q)\end{matrix}\right]. \tag{21}$$

Proof. Using (10) and (20) gives

$$B\left\{R_{x,r}^{\mu,\alpha}\left(t^{\vartheta-1}E_{\lambda,\beta}^{\gamma}\left(at^{\nu}\right)\right);c,d\right\} = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)}\frac{(a)^n}{n!} \times H_{p+2,q+1}^{m,n+2}\left[k \,\middle|\, \begin{matrix}(a_p,A_p),\left(1-\frac{(\mu+\vartheta+1+\nu n)}{r},\tau\right),(-\alpha,v)\\ \left(-\frac{(\mu+\vartheta+1+\nu n)}{r}-\alpha,\tau+v\right),(b_q,B_q)\end{matrix}\right] \times\int_0^1 t^{c+\vartheta+\nu n-1-1}(1-t)^{d-1}dt \tag{22}$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\lambda n+\beta)} \frac{(a)^n}{n!} \frac{\Gamma(c+\vartheta+\nu n-1)\Gamma(d)}{\Gamma(c+d+\vartheta+\nu n-1)} \times H^{m,n+2}_{p+2,q+1}\left[k \,\middle|\, \begin{array}{l} (a_p,A_p),\left(1-\frac{(\mu+\vartheta+1+\nu n)}{r},\tau\right),(-\alpha,\upsilon) \\ \left(-\frac{(\mu+\vartheta+1+\nu n)}{r}-\alpha,\tau+\upsilon\right),(b_q,B_q) \end{array}\right]. \quad (23)$$

Now, we obtain the result (23). This completes the proof of the theorem. □

Theorem 4. *Let* $\lambda,\beta,\vartheta,\gamma,c,d\in\mathbb{C}$, $a>0$, $\Re(c)>0$, $\Re(d)>0$, $\Re(\lambda)>0$, $\Re(1-\vartheta)<1$, $p^{-1}+q^{-1}=1$; $f(x)\in L_P(0,\infty)$, $1\le p\le 2$, $|\arg k|<\lambda\pi/2$, $\lambda>0$, $p^{-1}+q^{-1}=1$; $\Re(\varepsilon+\alpha+r\tau(b_j/B_j))>-p^{-1}$; $\Re(\alpha+r\upsilon(b_j/B_j))>-q^{-1}$; $(j=1,\dots,m)$; *then*

$$B\left\{K^{\varepsilon,\alpha}_{x,r}\left(t^{-\vartheta}E^{\gamma}_{\lambda,\beta}\left(at^{-\nu}\right)\right);c,d\right\}=\Gamma(d) \cdot \sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)\,n!}(a^n)\times\frac{\Gamma(c-\vartheta-\nu n)}{\Gamma(c+d-\vartheta-\nu n)} \cdot H^{m,n+2}_{p+2,q+1}\left[k \,\middle|\, \begin{array}{l} (a_p,A_p),\left(1-\frac{(\varepsilon+\vartheta+\nu n)}{r},\tau\right),(-\alpha,\upsilon) \\ \left(-\alpha-\frac{(\varepsilon+\vartheta+\nu n)}{r},\tau+\upsilon\right),(b_q,B_q) \end{array}\right]. \quad (24)$$

Proof. In similar manner, in proof of Theorem 3, we obtain the result (24). □

Mellin Transform (Debnath and Bhatta [15]). The Mellin transform of a function $f(t)$ is defined as

$$M\{f(t)\}(s)=\int_0^{\infty}t^{s-1}f(t)\,dt,\quad \Re(s)>0. \quad (25)$$

Theorem 5. *All conditions follow from that stated in Theorem 1 with* $\Re(s)>\Re(\nu)$; *the following result holds:*

$$M\left\{R^{\mu,\alpha}_{x,r}\left(t^{\vartheta-1}E^{\gamma}_{\lambda,\beta}\left(at^{\nu}\right)\right)\right\}(s)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)\,n!}(a^n) \times H^{m,n+2}_{p+2,q+1}\left[k \,\middle|\, \begin{array}{l} (a_p,A_p),\left(1-\frac{(\mu+\vartheta+1+\nu n)}{r},\tau\right),(-\alpha,\upsilon) \\ \left(-\frac{(\mu+\vartheta+1+\nu n)}{r}-\alpha,\tau+\upsilon\right),(b_q,B_q) \end{array}\right] \cdot \frac{1}{(s+\vartheta+\nu n-1)}. \quad (26)$$

