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Research Article

Generalized Asymptotically Almost Periodic and Generalized Asymptotically Almost Automorphic Solutions of Abstract Multiterm Fractional Differential Inclusions

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The main aim of this paper is to investigate generalized asymptotical almost periodicity and generalized asymptotical almost automorphy of solutions to a class of abstract (semilinear) multiterm fractional differential inclusions with Caputo derivatives. We illustrate our abstract results with several examples and possible applications.

1. Introduction and Preliminaries

Almost periodic and asymptotically almost periodic solutions of differential equations in Banach spaces have been considered by many authors so far (for the basic information on the subject, we refer the reader to the monographs [1–10]). Concerning almost automorphic and asymptotically almost automorphic solutions of abstract differential equations, one may refer, for example, to the monographs by Diagana [4], N'Guérékata [5], and references cited therein.

Of concern is the following abstract multiterm fractional differential inclusion:

$$\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) \in \mathscr{A} \mathbf{D}_t^{\alpha} u(t) + f(t), \quad t \geq 0,$$
$$u^{(k)}(0) = u_k, \quad k = 0, \ldots, \lceil \alpha_n \rceil - 1, \tag{1}$$

where $n \in \mathbb{N} \setminus \{1\}$, $A_1, \ldots, A_{n-1}$ are bounded linear operators on a Banach space X, $\mathscr{A}$ is a closed multivalued linear operator on X, $0 \leq \alpha_1 < \cdots < \alpha_n$, $0 \leq \alpha < \alpha_n$, $f(\cdot)$ is an X-valued function, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α ([11, 12]). In this paper, we provide the notions of k-regularized (C_1, C_2)-existence and uniqueness propagation families for (1) and k-regularized C-propagation families for (1). In Section 4, we profile these solution operator families in terms of vector-valued Laplace transform, while in Section 5 we consider asymptotical behaviour of analytic integrated solution operator families for (1). The main result of paper, Theorem 18, enables one to consider asymptotically periodic solutions, asymptotically almost periodic solutions, and asymptotically almost automorphic solutions of certain classes of abstract integrodifferential equations in Banach spaces. In a similar way, we can give the basic information about the following abstract semilinear multiterm fractional differential inclusion:

$$\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) \in \mathscr{A} \mathbf{D}_t^{\alpha} u(t) + f(t, u(t)),$$
$$t \geq 0, \tag{2}$$
$$u^{(k)}(0) = u_k, \quad k = 0, \ldots, \lceil \alpha_n \rceil - 1,$$

where $n \in \mathbb{N} \setminus \{1\}$, $A_1, \ldots, A_{n-1}$ are bounded linear operators on a Banach space X, $\mathscr{A}$ is a closed multivalued linear operator on X, $0 \leq \alpha_1 < \cdots < \alpha_n$, $0 \leq \alpha < \alpha_n$, and $f(\cdot, \cdot)$ is an X-valued function satisfying certain assumptions.

Since we essentially follow the method proposed by Kostić et al. [13] (see also [12, Subsection 2.10.1]), the boundedness of linear operator $A_1, \ldots, A_{n-1}$ is crucial for applications of vector-valued Laplace transform and therefore will be the starting point in our work.

The organization and main ideas of this paper can be briefly described as follows. In Section 2, we present the basic information about Stepanov and Weyl generalizations of asymptotically almost periodic functions and asymptotically almost automorphic functions (Proposition 4 is the only new contribution in this section). The main aim of third section is to give a brief recollection of results and definitions about multivalued linear operators in Banach spaces; in a separate Section 3.1, we analyze degenerate (a, k)-regularized C-resolvent families subgenerated by multivalued linear operators. Section 4, which is written almost in an expository manner, is devoted to the study of k-regularized C-propagation families for (1). The main result of fifth section is Theorem 18, where we investigate the asymptotic behaviour of k_i-regularized C-propagation families for (1). In the proof of this theorem, we use the well-known results on analytical properties of vector-valued Laplace transform established by Sova in [14] (see, e.g., [2, Theorem 2.6.1]) in place of Cuesta's method established in the proof of [15, Theorem 2.1]. The proof of Theorem 18 is much simpler and transparent than that of [15, Theorem 2.1] because of the simplicity of contour Γ in our approach. We will essentially use this fact for improvement of some known results on the asymptotic behaviour of solution operator families governing solutions of abstract two-term fractional differential equations, established recently by Keyantuo et al. [16] and Luong [17]. Contrary to a great number of papers from the existing literature, Theorem 18 is applicable to the almost sectorial operators, generators of integrated or C-regularized semigroups, and multivalued linear operators employing in the analysis of (fractional) Poisson heat equations in L^p-spaces ([18, 19]). For more details, see Section 6.

We use the standard notation throughout the paper. By X we denote a complex Banach space. If Y is also such a space, then by $L(X, Y)$ we denote the space of all continuous linear mappings from X into Y; $L(X) \equiv L(X, X)$. If A is a linear operator acting on X, then the domain, kernel space, and range of A will be denoted by $D(A)$, $N(A)$, and $R(A)$, respectively. The symbol I denotes the identity operator on X. By $C_b([0,\infty) : X)$ we denote the space consisted of all bounded continuous functions from $[0,\infty)$ into X; the symbol $C_0([0,\infty) : X)$ denotes the closed subspace of $C_b([0,\infty) : X)$ consisting of functions vanishing at infinity. By $BUC([0,\infty) : X)$ we denote the space consisted of all bounded uniformly continuous functions from $[0,\infty)$ to X. This space becomes one of Banach's spaces when equipped with the sup-norm. Let us recall that a subset X' of X is said to be total in X iff its linear span is dense in X.

Let $f \in L^1_{\mathrm{loc}}([0,\infty) : X)$. Consider the Laplace integral

$$(\mathscr{L} f)(\lambda) := \tilde{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\, dt \\ := \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} f(t)\, dt, \tag{3}$$

for $\lambda \in \mathbb{C}$. If $\tilde{f}(\lambda_0)$ exists for some $\lambda_0 \in \mathbb{C}$, then we define the abscissa of convergence of $\tilde{f}(\cdot)$ by

$$\mathrm{abs}_X(f) := \inf \left\{ \Re\lambda : \tilde{f}(\lambda) \text{ exists} \right\}; \tag{4}$$

otherwise, $\mathrm{abs}_X(f) := +\infty$. It is said that $f(\cdot)$ is Laplace transformable or equivalently that $f(\cdot)$ belongs to the class (P1)-X, iff $\mathrm{abs}_X(f) < \infty$; in scalar-valued case, we write $(\mathrm{P1}) := (\mathrm{P1})\text{-}\mathbb{C}$ and $\mathrm{abs}(f) := \mathrm{abs}_{\mathbb{C}}(f)$.

If $\zeta > 0$, then we define $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $t > 0$; $g_0(t) \equiv$ the Dirac delta distribution. Here, $\Gamma(\cdot)$ denotes the Gamma function. Set $\Sigma_\beta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \beta\}$ $(\beta \in (0, \pi])$, $\mathbb{N}_n := \{1, \ldots, n\}$, and $\mathbb{N}_n^0 := \{0, 1, \ldots, n\}$ $(n \in \mathbb{N})$.

During the past few decades, considerable interest in fractional calculus and fractional differential equations has been stimulated due to their numerous applications in many areas of physics and engineering. A great number of important phenomena in electromagnetics, acoustics, viscoelasticity, aerodynamics, electrochemistry, and cosmology are well described and modelled by fractional differential equations. For basic information about fractional calculus and nondegenerate fractional differential equations, one may refer, for example, to [11, 12, 20–25] and the references cited therein.

We will use only the Caputo fractional derivatives. Let $\zeta > 0$. Then the Caputo fractional derivative $\mathbf{D}_t^\zeta u$ ([11, 12]) is defined for those functions $u \in C^{\lceil \zeta \rceil - 1}([0,\infty) : X)$ for which $g_{\lceil \zeta \rceil - \zeta} * (u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1}) \in C^{\lceil \zeta \rceil}([0,\infty) : X)$, by

$$\mathbf{D}_t^\zeta u(t) := \frac{d^{\lceil \zeta \rceil}}{dt^{\lceil \zeta \rceil}} \left[g_{\lceil \zeta \rceil - \zeta} * \left(u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0)\, g_{j+1} \right) \right]. \tag{5}$$

Assuming that the Caputo fractional derivative $\mathbf{D}_t^\zeta u(t)$ exists, then for each number $\nu \in (0, \zeta)$ the Caputo fractional derivative $\mathbf{D}_t^\nu u(t)$ exists, as well.

The Mittag-Leffler function $E_{\beta,\gamma}(z)$ $(\beta > 0,\ \gamma \in \mathbb{R})$ is defined by

$$E_{\beta,\gamma}(z) := \sum_{k=0}^\infty \frac{z^k}{\Gamma(\beta k + \gamma)}, \quad z \in \mathbb{C}. \tag{6}$$

Set $E_\beta(z) := E_{\beta,1}(z)$, $z \in \mathbb{C}$.

The asymptotic behaviour of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is given in the following lemma (see, e.g., [12]):

Lemma 1. *Let $0 < \sigma < (1/2)\pi$. Then, for every $z \in \mathbb{C} \setminus \{0\}$ and $m \in \mathbb{N} \setminus \{1\}$,*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_s Z_s^{1-\beta} e^{Z_s} - \sum_{j=1}^{m-1} \frac{z^{-j}}{\Gamma(\beta - \alpha j)} + O(|z|^{-m}), \tag{7}$$

where Z_s is defined by $Z_s := z^{1/\alpha} e^{2\pi i s/\alpha}$ and the first summation is taken over all those integers s satisfying $|\arg(z) + 2\pi s| < \alpha(\pi/2 + \sigma)$.

If $\alpha \in (0,2)\backslash\{1\}$, $\beta > 0$, and $N \in \mathbb{N}\backslash\{1\}$, then the following special cases of Lemma 1 hold good:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} + \varepsilon_{\alpha,\beta}(z), \quad |\arg(z)| < \frac{\alpha\pi}{2},$$
$$E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z), \quad |\arg(-z)| < \pi - \frac{\alpha\pi}{2}, \tag{8}$$

where

$$\varepsilon_{\alpha,\beta}(z) = \sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O\left(|z|^{-N}\right), \quad |z| \longrightarrow \infty. \tag{9}$$

For further information about the Mittag-Leffler functions, compare [11, 12] and the references cited there.