Proof. From (10) and (25), it gives

$$M\left\{R^{\mu,\alpha}_{x,r}\left(t^{\vartheta-1}E^{\gamma}_{\lambda,\beta}\left(at^{\nu}\right)\right)\right\}(s)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)\,n!}(a^n) \times H^{m,n+2}_{p+2,q+1}\left[k \,\middle|\, \begin{array}{l} (a_p,A_p),\left(1-\frac{(\mu+\vartheta+1+\nu n)}{r},\tau\right),(-\alpha,\upsilon) \\ \left(-\frac{(\mu+\vartheta+1+\nu n)}{r}-\alpha,\tau+\upsilon\right),(b_q,B_q) \end{array}\right] \cdot M\left(t^{\vartheta+\nu n-1}\right). \quad (27)$$

Now, evaluating the Mellin transform of $t^{\vartheta+\nu n-1}$ using formula given by Mathai et al. [16]. we arrive at (26). □

Theorem 6. *All conditions follow from what is stated in Theorem 2 with* $\Re(1-\vartheta)<1$, $\Re(s)>\Re(\nu)$; *the following result holds:*

$$M\left\{K^{\varepsilon,\alpha}_{x,r}\left(t^{-\vartheta}E^{\gamma}_{\lambda,\beta}\left(at^{-\nu}\right)\right)\right\}(s) =\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)\,n!}(a^n)\times H^{m,n+2}_{p+2,q+1}\left[k \,\middle|\, \begin{array}{l} (a_p,A_p),\left(1-\frac{(\varepsilon+\vartheta+\nu n)}{r},\tau\right),(-\alpha,\upsilon) \\ \left(-\alpha-\frac{(\varepsilon+\vartheta+\nu n)}{r},\tau+\upsilon\right),(b_q,B_q) \end{array}\right] \cdot \frac{1}{(s-\varepsilon-\vartheta-\nu n)}. \quad (28)$$

Proof. In similar manner, in proof of Theorem 5, we obtain the result (28). □

Laplace Transform (Sneddon [14]). The Laplace transform of a function $f(t)$, denoted by $F(s)$, is defined by the equation

$$F(s) = (Lf)(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t)\, dt, \quad \Re(s) > 0. \tag{29}$$

Provided the integral (29) is convergent and that the function, $f(t)$, is continuous for $t > 0$ and of exponential order as $t \to \infty$, (29) may be symbolically written as

$$F(s) = L\{f(t); s\} \quad \text{or} \quad f(t) = L^{-1}\{F(s); t\}. \tag{30}$$

The following result is well known:

$$\int_0^\infty e^{-st} t^{p-1} dt = \frac{\Gamma(p)}{s^p}, \quad \Re(p) > 1, \ \Re(s) > 1. \tag{31}$$

Theorem 7. *All conditions follow from what is stated in Theorem 1 with $\Re(s) > 0$ and $\Re(\vartheta + \nu n) > 0$; the following result holds:*

$$L\left\{R_{x,r}^{\mu,\alpha}\left(t^{\vartheta-1}E_{\lambda,\beta}^{\gamma}(at^{\nu})\right); s\right\} = s^{-\vartheta}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)\, n!}(as^{-\nu})^n\,\Gamma(\vartheta+\nu n) \times H_{p+2,q+1}^{m,n+2}\left[k \,\middle|\, \begin{matrix}(a_p, A_p), \left(1-\frac{(\mu+\vartheta+1+\nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\frac{(\mu+\vartheta+1+\nu n)}{r}-\alpha, \tau+\upsilon\right), (b_q, B_q)\end{matrix}\right]. \tag{32}$$

Proof. we can develop similar line by using result of Laplace integral (31). □

Theorem 8. *All conditions follow from what is stated in Theorem 2 with $\Re(s) > 0$ and $\Re(1 - \vartheta - \nu n) > 0$; the following result holds:*

$$L\left\{K_{x,r}^{\varepsilon,\alpha}\left(t^{-\vartheta}E_{\lambda,\beta}^{\gamma}(at^{-\nu})\right)\right\}(s) = s^{1-\vartheta}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)\, n!}(as^{-\nu})^n\,\Gamma(1-\vartheta-\nu n) \times H_{p+2,q+1}^{m,n+2}\left[k \,\middle|\, \begin{matrix}(a_p, A_p), \left(1-\frac{(\varepsilon+\vartheta+\nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\alpha-\frac{(\varepsilon+\vartheta+\nu n)}{r}, \tau+\upsilon\right), (b_q, B_q)\end{matrix}\right]. \tag{33}$$

Proof. In a similar manner, in proof of Theorem 7, we obtain the result (33). □

Whittaker Transform (Whittaker and Watson [17]). Due to Whittaker transform, the following result holds:

$$\int_0^\infty e^{-t/2} t^{\zeta-1} W_{\chi,\omega}(t)\, dt = \frac{\Gamma(1/2+\omega+\zeta)\,\Gamma(1/2-\omega+\zeta)}{\Gamma(1-\chi+\zeta)}, \tag{34}$$

where $\Re(\omega \pm \zeta) > -1/2$ and $W_{\chi,\omega}(t)$ is the Whittaker confluent hypergeometric function:

$$W_{\omega,\zeta}(z) = \frac{\Gamma(-2\omega)}{\Gamma(1/2-\chi-\omega)} M_{\chi,\omega}(z) + \frac{\Gamma(2\omega)}{\Gamma(1/2+\chi+\omega)} M_{\chi,-\omega}(z), \tag{35}$$

where $M_{\chi,\omega}(z)$ is defined by

$$M_{\chi,\omega}(z) = z^{1/2+\omega} e^{-1/2z}\, {}_1F_1\left(\frac{1}{2}+\omega-\chi; 2\omega+1; z\right). \tag{36}$$

Theorem 9. *Following what is stated in Theorem 1 for conditions on parameters, with $\Re[\omega \pm (\vartheta + \zeta + \nu n - 1)] > 1/2$, then the following result holds:*

$$\int_0^\infty e^{-\varphi t/2} t^{\zeta-1} W_{\chi,\omega}(\varphi t)\left\{R_{x,r}^{\mu,\alpha}\left(t^{\vartheta-1}E_{\lambda,\beta}^{\gamma}(at^{\nu})\right)\right\} dt = \varphi^{1-\vartheta-\zeta}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\lambda n+\beta)\, n!}(a\varphi^{-\nu}) \times \frac{\Gamma(\omega+\vartheta+\zeta+\nu n-1/2)\,\Gamma(\vartheta-\omega+\zeta+\nu n-1/2)}{\Gamma(\vartheta-\chi+\zeta+\nu n)} \times H_{p+2,q+1}^{m,n+2}\left[k \,\middle|\, \begin{matrix}(a_p, A_p), \left(1-\frac{(\mu+\vartheta+1+\nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\frac{(\mu+\vartheta+1+\nu n)}{r}-\alpha, \tau+\upsilon\right), (b_q, B_q)\end{matrix}\right]. \tag{37}$$

Proof. Using (10) and (34), it gives

$$\int_0^\infty e^{-\varphi t/2} t^{\zeta-1} W_{\chi,\omega}(\varphi t)\left\{R_{x,r}^{\mu,\alpha}\left(t^{\vartheta-1}E_{\lambda,\beta}^{\gamma}(at^{\nu})\right)\right\} dt$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n}{\Gamma(\lambda n + \beta) n!} \times H^{m,n+2}_{p+2,q+1} \left[k \,\middle|\, \begin{matrix} (a_p, A_p), \left(1 - \frac{(\mu + \vartheta + 1 + \nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\frac{(\mu + \vartheta + 1 + \nu n)}{r} - \alpha, \tau + \upsilon\right), (b_q, B_q) \end{matrix} \right] \times \int_0^{\infty} e^{-\varphi t/2} t^{(\vartheta + \zeta + \nu n - 1) - 1} W_{\chi,\omega}(\varphi t)\, dt. \tag{38}$$