2. Stepanov and Weyl Generalizations of (Asymptotically) Almost Periodic and Almost Automorphic Functions

The class of almost periodic functions was introduced by H. Bohr in 1925 and later generalized by many other mathematicians. Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $f : I \rightarrow X$ be continuous, where X is a Banach space with the norm $\|\cdot\|$. For any number $\epsilon > 0$ given in advance, we say that a number $\tau > 0$ is an ϵ-period for $f(\cdot)$ iff $\|f(t+\tau) - f(t)\| \leq \epsilon$, $t \in I$. The set consisting of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f,\epsilon)$. We say that $f(\cdot)$ is almost periodic, a.p. for short, iff for each $\epsilon > 0$ the set $\vartheta(f,\epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f,\epsilon)$. For basic information about various classes of almost periodic functions and their generalizations, we refer the reader to [4–8, 10, 12, 13, 16, 19, 21, 26–34]. The space consisting of all almost periodic functions from the interval I into X will be denoted by $\mathrm{AP}(I : X)$.

It is well known that the vector space $\mathrm{P}_T([0,\infty) : X)$ consisting of all bounded continuous T-periodic functions, denoted by $\mathrm{P}_T([0,\infty) : X)$, $\mathrm{P}_T([0,\infty) : X) := \{f \in C_b([0,\infty)) : f(t+T) = f(t), t \geq 0\}$, is a vector subspace of $\mathrm{AP}([0,\infty) : X)$. Set $\mathrm{AP}_T([0,\infty) : X) := \mathrm{P}_T([0,\infty) : X) \oplus C_0([0,\infty) : X)$.

Suppose that $1 \leq p < \infty$, $l > 0$, and $f, g \in L^p_{\mathrm{loc}}(I : X)$, where $I = \mathbb{R}$ or $I = [0,\infty)$. Define the Stepanov "metric" by

$$D^p_{S_l}[f(\cdot), g(\cdot)] := \sup_{x\in I} \left[\frac{1}{l}\int_x^{x+l} \|f(t) - g(t)\|^p \, dt\right]^{1/p}. \tag{10}$$

Then, in scalar-valued case, there exists

$$D^p_W[f(\cdot), g(\cdot)] := \lim_{l\to\infty} D^p_{S_l}[f(\cdot), g(\cdot)] \tag{11}$$

in $[0,\infty]$. The distance appearing in (11) is called the Weyl distance of $f(\cdot)$ and $g(\cdot)$. The Stepanov and Weyl "norm" of $f(\cdot)$ are introduced by

$$\|f\|_{S^p_l} := D^p_{S_l}[f(\cdot), 0],$$
$$\|f\|_{W^p} := D^p_W[f(\cdot), 0], \tag{12}$$

respectively. We say that a function $f \in L^p_{\mathrm{loc}}(I : X)$ is Stepanov p-bounded, S^p-bounded shortly, iff

$$\|f\|_{S^p} := \sup_{t\in I}\left(\int_t^{t+1} \|f(s)\|^p \, ds\right)^{1/p} < \infty. \tag{13}$$

The space $L^p_S(I : X)$ consisting of all S^p-bounded functions becomes a Banach space when equipped with the above norm. A function $f \in L^p_S(I : X)$ is called Stepanov p-almost periodic, S^p-almost periodic shortly, iff the function $\hat{f} : I \rightarrow L^p([0,1] : X)$, defined by $\hat{f}(t)(s) := f(t+s)$, $t \in I$, $s \in [0,1]$ is almost periodic. It is said that $f \in L^p_S([0,\infty) : X)$ is asymptotically Stepanov p-almost periodic, asymptotically S^p-almost periodic for short, iff $\hat{f} : [0,\infty) \rightarrow L^p([0,1] : X)$ is asymptotically almost periodic.

It is a well-known fact that if $f(\cdot)$ is an almost periodic (resp., a.a.p.) function then $f(\cdot)$ is also S^p-almost periodic (resp., asymptotically S^p-a.a.p.) for $1 \leq p < \infty$. The converse statement is not true, in general.

By $\mathrm{APS}^p(I : X)$ we denote the space consisted of all S^p-almost periodic functions $I \mapsto X$. A function $f \in L^p_S([0,\infty) : X)$ is said to be asymptotically Stepanov p-almost periodic, asymptotically S^p-almost periodic for short, iff $\hat{f} : [0,\infty) \rightarrow L^p([0,1] : X)$ is asymptotically almost periodic. By $\mathrm{APS}^p([0,\infty) : X)$ and $\mathrm{AAPS}^p([0,\infty) : X)$ we denote the vector spaces consisting of all Stepanov p-almost periodic functions and asymptotically Stepanov p-almost periodic functions, respectively.

Let us recall that any asymptotically almost periodic function is also asymptotically Stepanov p-almost periodic ($1 \leq p < \infty$). The converse statement is clearly not true because an asymptotically Stepanov p-almost periodic function need not be continuous.

We are continuing by explaining the basic definitions and results about the (asymptotically) Weyl-almost periodic functions.

Definition 2 (see [35]). Assume that $I = \mathbb{R}$ or $I = [0,\infty)$. Let $1 \leq p < \infty$ and $f \in L^p_{\mathrm{loc}}(I : X)$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-p-almost periodic, $f \in e\text{–}W^p_{\mathrm{ap}}(I : X)$ for short, iff for each $\epsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l} \|f(t+\tau) - f(t)\|^p \, dt\right]^{1/p} \leq \epsilon,$$
$$\text{i.e., } D^p_{S_l}[f(\cdot+\tau), f(\cdot)] \leq \epsilon. \tag{14}$$

(ii) It is said that the function $f(\cdot)$ is Weyl-p-almost periodic, $f \in W^p_{\mathrm{ap}}(I : X)$ for short, iff for each $\epsilon > 0$

we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l\to\infty} \sup_{x\in I} \left[\frac{1}{l} \int_x^{x+l} \| f(t+\tau) - f(t) \|^p \, dt \right]^{1/p} \le \epsilon, \quad \text{i.e., } \lim_{l\to\infty} D_{S_l}^p [f(\cdot+\tau), f(\cdot)] \le \epsilon. \tag{15}$$

We know that $APS^p(I : X) \subseteq e - W_{ap}^p(I : X) \subseteq W_{ap}^p(I : X)$ in the set theoretical sense and that any of these two inclusions can be strict ([26]).

We refer the reader to [35] for basic definitions and results about asymptotically Weyl-almost periodic functions.

Definition 3. We say that $q \in L_{loc}^p([0,\infty) : X)$ is Weyl-p-vanishing iff

$$\lim_{t\to\infty} \lim_{l\to\infty} \sup_{x\ge 0} \left[\frac{1}{l} \int_x^{x+l} \| q(t+s) \|^p \, ds \right]^{1/p} = 0. \tag{16}$$

It is clear that for any function $q \in L_{loc}^p([0,\infty) : X)$ we can replace the limits in (16). It is said that $q \in L_{loc}^p([0,\infty) : X)$ is equi-Weyl-p-vanishing iff

$$\lim_{l\to\infty} \lim_{t\to\infty} \sup_{x\ge 0} \left[\frac{1}{l} \int_x^{x+l} \| q(t+s) \|^p \, ds \right]^{1/p} = 0. \tag{17}$$

If $q \in L_{loc}^p([0,\infty) : X)$ and $q(\cdot)$ is equi-Weyl-p-vanishing, then $q(\cdot)$ is Weyl-p-vanishing. The converse statement does not hold, in general ([35]). By $W_0^p([0,\infty) : X)$ and $e - W_0^p([0,\infty) : X)$ we denote the vector spaces consisting of all Weyl-p-vanishing functions and equi-Weyl-p-vanishing functions, respectively.

It can be simply proved that the limit of any uniformly convergent sequence of bounded continuous functions that are (asymptotically) almost periodic or automorphic, respectively (asymptotically), Stepanov almost periodic or automorphic, has again this property. The following result holds for the Weyl class.

Proposition 4. *Let (f_n) be a uniformly convergent sequence of functions from $e - W^p([0,\infty) : X) \cap C_b([0,\infty) : X)$, respectively, $W^p([0,\infty) : X) \cap C_b([0,\infty) : X)$, where $1 \le p < \infty$. If $f(\cdot)$ is the corresponding limit function, then $f \in e - W^p([0,\infty) : X) \cap C_b([0,\infty) : X)$, respectively, $f \in W^p([0,\infty) : X) \cap C_b([0,\infty) : X)$.*

Proof. We will prove the part (i) only for the equi-Weyl-p-almost periodic functions. It is clear that $f \in C_b([0,\infty) : X)$. Let $\epsilon > 0$ be given in advance. Then there exists an integer $n_0(\epsilon)$ such that for each $n \ge n_0(\epsilon)$ we have that

$$\| f_n(t) - f(t) \| \le \epsilon, \quad t \ge 0. \tag{18}$$

By definition, we know that there exist two real numbers $l_{n_0} > 0$ and $L_{n_0} > 0$ such that any interval $I' \subseteq I$ of length L_{n_0} contains a point $\tau_{n_0} \in I'$ such that

$$\sup_{x\in I} \left[\frac{1}{l_{n_0}} \int_x^{x+l_{n_0}} \| f_{n_0}(t+\tau) - f_{n_0}(t) \|^p \, dt \right]^{1/p} \le \epsilon. \tag{19}$$

Then, for the proof of equi-Weyl-p-almost periodicity of function $f(\cdot)$, we can choose the same $l := l_{n_0} > 0$ and $L = L_{n_0} > 0$, and the same $\tau := \tau_{n_0}$ from any subinterval $I' \subseteq [0,\infty)$; speaking-matter-of-factly, we have

$$\| f(t+\tau) - f(t) \| \le \| f(t+\tau) - f_{n_0}(t+\tau) \| + \| f_{n_0}(t+\tau) - f_{n_0}(t) \| + \| f(t) - f_{n_0}(t) \| \tag{20}$$

for all $t \ge 0$, so that a simple calculation involving (18) gives the existence of a finite constant $c_p > 0$ such that

$$\sup_{x\in I} \left[\frac{1}{l} \int_x^{x+l} \| f(t+\tau) - f(t) \|^p \, dt \right]^{1/p} \le c_p \left[\epsilon + \sup_{x\in I} \left[\frac{1}{l_{n_0}} \int_x^{x+l_{n_0}} \| f_{n_0}(t+\tau) - f_{n_0}(t) \|^p \, dt \right]^{1/p} \right] \le 2c_p\epsilon. \tag{21}$$

Then the final result simply follows from (19). □

And, just a few words about (generalized) automorphic extensions of introduced classes, where our results clearly apply. Let $f : \mathbb{R} \to X$ be continuous. As it is well known, $f(\cdot)$ is called almost automorphic, a.a. for short, iff for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \to X$ such that

$$\lim_{n\to\infty} f(t+a_n) = g(t), \quad \lim_{n\to\infty} g(t-a_n) = f(t), \tag{22}$$

pointwise for $t \in \mathbb{R}$. If this is the case, then it is well known that $f \in C_b(\mathbb{R} : X)$ and that the limit function $g(\cdot)$ must be bounded on $\mathbb{R}$ but not necessarily continuous on $\mathbb{R}$. Furthermore, it is clear that the uniform convergence of one of the limits appearing in (22) implies the convergence of the second one in this equation and that, in this case, the function $f(\cdot)$ has to be almost periodic and the function $g(\cdot)$ has to be continuous. If the convergence of limits appearing in (22) is uniform on compact subsets of $\mathbb{R}$, then we say that $f(\cdot)$ is compactly almost automorphic, c.a.a. for short. The vector space consisting of all almost automorphic, respectively, compactly almost automorphic functions, is denoted by $AA(\mathbb{R} : X)$, respectively, $AA_c(\mathbb{R} : X)$. By Bochner's criterion [4], any almost periodic function has to be compactly almost automorphic. The converse statement is not true, however [36]. It is also worth noting that P. Bender proved in doctoral dissertation that that a.a. function $f(\cdot)$ is c.a.a. iff it is uniformly continuous (1966, Iowa State University).

It is well-known that the reflexion at zero keeps the spaces $AA(\mathbb{R} : X)$ and $AA_c(\mathbb{R} : X)$ unchanged and that the function $g(\cdot)$ from (22) satisfies $\|f\|_\infty = \|g\|_\infty$ and $R(g) \subseteq \overline{R(f)}$, later needed to be a compact subset of X. An interesting example of an almost automorphic function that is not almost periodic has been constructed by W. A. Veech

$$f(t) := \frac{2 + e^{it} + e^{it\sqrt{2}}}{\left|2 + e^{it} + e^{it\sqrt{2}}\right|}, \quad t \in \mathbb{R}. \tag{23}$$

A continuous function $f : \mathbb{R} \to X$ is called asymptotically (compact) almost automorphic, a.(c.)a.a. for short, iff there exist a function $h \in C_0([0,\infty) : X)$ and a (compact) almost automorphic function $q : \mathbb{R} \to X$ such that $f(t) = h(t) + q(t)$, $t \geq 0$. Using Bochner's criterion again, it readily follows that any asymptotically almost periodic function $[0,\infty) \mapsto X$ is asymptotically (compact) almost automorphic. It is well known that the spaces of almost periodic, almost automorphic, compactly almost automorphic functions and asymptotically (compact) almost automorphic functions are closed subspaces of $C_b(\mathbb{R} : X)$ when equipped with the sup-norm.

We refer the reader to [28] for the notion of Stepanov-like almost automorphic functions. The concepts of Weyl-almost automorphy and Weyl pseudo almost automorphy, more general than those of Stepanov almost automorphy and Stepanov pseudo almost automorphy, were introduced by Abbas [37] in 2012. Besides the concepts of Stepanov-like almost automorphic functions, our results apply also to the classes of Weyl-almost automorphic functions and Besicovitch almost automorphic functions, introduced in [38] (cf. [7, 39] for more details).

3. Multivalued Linear Operators in Banach Spaces

In this section, we will present some necessary definitions and auxiliary results from the theory of multivalued linear operators in Banach spaces. For further information in this direction, the reader may consult the monographs by Cross [40] and Favini and Yagi [18].

Let X and Y be two Banach spaces over the field of complex numbers. A multivalued mapping $\mathcal{A} : X \to P(Y)$ is said to be a multivalued linear operator (MLO) iff the following two conditions hold:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x, y \in D(\mathcal{A})$, and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

In the case that $X = Y$, then we say that $\mathcal{A}$ is an MLO in X. It is well-known that the equality $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$ holds for every $x, y \in D(\mathcal{A})$ and for every $\lambda, \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is always a linear subspace of Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Put $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. Then the set $N(\mathcal{A}) := \mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the kernel of $\mathcal{A}$. The inverse $\mathcal{A}^{-1}$ of an MLO is generally defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It is checked at once that $\mathcal{A}^{-1}$ is an MLO in X, and that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, that is, if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is called injective. If $\mathcal{A}, \mathcal{B} : X \to P(Y)$ are two MLOs, then we define its sum $\mathcal{A}+\mathcal{B}$ by $D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A}+\mathcal{B})$. It is evident that $\mathcal{A}+\mathcal{B}$ is likewise an MLO. We write $\mathcal{A} \subseteq \mathcal{B}$ iff $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$.

Let $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : Y \to P(Z)$ be two MLOs, where Z is a complex Banach space. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. A simple proof shows that $\mathcal{B}\mathcal{A} : X \to P(Z)$ is an MLO and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A} : X \to P(Y)$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$. Then $z\mathcal{A} : X \to P(Y)$ is an MLO and $(\omega z)\mathcal{A} = \omega(z\mathcal{A}) = z(\omega\mathcal{A})$, $z, \omega \in \mathbb{C}$.

It is said that an MLO $\mathcal{A} : X \to P(Y)$ is closed iff for any two sequences (x_n) in $D(\mathcal{A})$ and (y_n) in Y such that $y_n \in \mathcal{A}x_n$; for all $n \in \mathbb{N}$ we have that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

We need the following lemma from [19].

Lemma 5. *Let Ω be a locally compact, separable metric space, and let μ be a locally finite Borel measure defined on Ω. Suppose that $\mathcal{A} : X \to P(Y)$ is a closed MLO. Let $f : \Omega \to X$ and $g : \Omega \to Y$ be μ-integrable, and let $g(x) \in \mathcal{A}f(x)$, $x \in \Omega$. Then $\int_\Omega f\,d\mu \in D(\mathcal{A})$ and $\int_\Omega g\,d\mu \in \mathcal{A}\int_\Omega f\,d\mu$.*

Henceforward, Ω will always be an appropriate subspace of $\mathbb{R}$ and μ will always be the Lebesgue measure defined on Ω.

Denote by (P1)-X the vector space consisting of all Laplace transformable functions $f : [0,\infty) \to X$; by $\tilde{f}(\cdot)$ we denote the Laplace transform of $f(\cdot)$, defined as in [2]. We need also the following lemma from [19].

Lemma 6. *Assume that $\mathcal{A} : X \to P(Y)$ is a closed MLO and that $f \in$ (P1)-X, $l \in$ (P1)-Y and $(\tilde{f}(\lambda), \tilde{l}(\lambda)) \in \mathcal{A}$, $\lambda \in \mathbb{C}$ for $\Re\lambda > \max(\operatorname{abs}(f), \operatorname{abs}(l))$. Then $l(t) \in \mathcal{A}f(t)$ for any $t \geq 0$ which is a point of continuity of both functions $f(t)$ and $l(t)$.*

Suppose that $\mathcal{A}$ is an MLO in X and that $C \in L(X)$ is possibly noninjective operator satisfying $C\mathcal{A} \subseteq \mathcal{A}C$. Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;

(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on X.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$); the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$).

We will use the following extension of [19, Theorem 1.2.4(i)], whose proof can be left to the reader as an easy exercise (see also the proof of [18, Theorem 1.7, p. 24]).

Lemma 7. *Let $B, D \in L(X)$ and let $\mathcal{A}$ be an MLO. If $BD = DB$, $B\mathcal{A} \subseteq \mathcal{A}B$, $D\mathcal{A} \subseteq \mathcal{A}D$, and $(B - \mathcal{A})^{-1}D \in L(X)$, then one has*

$$\begin{aligned}(B - \mathcal{A})^{-1} D\mathcal{A} &\subseteq B(B - \mathcal{A})^{-1} D - D \\ &\subseteq \mathcal{A}(B - \mathcal{A})^{-1} D.\end{aligned} \tag{24}$$

Suppose that $\mathscr{A}$ is an MLO in X. Then $\lambda \in \mathbb{C}$ is said to be an eigenvalue of $\mathscr{A}$ iff there exists an element $x \in X \setminus \{0\}$ such that $\lambda x \in \mathscr{A}x$; we call x an eigenvector of operator $\mathscr{A}$ corresponding to the eigenvalue λ. Let us recall that, in purely multivalued case, an element $x \in X\setminus\{0\}$ can be an eigenvector of operator $\mathscr{A}$ corresponding to different values of scalars λ. The point spectrum of $\mathscr{A}$, $\sigma_p(\mathscr{A})$ for short, is defined as the union of all eigenvalues of $\mathscr{A}$.

3.1. Degenerate (a,k)-Regularized C-Resolvent Operator Families. If it is not stated otherwise, we assume that $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\text{loc}}([0,\tau))$, $a \neq 0$, $\mathscr{A} : X \to P(X)$ is an MLO, $C_1 \in L(Y,X)$, $C_2 \in L(X)$ is injective, $C \in L(X)$ is injective, and $C\mathscr{A} \subseteq \mathscr{A}C$.

We need the following notions from [19].

Definition 8. Suppose $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\text{loc}}([0,\tau))$, $a \neq 0$, $\mathscr{A} : X \to P(X)$ is an MLO, $C_1 \in L(Y,X)$, and $C_2 \in L(X)$ is injective.

(i) Then it is said that $\mathscr{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)} \subseteq L(Y,X) \times L(X)$ iff the mappings $t \mapsto R_1(t)y$, $t \geq 0$, and $t \mapsto R_2(t)x$, $t \in [0,\tau)$, are continuous for every fixed $x \in X$ and $y \in Y$, and the following conditions hold:

$$\left(\int_0^t a(t-s) R_1(s) y\,ds, R_1(t) y - k(t) C_1 y\right) \in \mathscr{A}, \quad t \in [0,\tau), \ y \in Y, \tag{25}$$

$$\int_0^t a(t-s) R_2(s) y\,ds = R_2(t) x - k(t) C_2 x, \quad \text{whenever } t \in [0,\tau), \ (x,y) \in \mathscr{A}. \tag{26}$$

(ii) Let $(R_1(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ be strongly continuous. Then it is said that $\mathscr{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ iff (25) holds.

(iii) Let $(R_2(t))_{t\in[0,\tau)} \subseteq L(X)$ be strongly continuous. Then it is said that $\mathscr{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ iff (26) holds.

Definition 9. Suppose that $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{\text{loc}}([0,\tau))$, $a \neq 0$, $\mathscr{A} : X \to P(X)$ is an MLO, $C \in L(X)$ is injective, and $C\mathscr{A} \subseteq \mathscr{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (a,k)-regularized C-resolvent family with a subgenerator $\mathscr{A}$ iff $(R(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized C-uniqueness family having $\mathscr{A}$ as subgenerator, $R(t)C = CR(t)$, and $R(t)\mathscr{A} \subseteq \mathscr{A}R(t)$ $(t \in [0,\tau))$.

If $\tau = \infty$, $(R(t))_{t\geq 0}$ is said to be exponentially bounded (bounded) iff there exists $\omega \in \mathbb{R}$ $(\omega = 0)$ such that the family $\{e^{-\omega t}R(t) : t \geq 0\} \subseteq L(X)$ is bounded. If $k(t) = g_{\alpha+1}(t)$, where $\alpha \geq 0$, then it is also said that $(R(t))_{t\in[0,\tau)}$ is an α-times integrated (a,C)-resolvent family; 0-times integrated (a,C)-resolvent family is further abbreviated to (a,C)-resolvent family. We accept a similar terminology for the classes of mild (a,k)-regularized C_1-existence families and mild (a,k)-regularized C_2-uniqueness families.