Assume that $t = k, \Rightarrow dt = dk/\varphi$; we get

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n}{\Gamma(\lambda n + \beta) n!} H^{m,n+2}_{p+2,q+1} \left[k \,\middle|\, \begin{matrix} (a_p, A_p), \left(1 - \frac{(\mu + \vartheta + 1 + \nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\frac{(\mu + \vartheta + 1 + \nu n)}{r} - \alpha, \tau + \upsilon\right), (b_q, B_q) \end{matrix} \right] \times \varphi^{1-\vartheta-\zeta-\nu n} \int_0^{\infty} e^{-k/2} k^{(\vartheta + \zeta + \nu n - 1) - 1} W_{\chi,\omega}(k)\, dk. \tag{39}$$

Interpreting the right-hand side of (39), using (34), we arrive at the result (37). □

Theorem 10. *Following what is stated in Theorem 2 for conditions on parameters, with $\Re[\omega \pm (-\vartheta + \zeta - \nu n - 1)] > 1/2$, then the following result holds:*

$$\int_0^{\infty} e^{-\varphi t/2} t^{\zeta - 1} W_{\chi,\omega}(\varphi t) \left\{ K^{\varepsilon,\alpha}_{x,r} \left(t^{-\vartheta} E^{\gamma}_{\lambda,\beta}(a t^{-\nu}) \right) \right\} dt$$
$$= \varphi^{\vartheta - \zeta} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\lambda n + \beta) n!} (a \varphi^{\nu}) \times \frac{\Gamma(\omega - \vartheta + \zeta - \nu n + 1/2)\, \Gamma(-\vartheta - \omega + \zeta - \nu n + 1/2)}{\Gamma(1 - \vartheta - \chi + \zeta - \nu n)} \times H^{m,n+2}_{p+2,q+1} \left[k \,\middle|\, \begin{matrix} (a_p, A_p), \left(1 - \frac{(\varepsilon + \vartheta + \nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\alpha - \frac{(\varepsilon + \vartheta + \nu n)}{r}, \tau + \upsilon\right), (b_q, B_q) \end{matrix} \right]. \tag{40}$$

Proof. In a similar manner, in proof of Theorem 9, we obtain the result (40). □

K-Transform (Erdélyi et al. [18]). This transform is defined by the following integral equation:

$$\Re_{\upsilon}[f(x); p] = g[p; \upsilon] = \int_0^{\infty} (px)^{1/2} K_{\upsilon}(px) f(x)\, dx, \tag{41}$$

where $\Re(p) > 0$; $K_{\upsilon}(x)$ is the Bessel function of the second kind defined by ([18], p. 332)

$$K_{\upsilon}(z) = \left(\frac{\pi}{2z}\right)^{1/2} W_{0,\upsilon}(2z), \tag{42}$$

where $W_{0,\upsilon}(\cdot)$ is the Whittaker function defined in Erdélyi et al. [18].

The following result given in Mathai et al. ([16], p. 54, eq. 2.37) will be used in evaluating the integrals:

$$\int_0^{\infty} t^{\rho - 1} K_{\upsilon}(ax)\, dx = 2^{\rho - 2} a^{-\rho} \Gamma\left(\frac{\rho \pm \upsilon}{2}\right); \quad \Re(a) > 0;\ \Re(\rho \pm \upsilon) > 0. \tag{43}$$

Theorem 11. *Following what is stated in Theorem 1 for conditions on parameters, with $\Re(\omega) > 0$; $\Re((\rho + \vartheta + \nu n - 1) \pm \ell) > 0$, then the following result holds:*

$$\int_0^{\infty} t^{\rho - 1} K_{\ell}(\omega t) \left\{ R^{\mu,\alpha}_{x,r} \left(t^{\vartheta - 1} E^{\gamma}_{\lambda,\beta}(a t^{\nu}) \right) \right\} dt$$
$$= 2^{\rho + \vartheta - 3} \omega^{(1 - \rho - \vartheta)} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\lambda n + \beta) n!} \left(a \left(\frac{2}{\omega}\right)^{\nu} \right) \cdot \Gamma\left(\frac{(\rho + \vartheta + \nu n - 1) \pm \ell}{2} \right) \times H^{m,n+2}_{p+2,q+1} \left[k \,\middle|\, \begin{matrix} (a_p, A_p), \left(1 - \frac{(\mu + \vartheta + 1 + \nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\frac{(\mu + \vartheta + 1 + \nu n)}{r} - \alpha, \tau + \upsilon\right), (b_q, B_q) \end{matrix} \right]. \tag{44}$$