The integral generator of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ (mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)}$) is defined through

$$\mathscr{A}_{\text{int}} := \left\{(x,y) \in X \times X : R_2(t) x - k(t) C_2 x = \int_0^t a(t-s) R_2(s) y\,ds, \ t \in [0,\tau)\right\}; \tag{27}$$

the integral generator of an (a,k)-regularized C-regularized family $(R(t))_{t\in[0,\tau)}$ is defined in a similar fashion. The integral generator $\mathscr{A}_{\text{int}}$ is a closed MLO in X which is, in fact, the maximal subgenerator of $(R_2(t))_{t\in[0,\tau)}$ $((R(t))_{t\in[0,\tau)})$ with respect to the set inclusion. We refer the reader to [19] for the notion of an exponentially bounded, analytic (a,k)-regularized C-resolvent operator family.

Unless stated otherwise, we will always assume henceforth that the function $k(\cdot)$ is a scalar-valued kernel on $[0,\tau)$ and that the operator $C \in L(X)$ is injective. For more details about abstract degenerate differential equations, the reader may consult the monographs [18, 41–43].

4. k-Regularized C-Propagation Families for (1)

Recall that $n \in \mathbb{N} \setminus \{1\}$, $A_1, \ldots, A_{n-1}$ are bounded linear operators on a Banach space X, $\mathscr{A}$ is a closed multivalued linear operator on X, $0 \leq \alpha_1 < \cdots < \alpha_n$, $0 \leq \alpha < \alpha_n$, and $f(t)$ is an X-valued function. Henceforth, we always assume that $k, k_1, k_2, \ldots$ are scalar-valued kernels and $a \neq 0$ in $L^1_{\text{loc}}([0,\tau))$. Set $m_j := \lceil \alpha_j \rceil$, $1 \leq j \leq n$, $m := m_0 := \lceil \alpha \rceil$, $A_0 := \mathscr{A}$, and $\alpha_0 := \alpha$.

We will use the following definition.

Definition 10. A function $u \in C^{m_n-1}([0,\infty) : X)$ is called a (strong) solution of (1) iff $A_i \mathbf{D}_t^{\alpha_i} u \in C([0,\infty) : X)$ for $0 \leq i \leq n-1$, $g_{m_n-\alpha_n} * (u - \sum_{k=0}^{m_n-1} u_k g_{k+1}) \in C^{m_n}([0,\infty) : X)$, and (1) holds.

Integrating both sides of (1) α_n-times and employing the closedness of $\mathscr{A}$, Lemma 5, and the equality [11, (1.21)], it readily follows that any strong solution $u(t)$, $t \geq 0$ of (1) satisfies the following:

$$u(\cdot) - \sum_{k=0}^{m_n-1} u_k g_{k+1}(\cdot) + \sum_{j=1}^{n-1} g_{\alpha_n-\alpha_j} * A_j \left[u(\cdot) - \sum_{k=0}^{m_j-1} u_k g_{k+1}(\cdot)\right] \in g_{\alpha_n-\alpha} * \mathscr{A}\left[u(\cdot) - \sum_{k=0}^{m-1} u_k g_{k+1}(\cdot)\right]. \tag{28}$$

If $i \in \mathbb{N}^0_{m_n-1}$, then we define $D_i := \{j \in \mathbb{N}_{n-1} : m_j - 1 \geq i\}$. Plugging $u_j = 0$, $0 \leq j \leq m_n - 1$, $j \neq i$, in (28), we get

$$\begin{aligned}
&[u(\cdot;0,\ldots,u_i,\ldots,0) - u_i g_{i+1}(\cdot)] + \sum_{j\in D_i} g_{\alpha_n-\alpha_j} \\
&\quad * A_j [u(\cdot;0,\ldots,u_i,\ldots,0) - u_i g_{i+1}(\cdot)] \\
&\quad + \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} \left[g_{\alpha_n-\alpha_j} * A_j u(\cdot;0,\ldots,u_i,\ldots,0)\right] \\
&\in \begin{cases} g_{\alpha_n-\alpha} * \mathscr{A} u(\cdot;0,\ldots,u_i,\ldots,0), & m-1<i, \\ g_{\alpha_n-\alpha} * \mathscr{A}[u(\cdot;0,\ldots,u_i,\ldots,0) - u_i g_{i+1}(\cdot)], & m-1\geq i, \end{cases}
\end{aligned} \tag{29}$$

where u_i appears in the ith place ($0 \leq i \leq m_n - 1$) starting from 0. Proceeding as in nondegenerate case [12], this inclusion motivates us to introduce the following extension of [12, Definition 2.10.2] (cf. also [34, Definition 2.1] and [32, Definitions 3.6 and 3.7] for similar notions).

Definition 11. Suppose that $0 < \tau \leq \infty$, $k \in C([0,\tau))$, $C, C_1, C_2 \in L(X)$, and C and C_2 are injective. A sequence $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ of strongly continuous operator families in $L(X)$ is called a (local, if $\tau < \infty$):

(i) k-regularized C_1-existence propagation family for (1) iff the following holds:

$$\begin{aligned}
&[R_i(\cdot)x - (k*g_i)(\cdot)C_1x] + \sum_{j\in D_i} A_j\left[g_{\alpha_n-\alpha_j} * (R_i(\cdot)x - (k*g_i)(\cdot)C_1x)\right] \\
&\quad + \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} A_j\left(g_{\alpha_n-\alpha_j} * R_i\right)(\cdot)x \in \begin{cases} \mathscr{A}\left(g_{\alpha_n-\alpha} * R_i\right)(\cdot)x, & m-1<i,\ x\in X, \\ \mathscr{A}\left[g_{\alpha_n-\alpha} * \left(R_i(\cdot)x - (k*g_i)(\cdot)C_1x\right)\right](\cdot), & m-1\geq i,\ x\in X, \end{cases}
\end{aligned} \tag{30}$$

for any $i = 0, \ldots, m_n - 1$.

(ii) k-regularized C_2-uniqueness propagation family for (1) iff the following holds:

$$\begin{aligned}
&[R_i(\cdot)x - (k*g_i)(\cdot)C_2x] + \sum_{j\in D_i} g_{\alpha_n-\alpha_j} \\
&\quad * \left[R_i(\cdot)A_jx - (k*g_i)(\cdot)C_2A_jx\right] \\
&\quad + \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} \left(g_{\alpha_n-\alpha_j} * R_i(\cdot)A_jx\right)(\cdot) \\
&= \begin{cases} \left(g_{\alpha_n-\alpha} * R_i(\cdot)y\right)(\cdot), & m-1<i, \\ g_{\alpha_n-\alpha} * [R_i(\cdot)y - (k*g_i)(\cdot)C_2y](\cdot), & m-1\geq i, \end{cases}
\end{aligned} \tag{31}$$

provided $(x, y) \in \mathscr{A}$ and $i \in \mathbb{N}^0_{m_n-1}$.

(iii) k-regularized C-resolvent propagation family for (1), in short k-regularized C-propagation family for (1), iff $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a k-regularized C-uniqueness propagation family for (1), and if for every $t \in [0,\tau)$, $i \in \mathbb{N}^0_{m_n-1}$, and $j \in \mathbb{N}^0_{n-1}$, one has $R_i(t)A_j \subseteq A_jR_i(t)$, $R_i(t)C = CR_i(t)$, and $CA_j \subseteq A_jC$.

In the case that $k(t) = g_{\zeta+1}(t)$, where $\zeta \geq 0$, then we also say that $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is a ζ-times integrated C-resolvent propagation family for (1); 0-times integrated C-resolvent propagation family for (1) is simply called C-resolvent propagation family for (1). For a k-regularized (C_1, C_2)-existence and uniqueness family $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$, it is said that is exponentially bounded iff each single operator family $(R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)}$ is. The above terminological agreement is accepted for all other classes of k-regularized C-propagation families introduced so far.

If $A_j = c_jI$, where $c_j \in \mathbb{C}$ for $1 \leq j \leq n-1$, then it is also said that $\mathscr{A}$ is a subgenerator of $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$. The notion of integral generator of $((R_0(t))_{t\in[0,\tau)}, \ldots, (R_{m_n-1}(t))_{t\in[0,\tau)})$ is introduced as in nondegenerate case [12].

Hereafter, the following equality will play an important role in our analysis:

$$\begin{aligned}
&[R_i(\cdot)x - (k*g_i)(\cdot)Cx] + \sum_{j=1}^{n-1} A_j g_{\alpha_n-\alpha_j} \\
&\quad * [R_i(\cdot)x - (k*g_i)(\cdot)Cx] + \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} A_j\left[g_{\alpha_n-\alpha_j+i} * k\right](\cdot)Cx \\
&\in \begin{cases} \mathscr{A}\left[g_{\alpha_n-\alpha} * R_i\right](\cdot)x, & m-1<i,\ x\in X, \\ \mathscr{A}\left[g_{\alpha_n-\alpha} * (R_i(\cdot)x - (k*g_i)(\cdot)Cx)\right], & m-1\geq i,\ x\in X, \end{cases}
\end{aligned} \tag{32}$$

for any $i = 0, \ldots, m_n - 1$. The basic properties of subgenerators and integral generators continue to hold, with appropriate changes, in degenerate case; compare [12] and [19, Section 3.2] for more details. We leave to the interested reader the problem of transferring the assertions of [12, Propositions 2.10.3–2.10.5, Theorem 2.10.7] to degenerate case.

The following is a degenerate version of [12, Definition 2.10.6].

Definition 12. Let $f \in C([0,\infty) : X)$. Consider the following inhomogeneous Cauchy inclusion:

$$\begin{aligned}
&u(t) + \sum_{j=1}^{n-1}\left(g_{\alpha_n-\alpha_j} * A_ju\right)(t) \\
&\quad \in f(t) + \left(g_{\alpha_n-\alpha} * \mathscr{A}u\right)(t), \quad t\geq 0.
\end{aligned} \tag{33}$$

A function $u \in C([0,\infty) : X)$ is said to be

(i) a strong solution of (33) iff there exists a continuous function $u_{\mathscr{A}} \in C([0,\infty) : X)$ such that $u_{\mathscr{A}}(t) \in \mathscr{A}u(t)$ for all $t \geq 0$ and

$$u(t) + \sum_{j=1}^{n-1} \left(g_{\alpha_n-\alpha_j} * A_j u\right)(t) = f(t) + \left(g_{\alpha_n-\alpha} * u_{\mathscr{A}}\right)(t), \quad t \geq 0, \tag{34}$$

(ii) a mild solution of (33) iff

$$u(t) + \sum_{j=1}^{n-1} A_j \left(g_{\alpha_n-\alpha_j} * u\right)(t) \in f(t) + \mathscr{A}\left(g_{\alpha_n-\alpha} * u\right)(t), \quad t \in [0,T]. \tag{35}$$

Clearly, every strong solution of (33) is also a mild solution of the same problem while the converse statement is not true, in general. We similarly define the notion of a strong (mild) solution of problem (28).