Proof. Using (10) and (44), it gives

$$\int_0^{\infty} t^{\rho - 1} K_{\ell}(\omega t) \left\{ R^{\mu,\alpha}_{x,r} \left(t^{\vartheta - 1}\ {}_{p}^{\mu,\xi,\gamma}K_q (a t^{\nu}) \right) \right\} dt$$
$$= \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n}{\Gamma(\lambda n + \beta) n!} \times H^{m,n+2}_{p+2,q+1} \left[k \,\middle|\, \begin{matrix} (a_p, A_p), \left(1 - \frac{(\mu + \vartheta + 1 + \nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\frac{(\mu + \vartheta + 1 + \nu n)}{r} - \alpha, \tau + \upsilon\right), (b_q, B_q) \end{matrix} \right] \times \int_0^{\infty} t^{(\rho + \vartheta + \nu n - 1) - 1} K_{\ell}(\omega t)\, dt, \tag{45}$$

and we get

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n}{\Gamma(\lambda n + \beta) n!} H^{m,n+2}_{p+2,q+1} \left[k \,\middle|\, \begin{matrix} (a_p, A_p), \left(1 - \frac{(\mu + \vartheta + 1 + \nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\frac{(\mu + \vartheta + 1 + \nu n)}{r} - \alpha, \tau + \upsilon\right), (b_q, B_q) \end{matrix} \right] \times 2^{\rho+\vartheta+\nu n-3} \omega^{(1-\rho-\vartheta-\nu n)} \Gamma\left(\frac{(\rho + \vartheta + \nu n - 1) \pm \ell}{2}\right). \tag{46}$$

Interpreting the right-hand side of (46), we arrive at the result (44). □

Theorem 12. *Following what is stated in Theorem 2 for conditions on parameters, with $\Re(\omega) > 0$; $\Re((\rho - \vartheta - \nu n) \pm \ell) > 0$, then the following result holds:*

$$\int_0^{\infty} t^{\rho-1} K_\ell(\omega t) \left\{ K^{\varepsilon,\alpha}_{x,r} \left(t^{-\vartheta} E^{\gamma}_{\lambda,\beta} (a t^{-\nu}) \right) \right\} dt = 2^{\rho-\vartheta-2} \omega^{(\vartheta-\rho)} \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\lambda n + \beta) n!} \left(a \left(\frac{\omega}{2}\right)^{\nu} \right) \cdot \Gamma\left(\frac{(\rho - \vartheta - \nu n) \pm \ell}{2}\right) \times H^{m,n+2}_{p+2,q+1} \left[k \,\middle|\, \begin{matrix} (a_p, A_p), \left(1 - \frac{(\varepsilon + \vartheta + \nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\alpha - \frac{(\varepsilon + \vartheta + \nu n)}{r}, \tau + \upsilon\right), (b_q, B_q) \end{matrix} \right]. \tag{47}$$

Proof. In a similar manner, in proof of Theorem 11, we obtain the result (47). □

4. Properties of Integral Operators

Here, we established some properties of the operators as consequences of Theorems 1 and 2. These properties show compositions of power function.

Theorem 13. *Following all the conditions on parameters as stated in Theorem 1 with $\Re(\psi + \vartheta) > 0$, then the following result holds true:*

$$x^{\psi} R^{\mu,\alpha}_{x,r} \left[t^{\vartheta-1} E^{\gamma}_{\lambda,\beta} (a t \nu) \right](x) = R^{\mu-\psi,\alpha}_{x,r} \left[t^{\psi+\vartheta-1} E^{\gamma}_{\lambda,\beta} (a t^{\nu}) \right](x). \tag{48}$$