We have the following:

(a) If $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is a C_1-existence propagation family for (1), then the function $u(t) := \sum_{i=0}^{m_n-1} R_i(t)x_i$, $t \geq 0$, is a mild solution of (28) with $u_i = C_1 x_i$ for $0 \leq i \leq m_n - 1$.

(b) If $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is a C_2-uniqueness propagation family for (1), and $R_i(t)A_j \subseteq A_j R_i(t)x$, $t \geq 0$, $x \in \bigcap_{j=0}^{n-1} D(A_j)$, $i \in \mathbb{N}^0_{m_n-1}$, and $j \in \mathbb{N}^0_{n-1}$, then the function $u(t) := \sum_{i=0}^{m_n-1} R_i(t)C_2^{-1}u_i$, $t \geq 0$, is a strong solution of (28), provided $u_i \in C_2(\bigcap_{j=0}^{n-1} D(A_j))$ for $0 \leq i \leq m_n - 1$.

For our later purposes, it will be sufficient to characterize the introduced classes of k-regularized propagation families by vector-valued Laplace transform; keeping in mind Lemmas 5–7, the proofs are almost the same as in nondegenerate case and we will only notify some details of the proof of Theorem 14 below because the formulation of [12, Theorem 2.10.9] is slightly misleading since the injectivity of operator P_λ for $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ has not been clarified in a proper way and property (ii) in the formulation of this theorem is required to hold for all $i \in \mathbb{N}^0_{m_n-1}$.

Theorem 13. *Suppose $k(t)$ satisfies (P1), $\omega \geq \max(0, \mathrm{abs}(k))$, $(R_i(t))_{t\geq 0}$ is strongly continuous, and the family $\{e^{-\omega t}R_i(t) : t \geq 0\} \subseteq L(X)$ is bounded, provided $0 \leq i \leq m_n - 1$. Let $\mathscr{A}$ be a closed MLO on X, let $C_1, C_2 \in L(X)$, and let C_2 be injective. Set*

$$P_\lambda := \lambda^{\alpha_n-\alpha} + \sum_{j=1}^{n-1} \lambda^{\alpha_j-\alpha} A_j - \mathscr{A}, \quad \lambda \in \mathbb{C} \setminus \{0\}. \tag{36}$$

(i) *Suppose $A_j \in L(X)$, $j \in \mathbb{N}_{n-1}$. Then $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is a global k-regularized C_1-existence propagation family for (1) iff the following conditions hold:*

(a) *The inclusion*

$$P_\lambda \int_0^\infty e^{-\lambda t} R_i(t)\, x\, dt \ni \lambda^{\alpha_n-\alpha-i} \tilde{k}(\lambda)\, C_1 x + \sum_{j\in D_i} \lambda^{\alpha_j-\alpha-i} \tilde{k}(\lambda)\, A_j C_1 x \tag{37}$$

holds provided $x \in X$, $i \in \mathbb{N}^0_{m_n-1}$, $m - 1 < i$, and $\Re\lambda > \omega$.

(b) *The inclusion*

$$P_\lambda \int_0^\infty e^{-\lambda t} \left[R_i(t)\, x - (k * g_i)(t)\, C_1 x\right] dt \ni - \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} \lambda^{\alpha_j-\alpha-i} \tilde{k}(\lambda)\, A_j C_1 x \tag{38}$$

holds provided $x \in X$, $i \in \mathbb{N}^0_{m_n-1}$, $m - 1 \geq i$, and $\Re\lambda > \omega$.

(ii) *Suppose $A_j \in L(X)$, $j \in \mathbb{N}_{n-1}$. Then $((R_0(t))_{t\geq 0}, \ldots, (R_{m_n-1}(t))_{t\geq 0})$ is a global k-regularized C_2-uniqueness propagation family for (1) iff, for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$, $x \in D(\mathscr{A})$, and $y \in \mathscr{A}x$, the following equality holds:*

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} \left[R_i(t)\, x - (k * g_i)(t)\, C_2 x\right] dt \\
&+ \sum_{j\in D_i} \lambda^{\alpha_j-\alpha_n} \int_0^\infty e^{-\lambda t} \left[R_i(t)\, A_j x - (k * g_i)(t)\, C_2 A_j x\right] dt \\
&+ \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} \lambda^{\alpha_j-\alpha_n} \int_0^\infty e^{-\lambda t} R_i(t)\, A_j x\, dt \\
&= \begin{cases} \lambda^{\alpha-\alpha_n} \displaystyle\int_0^\infty e^{-\lambda t} R_i(t)\, y\, dt, & m - 1 < i, \\ \lambda^{\alpha-\alpha_n} \displaystyle\int_0^\infty e^{-\lambda t} \left[R_i(t)\, y - (k * g_i)(t)\, C_2 y\right] dt, & m - 1 \geq i. \end{cases}
\end{aligned} \tag{39}$$

Theorem 14. *Suppose $k(t)$ satisfies (P1), $\omega \geq \max(0, \mathrm{abs}(k))$, $(R_i(t))_{t\geq 0}$ is strongly continuous, and the family $\{e^{-\omega t}R_i(t) : t \geq 0\} \subseteq L(X)$ is bounded, provided $0 \leq i \leq m_n - 1$.*

(I) *Let the following two conditions hold:*

(i) *$CA_j \subseteq A_j C$, $j \in \mathbb{N}^0_{n-1}$, $A_j \in L(X)$, $j \in \mathbb{N}_{n-1}$, $A_i A_j = A_j A_i$, $i, j \in \mathbb{N}_{n-1}$, and $A_j \mathscr{A} \subseteq \mathscr{A} A_j$, $j \in \mathbb{N}_{n-1}$.*

(ii) *There exists an index $i \in \mathbb{N}^0_{m_n-1}$ satisfying exactly one of the following two conditions:*

(a) *$m - 1 < i$ and the operator $\lambda^{\alpha_n-i} + \sum_{j\in D_i} \lambda^{\alpha_j-i} A_j$ is injective for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$,*

(b) *$m - 1 \geq i$, $\mathbb{N}_{n-1} \setminus D_i \neq \emptyset$, and the operator $\sum_{j\in\mathbb{N}_{n-1}\setminus D_i} \lambda^{\alpha_j-i} A_j$ is injective for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $\tilde{k}(\lambda) \neq 0$.*

If $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ is a global k-regularized C-resolvent propagation family for (1) and (30) holds, then P_λ is injective for every $\lambda\in\mathbb{C}$ with $\Re\lambda>\omega$ and $\tilde{k}(\lambda)\neq 0$ and equalities (37)-(38) are fulfilled.

(II) *Suppose that P_λ is injective for every $\lambda\in\mathbb{C}$ with $\Re\lambda>\omega$ and $\tilde{k}(\lambda)\neq 0$ and equalities (37)-(38) are fulfilled. If condition (I)(i) holds, then $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ is a global k-regularized C-resolvent propagation family for (1).*

Proof. Concerning assertion (I), we will only sketch the main details of the proof of the injectivity of operator P_λ for every $\lambda\in\mathbb{C}$ with $\Re\lambda>\omega$ and $\tilde{k}(\lambda)\neq 0$ (we know that (37)-(38) hold on account of Theorem 13). Observe that we do not need the condition (I)(ii) for the proof of (II), where we only use an elementary argumentation as well as Lemmas 5–7 (the composition property (31) follows by applying the Laplace transform and Lemma 7, while the commutation of operator families $R_i(\cdot)$ with the operators C and A_j for $0\leq j\leq n-1$ is much simpler to show). The consideration is quite similar in the case that the condition (II) holds and, because of that, we will consider only the first case. Let $\lambda_0\in\mathbb{C}$ with $\Re\lambda_0>\omega$ and $\tilde{k}(\lambda_0)\neq 0$ be fixed, and let $0\in P_{\lambda_0}x$ for some $x\in X$. Using the fact that $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ is a global k-regularized C-uniqueness propagation family for (1), we can simply prove that

$$\lambda_0^{\alpha_n-i}x+\sum_{j\in D_i}\lambda_0^{\alpha_j-i}A_jx=0 \tag{40}$$

by performing the Laplace transform at the both sides of the composition property (31). By the injectivity of the operator $\lambda^{\alpha_n-i}+\sum_{j\in D_i}\lambda^{\alpha_j-i}A_j$ for $\lambda=\lambda_0$, we obtain that $x=0$ and the claimed assertion follows. □

These results enable one to simply clarify the Hille-Yosida type theorems for exponentially bounded k-regularized C-resolvent propagation families. The analytical properties of k-regularized C-resolvent propagation families can be analyzed similarly as in nondegenerate case [12]. We will use the following definition.

Definition 15. (i) Let $\beta\in(0,\pi]$, and let $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ be a k-regularized C-resolvent propagation family for (1). Then it is said that $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ is an analytic k-regularized C-resolvent propagation family of angle β, iff for each $i\in\mathbb{N}^0_{m_n-1}$ there exists a function $\mathbf{R}_i:\Sigma_\alpha\to L(X)$ which satisfies that, for every $x\in X$, the mapping $z\mapsto\mathbf{R}_i(z)x$, $z\in\Sigma_\beta$ is analytic and that

(a) $\mathbf{R}_i(t)=R_i(t)$, $t>0$, and

(b) $\lim_{z\to 0,z\in\Sigma_\gamma}\mathbf{R}_i(z)x=R_i(0)x$ for all $\gamma\in(0,\beta)$ and $x\in X$.

(ii) Suppose that $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ is an analytic k-regularized C-resolvent propagation family of angle β. Then it is said that $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ is an exponentially bounded, analytic k-regularized C-resolvent propagation family of angle β, respectively, bounded k-regularized C-resolvent propagation family, iff for every $\gamma\in(0,\beta)$, there exists $\omega_\gamma\geq 0$, respectively, $\omega_\gamma=0$, such that the family $\{e^{-\omega_\gamma\Re z}\mathbf{R}_i(z):z\in\Sigma_\gamma\}\subseteq L(X)$ is bounded for all $i\in\mathbb{N}^0_{m_n-1}$. Since there is no risk for confusion, we will identify in the sequel $R_i(\cdot)$ and $\mathbf{R}_i(\cdot)$.

For our purposes, the following result will be sufficiently enough (cf. Theorem 14 and [2, Theorem 2.6.1, Proposition 2.6.3 b]); we feel duty bound to say that the small inconsistencies in the formulation of [12, Theorem 2.10.11] have been made; see also [34].

Theorem 16. *Assume $k(t)$ satisfies (P1), $\omega\geq\max(0,\mathrm{abs}(k))$, $\beta\in(0,\pi/2]$, and, for every $i\in\mathbb{N}^0_{m_n-1}$, the function $(k*g_i)(t)$ can be analytically extended to a function $k_i:\Sigma_\beta\to\mathbb{C}$ satisfying that, for every $\gamma\in(0,\beta)$, the set $\{e^{-\omega z}k_i(z):z\in\Sigma_\gamma\}$ is bounded.*

Let the following three conditions hold:

(i) *$CA_j\subseteq A_jC$, $j\in\mathbb{N}^0_{n-1}$, $A_j\in L(X)$, $j\in\mathbb{N}_{n-1}$, $A_iA_j=A_jA_i$, $i,j\in\mathbb{N}_{n-1}$, and $A_j\mathscr{A}\subseteq\mathscr{A}A_j$, $j\in\mathbb{N}_{n-1}$.*

(ii) *The operator P_λ is injective for all $\omega+\Sigma_{\beta+\pi/2}$.*

(iii) *Let $q_i:\omega+\Sigma_{\pi/2+\beta}\to L(X)$ $(0\leq i\leq m_n-1)$ satisfy that, for every $x\in X$, the mapping $\lambda\mapsto q_i(\lambda)x$, $\lambda\in\omega+\Sigma_{\pi/2+\beta}$ is analytic and that for each $i\in\mathbb{N}^0_{m_n-1}$ there exists an operator $D_i\in L(X)$ such that*

$$q_i(\lambda)x=\tilde{k}_i(\lambda)P_\lambda^{-1}\left(\lambda^{\alpha_n-\alpha}C+\sum_{j\in D_i}\lambda^{\alpha_j-\alpha}A_jC\right)x,\qquad x\in X,\ \lambda\in V_i, \tag{41}$$

provided $m-1<i$,

$$q_i(\lambda)x=-\tilde{k}_i(\lambda)P_\lambda^{-1}\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\lambda^{\alpha_j-\alpha}A_jCx,\qquad x\in X,\ \lambda\in V_i, \tag{42}$$

provided $m-1\geq i$,

$$\text{the family }\left\{(\lambda-\omega)q_i(\lambda):\lambda\in\omega+\Sigma_{\pi/2+\gamma}\right\}\ \text{is equicontinuous}\quad\forall\gamma\in(0,\beta), \tag{43}$$

and, in the case $\overline{D(\mathscr{A})}\neq X$,

$$\lim_{\lambda\to+\infty}\lambda q_i(\lambda)x=D_ix,\quad x\notin\overline{D(\mathscr{A})}. \tag{44}$$

Then there exists an exponentially bounded, analytic k-regularized C-resolvent propagation family $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$ for (1), of angle β. Furthermore, (30) holds, the family $\{e^{-\omega z}R_i(z):z\in\Sigma_\gamma\}$ is bounded for all $i\in\mathbb{N}^0_{m_n-1}$ and $\gamma\in(0,\beta)$, and $R_i(z)A_j\subseteq A_jR_i(z)$, $z\in\Sigma_\beta$, and $j\in\mathbb{N}^0_{n-1}$.

Remark 17. For the sequel, it will be very important to note that the notion introduced in Definition 11(iii) and Definition 15 can be introduced for any single operator family $(R_i(t))_{t\geq 0}$ of the tuple $((R_0(t))_{t\geq 0},\ldots,(R_{m_n-1}(t))_{t\geq 0})$. The assertions of Theorems 14 and 16 can be simply reformulated for $(R_i(t))_{t\geq 0}$; for example, if the index $i\in\mathbb{N}^0_{m_n-1}$ is given in advance, then in the formulation of Theorem 16 it suffices to assume that the function $(k*g_i)(\cdot)$ can be analytically extended to a function $k_i:\Sigma_\beta\to\mathbb{C}$ satisfying that, for every $\gamma\in(0,\beta)$, the set $\{e^{-\omega z}k_i(z):z\in\Sigma_\gamma\}$ is bounded, and that (i)-(ii) hold and (iii) holds only for this specified index i. It will be said that $(R_i(t))_{t\geq 0}$ is an (exponentially bounded, analytic/analytic) k_i-regularized C-resolvent propagation family. All terminological agreements explained before will be accepted for k_i-regularized C-resolvent propagation families; the classes of k_i-regularized C_1-existence propagation families and k_i-regularized C_2-uniqueness propagation families are introduced similarly.

5. Asymptotical Behaviour of k_i-Regularized C-Propagation Families for (1)

The main aim of this section is to investigate polynomial decaying of k_i-regularized C-propagation families for (1) as time goes to infinity. Applications of Theorem 16 (see also Remark 17) will be crucial in our work and we start by observing that it is not clear how one can prove the injectivity of operator P_λ, given by (36), in general case. Because of that, we will first focus our attention to the case that $A_j=c_jI$, where $c_j\in\mathbb{C}$ for $1\leq j\leq n-1$, by exploring the generation of fractionally integrated C-propagation families for (1) only. Moreover, we will assume that the numbers c_j are nonnegative for $1\leq j\leq n-1$ and that $m-1<i$ (the case $m-1\geq i$ can be analyzed similarly) and the multivalued linear operator $\mathcal{A}$ under our consideration is possibly not densely defined.

Theorem 18. *Suppose that $c_j\geq 0$ for $1\leq j\leq n-1$, $\zeta'\geq 0$, $\mathcal{A}:X\to P(X)$ is a closed MLO, $C\in L(X)$ is injective, $C\mathcal{A}\subseteq\mathcal{A}C$, and the following condition holds:*

(H) *There exist finite constants $c<0$, $M>0$, $0<\theta<\pi$, and $\beta\in(0,1]$ such that*

$$\overline{c+\Sigma_{\pi-\theta}}\subseteq\rho_C(\mathcal{A}),\qquad \left\|(\lambda-\mathcal{A})^{-1}C\right\|\leq\frac{M}{|\lambda-c|^\beta},\quad \lambda\in\overline{c+\Sigma_{\pi-\theta}}. \tag{45}$$

Assume that the mapping $\lambda\mapsto(\lambda-\mathcal{A})^{-1}C$, $\lambda\in\overline{c+\Sigma_{\pi-\theta}}$ is strongly continuous. Assume also that $i\in\mathbb{N}^0_{m_n-1}$ satisfies $m-1<i$, and

$$\alpha_n-\alpha-i-\zeta'-(\alpha_n-\alpha)\beta\leq 0, \tag{46}$$

$$\nu':=\frac{\pi-\theta}{\alpha_n-\alpha}-\frac{\pi}{2}>0. \tag{47}$$

Set $\zeta:=\zeta'$ if $\mathcal{A}$ is densely defined, $\zeta>\zeta'$ otherwise, and $k_i(\cdot):=g_{\zeta+1}(\cdot)$. Then there exists an exponentially bounded, analytic k_i-regularized C-propagation family $(R_i(t))_{t\geq 0}$ for (1), of angle $\nu:=\min(\nu',\pi/2)$. Moreover, (30) holds and there exists a finite constant $M'>0$ such that

$$\left\|R_i(t)\right\|\leq M'\left[t^{\alpha+\zeta+i-\alpha_n}+\sum_{j\in D_i,c_j\neq 0}t^{\alpha+\zeta+i-\alpha_j}\right],\qquad t>0. \tag{48}$$

Proof. Since we have assumed that the mapping $\lambda\mapsto(\lambda-\mathcal{A})^{-1}C$, $\lambda\in\overline{c+\Sigma_{\pi-\theta}}$ is strongly continuous, and its restriction to $c+\Sigma_{\pi-\theta}$ is strongly analytic on this region; see [19, Proposition 1.2.6(iii)]. Taking into account (47) and $c_j\geq 0$ for $1\leq j\leq n-1$, we get

$$\lambda^{\alpha_n-\alpha}+\sum_{j\in D_i}c_j\lambda^{\alpha_j-\alpha}\in\Sigma_{(\alpha_n-\alpha)(\pi/2+\nu)}\subseteq\Sigma_{(\alpha_n-\alpha)(\pi/2+\nu)}\subseteq\Sigma_{\pi-\theta},\quad\lambda\in\Sigma_{\nu+\pi/2}. \tag{49}$$

It is clear that (46) implies

$$\alpha_j-\alpha-i-\zeta'-(\alpha_n-\alpha)\beta\leq 0\qquad\forall j\in D_i\text{ with }c_j\neq 0. \tag{50}$$

Using an elementary argumentation, (46), (49), and (50), we can simply verify that the conditions of Theorem 16 hold with $\omega>0$ sufficiently large, $k(t)=g_{\zeta+1}(t)$, $t\geq 0$, and $D_i=0$, in the case that the operator $\mathcal{A}$ is not densely defined. Hence, $\mathcal{A}$ is a subgenerator of an exponentially bounded, analytic ζ-times integrated C-propagation family $(R_i(t))_{t\geq 0}$ for (1), of angle $\nu:=\min(\nu',\pi/2)$, as claimed. Estimate (48) remains to be proved. Fix the numbers $t>0$ and $0<\gamma<\nu$. By the proof of [2, Theorem 2.6.1] and Cauchy theorem, we have that, for every $x\in X$,

$$R_i(t)x=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\lambda^{-\zeta-1}P_\lambda^{-1}\left[\lambda^{\alpha_n-\alpha-i}Cx+\sum_{j\in D_i,c_j\neq 0}c_j\lambda^{\alpha_j-\alpha-i}Cx\right]d\lambda, \tag{51}$$

where Γ is oriented counterclockwise and consists of $\Gamma_\pm:=\{re^{i(\pi/2+\gamma)}:r\geq t^{-1}\}$ and $\Gamma_0:=\{t^{-1}e^{i\theta}:|\theta|\leq\pi/2+\gamma\}$. Keeping in mind (49) and the estimate from the condition (H), it readily follows that for each $\lambda\in\Gamma$ we have that $\|P_\lambda^{-1}C\|\leq M/|c\sin\theta|^\beta$, so that

$$\left\|R_i(t)\right\|\leq\frac{M/|c\sin\theta|^\beta}{2\pi}\times\int_\Gamma e^{\Re(\lambda t)}|\lambda|^{-\zeta-1}\cdot\left[|\lambda|^{\alpha_n-\alpha-i}+\sum_{j\in D_i,c_j\neq 0}\left|c_j\right||\lambda|^{\alpha_j-\alpha-i}\right]|d\lambda|. \tag{52}$$

But, then estimate (48) follows from a simple integral computation that is very similar to that appearing in the proof of [2, Theorem 2.6.1]. □

Remark 19. (i) It is worth noting that the value of exponent β in (H) does not depend on the final estimate (48). In our proof, we only use the estimate $\|P_\lambda^{-1}C\| \leq M/|c\sin\theta|^\beta$, $\lambda \in \Gamma$.