Proof. From (10), the left-hand side of (48), we have

$$x^{\psi} R^{\mu,\alpha}_{x,r} \left[t^{\vartheta-1} E^{\gamma}_{\lambda,\beta} (a t^{\nu}) \right](x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n}{\Gamma(\lambda n + \beta) n!} x^{\vartheta+\psi+\nu n-1} \times H^{m,n+2}_{p+2,q+1} \left[k \,\middle|\, \begin{matrix} (a_p, A_p), \left(1 - \frac{(\mu + \vartheta + 1 + \nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\frac{(\mu + \vartheta + 1 + \nu n)}{r} - \alpha, \tau + \upsilon\right), (b_q, B_q) \end{matrix} \right], \tag{49}$$

and again, by (10), the right-hand side of (48) follows:

$$R^{\mu-\psi,\alpha}_{x,r} \left[t^{\psi+\vartheta-1} E^{\gamma}_{\lambda,\beta} (a t^{\nu}) \right](x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n}{\Gamma(\lambda n + \beta) n!} x^{\vartheta+\psi+\nu n-1} \times H^{m,n+2}_{p+2,q+1} \left[k \,\middle|\, \begin{matrix} (a_p, A_p), \left(1 - \frac{(\mu + \vartheta + 1 + \nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\frac{(\mu + \vartheta + 1 + \nu n)}{r} - \alpha, \tau + \upsilon\right), (b_q, B_q) \end{matrix} \right]. \tag{50}$$

It seems that Theorem 13 readily follows due to (49) and (50). □

Theorem 14. *Following all the conditions on parameters as stated in Theorem 2 with $\Re(\beta + \vartheta) > 0$, then the following result holds true:*

$$x^{-\psi} K^{\varepsilon,\alpha}_{x,r} \left[t^{-\vartheta} E^{\gamma}_{\lambda,\beta} (a t^{-\nu}) \right](x) = K^{\varepsilon-\psi,\alpha}_{x,r} \left[t^{-\vartheta-\psi} E^{\gamma}_{\lambda,\beta} (a t^{-\nu}) \right](x). \tag{51}$$

Proof. From (12), the left-hand side of (51), we have

$$x^{-\psi} K^{\varepsilon,\alpha}_{x,r} \left[t^{-\vartheta} E^{\gamma}_{\lambda,\beta} (a t^{-\nu}) \right](x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n}{\Gamma(\lambda n + \beta) n!} x^{-\psi-\vartheta-\nu n} \times H^{m,n+2}_{p+2,q+1} \left[k \,\middle|\, \begin{matrix} (a_p, A_p), \left(1 - \frac{(\varepsilon + \vartheta + \nu n)}{r}, \tau\right), (-\alpha, \upsilon) \\ \left(-\alpha - \frac{(\varepsilon + \vartheta + \nu n)}{r}, \tau + \upsilon\right), (b_q, B_q) \end{matrix} \right]. \tag{52}$$

Again by (12), the right-hand side of (51) follows:

$$K_{x,r}^{\varepsilon-\psi,\alpha}\left[t^{-\vartheta-\psi}E_{\lambda,\beta}^{\gamma}\left(at^{-\nu}\right)\right](x)$$
$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n(a)^n}{\Gamma(\lambda n+\beta)\,n!}x^{-\psi-\vartheta-\nu n}\times H_{p+2,q+1}^{m,n+2}\left[k\left|\begin{matrix}(a_p,A_p),\left(1-\dfrac{(\varepsilon+\vartheta+\nu n)}{r},\tau\right),(-\alpha,\upsilon)\\ \left(-\alpha-\dfrac{(\varepsilon+\vartheta+\nu n)}{r},\tau+\upsilon\right),(b_q,B_q)\end{matrix}\right.\right]. \quad (53)$$

It seems that Theorem 14 readily follows due to (52) and (53). □

5. Conclusions

In this article, we have investigated and studied two classes of generalized fractional integral operators involving Fox's H-function as kernel due to Saxena and Kumbhat which are applied on M-L function. We discussed the actions of fractional integral operators under Euler, Mellin, Laplace, Whittaker, and K-transforms and results are given in better pragmatic series solutions. The majority of the results derived here are general in nature and compact forms are fairly helpful in deriving a variety of integral formulas in the theory of integral operators which arises in a range of problems of applied sciences like kinematics, diffusion equation, kinetic equation, fractal geometry, anomalous diffusion, propagation of seismic waves, turbulence, etc. We may obtain other special functions such as M-L function and Bessel-Maitland function (see, e.g., ([19–21]) as its special cases and, therefore, various unified fractional integral presentations can be obtained as special cases of our results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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