(ii) As mentioned in the introductory part, the proof of important result of Cuesta [15, Theorem 2.1] for the classical fractional oscillation resolvent families generated by densely defined linear operators [11], satisfying the condition (H) with $\beta = 1$, $C = I$, and $\omega = c < 0$, follows completely different lines. Furthermore, in our approach, the case in which $\pi/2 < \theta < \pi$ or $\alpha_n - \alpha < 1$ can occur, Theorem 18 is applicable in the qualitative analysis of fractional relaxation multiterm differential inclusions (but not in the analysis of generalized asymptotical almost periodicity and generalized asymptotical almost automorphy of solutions; see (53)).

Let $\mathcal{F} \in \{AP_T([0,\infty) : \mathbb{C}), AAP([0,\infty) : \mathbb{C}), AAA([0,\infty) : \mathbb{C})\}$, where the symbol $AAA([0,\infty) : \mathbb{C})$ denotes the space of scalar-valued asymptotically almost automorphic functions, and let $k_i(\cdot)$ be defined as above. If $(R_i(t))_{t\geq 0}$ is the k_i-regularized C-propagation family for (1), constructed with the help of Theorem 18, and $f \in \mathcal{F}$, then it can be easily checked that $((R_i * f)(t))_{t\geq 0}$ is a k_i-regularized C-propagation family $(R_i(t))_{t\geq 0}$ for (1), satisfying additionally (30), where $k_i(\cdot) = (g_{\zeta+1} * f)(\cdot)$. By Theorem 18, some known assertions concerning inheritance of asymptotical periodicity, almost asymptotical almost periodicity and asymptotical almost automorphy under the action of finite convolution products ([5, 16]), and the assertions (a)-(b) clarified above, this yields the following result (the uniqueness of solutions follows from the fact that [12, Theorem 2.10.7] holds in degenerate case and that the condition $(\diamond)$ stated in the formulation of [12, Proposition 2.10.3] holds true, which can be verified by performing the Laplace transform).

Corollary 20. *Let the requirements of Theorem 18 hold, let $f \in \mathcal{F}$, and let $k_i(\cdot) = (g_{\zeta+1} * f)(\cdot)$. Set $u_x(t) := (R_i * f)(t)x$, $t \geq 0$, $x \in X$. Assume that*

$$\begin{aligned} &\alpha + \zeta + i - \alpha_n < -1, \\ &\alpha + \zeta + i - \alpha_j < -1 \quad \forall j \in D_i \text{ with } c_j \neq 0. \end{aligned} \tag{53}$$

Then, for every $x \in X$, $u_x(\cdot) \in \mathcal{F}$ is a unique mild solution of the abstract Cauchy inclusion

$$\begin{aligned} &\left[u(\cdot) - \left(g_{\zeta+1+i} * f\right)(\cdot)Cx\right] + \sum_{j=1}^{n-1} c_j g_{\alpha_n-\alpha_j} \\ &\quad * \left[u(\cdot) - \left(g_{\zeta+1+i} * f\right)(\cdot)Cx\right] \\ &\quad + \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} c_j \left[g_{\alpha_n-\alpha_j+i+\zeta+1} * f\right](\cdot)Cx \\ &\in \mathcal{A}\left[g_{\alpha_n-\alpha} * u\right](\cdot). \end{aligned} \tag{54}$$

Furthermore, $u_x(\cdot)$ is a strong solution of (54) provided that $x \in D(\mathcal{A})$.

Concerning the Stepanov asymptotical almost periodicity and (equi-) Weyl asymptotical almost periodicity, some extra conditions on the vanishing part of function $f(\cdot)$ must be imposed if we want that the solution $u_x(\cdot)$ defined above belongs to the same class of functions as $f(\cdot)$. For example, we have the following:

(i) Stepanov class: suppose that $1 \leq p < \infty$ and $f : [0,\infty) \to X$ is asymptotically S^p-almost periodic and that the locally p-integrable functions $g : \mathbb{R} \to X$, $q : [0,\infty) \to X$ satisfy the conditions from [30, Lemma 1.1], and $\lim_{t\to+\infty}\int_t^{t+1}(\int_{s/2}^s \|q(r)\|dr)^p ds = 0$. Then the function $(R_i * f)(\cdot)$ is asymptotically S^p-almost periodic (see [33, Proposition 2.13, Remark 2.14]).

(ii) Weyl classes: if $g : \mathbb{R} \to X$ is bounded and Weyl-almost periodic and $q \in e - W_0^1([0,\infty) : X)$ satisfies the following conditions:

  (i) the mapping $t \mapsto \int_0^t \|q(r)\|dr$, $t > 0$, is bounded as $t \to +\infty$,

  (ii) $\lim_{t\to+\infty}\int_0^t (t+1-r)^{-\nu}\|q(r)\|dr = 0$.

then the function $(R_i * (g+q))(\cdot)$ is in class $W^1_{eaap}([0,\infty) : X)$, where $W^1_{eaap}([0,\infty) : X) := W^1_{ap}([0,\infty) : X) + e - W_0^1([0,\infty) : X)$; see [35, Proposition 2.3(i), Example 5.5]. Here, $\nu > 1$ is chosen so that $\|R_i(t)\| = O(t^{-\nu})$, $t \geq 1$.

Denote by $\mathcal{F}'$ the set consisting of all generalized (asymptotically) almost periodic function spaces and all generalized (asymptotically) almost automorphic function spaces considered so far. Let $\mathcal{F}''$ denote the collection of all spaces from $\mathcal{F}'$ that are not in any class of functions obtained as a sum of some spaces of (equi-) Weyl almost periodic (automorphic) functions and some space of (equi-) Weyl almost vanishing functions.

The second part of the following proposition is very similar to [7, Proposition 2.5.1].

Proposition 21. *Suppose that $k(t)$ satisfies (P1) and $(R_i(t))_{t\geq 0}$ is a strongly Laplace transformable k_i-regularized C-propagation family for (1).*

(i) *For every $\lambda \in \mathbb{C}$, there exists a function $f_\lambda(\cdot)$ satisfying (P1)-$L(X)$ and*

$$\begin{aligned} f_\lambda(t) := \mathcal{L}^{-1}\Bigg(&\left[\left(1 - \frac{\lambda}{z^{\alpha_n-\alpha}}\right)I + \sum_{j=1}^{n-1}\frac{A_j}{z^{\alpha_n-\alpha_j}}\right]^{-1} \\ &\cdot \sum_{j\in D_i}\frac{\widetilde{k}(z)}{z^i}CA_j\Bigg)(t), \quad t \geq 0, \end{aligned} \tag{55}$$

provided that $m-1<i$, respectively,

$$f_\lambda(t) := \mathscr{L}^{-1}\left(\left[\left(1-\frac{\lambda}{z^{\alpha_n-\alpha}}\right)I+\sum_{j=1}^{n-1}\frac{A_j}{z^{\alpha_n-\alpha_j}}\right]^{-1}\right.$$
$$\left.\times\left(\sum_{j\in D_i}\frac{\tilde{k}(z)}{z^i}CA_j-\lambda\frac{\tilde{k}(z)}{z^{\alpha_n-\alpha+i}}C\right)\right)(t),\quad t\geq 0, \tag{56}$$

provided that $m-1\geq i$.

(ii) *Denote by D the set consisting of all eigenvectors x of operator $\mathscr{A}$ which corresponds to eigenvalues $\lambda\in\mathbb{C}$ of operator $\mathscr{A}$ for which the mapping*

$$f_{\lambda,x}(t) := f_\lambda(t)\,x,\quad t\geq 0, \tag{57}$$

belongs to the space $\mathscr{F}'$. Then the mapping $t\mapsto R_i(t)x$, $t\geq 0$, belongs to the space $\mathscr{F}'$ for all $x\in\operatorname{span}(D)$; furthermore, the mapping $t\mapsto R_i(t)x$, $t\geq 0$, belongs to the space $\mathscr{F}''$ for all $x\in\overline{\operatorname{span}(D)}$ provided additionally that $(R_i(t))_{t\geq 0}$ is bounded.

Proof. We will examine the case $m-1<i$ only. The proof of (i) can be given following the lines of the proof of [9, Theorem 1.1.11], with appropriate changes briefly described as follows. Since the operator $(1-\lambda/z^{\alpha_n-\alpha})I+\sum_{j=1}^{n-1}(A_j/z^{\alpha_n-\alpha_j})$ is invertible in $L(X)$ for all $z\in\mathbb{C}$ with $|z|$ sufficiently large, because the norm of bounded linear operator $(\lambda/z^{\alpha_n-\alpha})I-\sum_{j=1}^{n-1}(A_j/z^{\alpha_n-\alpha_j})$ for such values of z is strictly less than 1, we get that the term

$$\left[\left(1-\frac{\lambda}{z^{\alpha_n-\alpha}}\right)I+\sum_{j=1}^{n-1}\frac{A_j}{z^{\alpha_n-\alpha_j}}\right]^{-1}\sum_{j\in D_i}\frac{\tilde{k}(z)}{z^i}CA_j \tag{58}$$

is well-defined for all $z\in\mathbb{C}$ with $\Re z>\omega_0$, for some $\omega_0>\max(0,\operatorname{abs}(k))$. Set

$$H_0(t) := \lambda g_{\alpha_n-\alpha_j}(t)+\sum_{j=1}^{n-1}g_{\alpha_n-\alpha_j}(t)\,A_j,\quad t>0, \tag{59}$$

and

$$\begin{aligned} a &:= \min\left(\alpha_n-\alpha_0,\dots,\alpha_n-\alpha_{n-1}\right),\\ b &:= \max\left(\alpha_n-\alpha_0,\dots,\alpha_n-\alpha_{n-1}\right). \end{aligned} \tag{60}$$

Then it is clear that

$$\int_0^\infty e^{-zt}H_0(t)\,dt=\frac{\lambda}{z^{\alpha_n-\alpha}}I-\sum_{j=1}^{n-1}\frac{A_j}{z^{\alpha_n-\alpha_j}},\quad \Re z>\omega_0, \tag{61}$$

and that there exists a finite constant $c>0$ such that

$$\begin{aligned} \left\|H_0^{*k}(t)\right\| &\leq c^n g_{k(\alpha_n-b)}(t),\quad t\in(0,1],\\ \left\|H_0^{*k}(t)\right\| &\leq c^n g_{k(\alpha_n-a)}(t),\quad t\geq 1, \end{aligned} \tag{62}$$

where $H_0^{*k}(\cdot)$ denotes the kth convolution power of $H_0(\cdot)$. The function $H(t) := \sum_{k=1}^\infty H_0^{*k}(t)$, $t>0$, is well-defined since there exists a finite constant $c'>0$ such that

$$\begin{aligned} \|H(t)\| &\leq \sum_{k=1}^\infty c^k\frac{t^{k(\alpha_n-b)-1}}{\Gamma(k(\alpha_n-b))}\\ &\leq ct^{\alpha_n-b-1}\sum_{k=0}^\infty\frac{\left(ct^{\alpha_n-b}\right)^k}{\Gamma(k(\alpha_n-b)+\alpha_n-b)}\\ &\leq ct^{\alpha_n-b-1}E_{\alpha_n-b,\alpha_n-b}\left(ct^{\alpha_n-b}\right)\leq c't^{\alpha_n-b-1},\\ &\qquad t\in(0,1] \end{aligned} \tag{63}$$

and, due to Lemma 1,

$$\begin{aligned} \|H(t)\| &\leq \sum_{k=1}^\infty c^k\frac{t^{k(\alpha_n-a)-1}}{\Gamma(k(\alpha_n-a))}\\ &\leq ct^{\alpha_n-a-1}\sum_{k=0}^\infty\frac{\left(ct^{\alpha_n-a}\right)^k}{\Gamma(k(\alpha_n-a)+\alpha_n-a)}\\ &\leq ct^{\alpha_n-a-1}E_{\alpha_n-a,\alpha_n-a}\left(ct^{\alpha_n-a}\right)\\ &\leq c't^{\alpha_n-a-1}\left[1+\left(ct^{\alpha_n-a}\right)^{1-(\alpha_n-a)/\alpha_n-a}e^{c^{1/\alpha_n-a}t}\right],\\ &\qquad t\geq 1. \end{aligned} \tag{64}$$

For the remaining part of proof of (i), it suffices to repeat literally the arguments from the proof of [9, Theorem 1.1.11]. For the proof of (ii), observe first that, if $\lambda x\in\mathscr{A}x$ for some $\lambda\in\mathbb{C}$, then performing the Laplace transform at the both sides of the composition property (31), as it has been done as in our previous examinations, immediately yields that

$$\begin{aligned} &\left[\left(1-\frac{\lambda}{z^{\alpha_n-\alpha}}\right)I+\sum_{j=1}^{n-1}\frac{A_j}{z^{\alpha_n-\alpha_j}}\right]\int_0^\infty e^{-zt}R_i(t)\,x\,dt\\ &\quad=\sum_{j\in D_i}\frac{\tilde{k}(z)}{z^i}CA_jx, \end{aligned} \tag{65}$$

for $\Re z>0$ suff. large, and therefore $R_i(t)x=f_{\lambda,x}(t)$, $t\geq 0$. As a consequence, we have that the mapping $t\mapsto R_i(t)x$, $t\geq 0$, belongs to the space $\mathscr{F}'$ for all $x\in\operatorname{span}(D)$. The boundedness of $(R_i(t))_{t\geq 0}$ implies the uniform convergence of $R_i(t)x_n$ to $R_i(t)x$ ($t\geq 0$) for any sequence $(x_n)\in\operatorname{span}(D)$ converging to some element $x\in\overline{\operatorname{span}(D)}$; then the final result follows by combining the previously proved statement and the fact that the limit of a uniform convergent sequence of bounded continuous functions belonging to any space from $\mathscr{F}'$ belongs to this space again (see Proposition 4 for the class of (equi-) Weyl-almost periodic functions). □

Remark 22. If $m-1<i$ and $A_j=c_jI$ for some $c_j\in\mathbb{C}$ ($1\leq i\leq n-1$), then a simple calculation shows that

$$f_{\lambda,x}(t)=\sum_{j\in D_i}\frac{\tilde{k}(z)\,z^{\alpha_n-i}}{z^{\alpha_n}-\lambda z^{-\alpha}+\sum_{j=1}^{n-1}c_jz^{-\alpha_j}}CA_jx, \tag{66}$$

for $x \in X$ satisfying $\lambda x \in \mathscr{A}x$ ($\lambda \in \mathbb{C}$). To the best knowledge of the authors, in the handbooks containing tables of Laplace transforms, the explicit forms of functions like $f_{\lambda,x}(\cdot)$ are not known, with the exception of some very special cases of the coefficients α_j, c_j (see, e.g., [12, Remark 3.3.10(vi)]).

The following theorem is motivated by some pioneering results of Ruess and Summers concerning integration of asymptotically almost periodic functions [24].

Theorem 23. *Assume that $(R_i(t))_{t\geq 0}$ is an exponentially bounded k_i-regularized C-propagation family $(R_i(t))_{t\geq 0}$ for (1). Let $m-1 < i$, and let there exist a number $\nu > 1$ such that $\|R_i(t)\| = O(t^{-\nu})$, $t \geq 1$. Assume, further, that the following conditions hold:*

(i) *Let $f \in C([0,\infty) : X)$ satisfy that there exists a function $g \in AAP([0,\infty) : X)$ such that $(C^{-1}f)(t) = (g_i * k * g)(t)$, $t \geq 0$.*

(ii) *Assume that $g(t) = g_{ap}(t) + g_0(t)$, $t \geq 0$, where $g_{ap} \in AP([0,\infty) : X)$ and $g_0 \in C_0([0,\infty) : X)$.*

(iii) *Let $\alpha_n - \alpha_j \in \mathbb{N}$ for all $j \in D_i$.*

(iv) *Assume that $g_k * g \in L_\infty([0,\infty) : X)$ for all $k \in \mathbb{N}$ and $c_0 \nsubseteq X$ (i.e., X does not contain an isomorphic copy of c_0), or $R(g_k * g)$ is weakly relatively compact in X for all $k \in \mathbb{N}$.*

(v) *For every $k \in \mathbb{N}$, we have*

$$\begin{gathered}\int_0^\infty \|g_0(t)\|\,dt < \infty,\\ \int_0^\infty \int_t^\infty \|g_0(s)\|\,ds\,dt < \infty,\\ \vdots\\ \int_0^\infty \int_t^\infty \int_{x_k}^\infty \cdots \int_{x_1}^\infty \|g_0(s)\|\,ds\,dx_1\cdots dx_k dt < \infty.\end{gathered} \tag{67}$$

Then there exists a unique exponentially bounded mild solution $u(\cdot)$ of the abstract Cauchy inclusion (33). Furthermore, $u \in AAP([0,\infty) : X)$.

Proof. From our previous considerations of nondegenerate case, it is well known that any mild solution of the abstract Cauchy inclusion (33) has to satisfy the following equality:

$$\begin{aligned}(R_i * f)(t) &= (k * g_i * Cu)(t)\\ &\quad+ \sum_{j\in D_i} \left(g_{\alpha_n-\alpha_j+i} * k * CA_j u\right)(t),\\ &\qquad t \geq 0;\end{aligned} \tag{68}$$

See, for example, [12, Theorem 2.10.7]. Taking the Laplace transform, we get

$$\begin{aligned}\tilde{u}(z) &= \left[I + \sum_{j\in D_i} z^{\alpha_j-\alpha_n} A_j\right]^{-1} \widetilde{R_i}(z)\,\widetilde{C^{-1}f}(z)\, z^i \left[\tilde{k}(z)\right]^{-1},\\ &\qquad \mathfrak{R}z > \omega,\ \tilde{k}(z) \neq 0.\end{aligned} \tag{69}$$

Since $(C^{-1}f)(t) = (g_i * k * g)(t)$, $t \geq 0$, we get

$$\begin{aligned}\tilde{u}(z) &= \left[I + \sum_{j\in D_i} z^{\alpha_j-\alpha_n} A_j\right]^{-1} \widetilde{R_i}(z)\,\tilde{g}(z),\\ &\qquad \mathfrak{R}z > \omega,\ \tilde{k}(z) \neq 0.\end{aligned} \tag{70}$$

By the proofs of Proposition 21 and [9, Theorem 1.1.11], the right-hand side of above equality is really the Laplace transform of a continuous exponentially bounded function $u(\cdot)$ given by

$$\begin{aligned}u(t) &= \sum_{k=1}^\infty \left\{R_i * \left[-\sum_{j\in D_i} g_{\alpha_n-\alpha_j}(\cdot)A_j\right]^{*k} * g\right\}(t)\\ &\quad+ (R_i * g)(t), \quad t \geq 0.\end{aligned} \tag{71}$$

With the help of Laplace transform and a simple calculation, it can be simply verified that the function $u(\cdot)$, whose Laplace transform is given by (69), is a mild solution of abstract fractional inclusion (33). The growth order of $R_i(\cdot)$ implies that the function $t \mapsto (R_i * g)(t)$, $t \geq 0$, is in AAP$([0,\infty) : X)$. Since AAP$([0,\infty) : X)$ is closed in $C_b([0,\infty) : X)$ and $\{R_i * [-\sum_{j\in D_i} g_{\alpha_n-\alpha_j}(\cdot)A_j]^{*k} * g\}(t)$ converges uniformly to $\sum_{k=1}^\infty\{R_i * [-\sum_{j\in D_i} g_{\alpha_n-\alpha_j}(\cdot)A_j]^{*k} * g\}(t)$ for $t \geq 0$, the asymptotical almost periodicity of $u(\cdot)$ immediately follows if we prove that the function $G_k(t) = (g_k * g)(t)$, $t \geq 0$, is asymptotically almost periodic for all $k \in \mathbb{N}$; see (iii). But, this can be shown by making use of (iv)-(v) and applying successively [24, Theorem 2.2.2]; here, we only want to observe that the vanishing term of function $G_1(t)$ is given by $\int_t^\infty g_0(s)ds$ ($t \geq 0$), since $\int_0^\infty \|g_0(s)\|ds < \infty$. The uniqueness of exponentially bounded mild solutions of (33) can be proved as follows. Let $u(\cdot)$ be such a solution. Taking the Laplace transform and multiplying after that with $z^{\alpha_n-\alpha}$, we get

$$\begin{aligned}&\left(\tilde{u}(z), \left[z^{\alpha_n-\alpha} + \sum_{j=1}^{n-1} z^{\alpha_j-\alpha} A_j\right]\tilde{u}(z) - z^{\alpha_n-\alpha}\tilde{f}(z)\right)\\ &\quad\in \mathscr{A},\quad \mathfrak{R}z > \omega,\ \tilde{k}(z) \neq 0.\end{aligned} \tag{72}$$

Hence, $-z^{\alpha_n-\alpha}\tilde{f}(z) \in P_z\tilde{u}(z)$ and $\tilde{u}(z) = -P_z^{-1}z^{\alpha_n-\alpha}\tilde{f}(z)$ for $\mathfrak{R}z > \omega, \tilde{k}(z) \neq 0$. By the uniqueness theorem for the Laplace transform, $u(\cdot)$ must be uniquely determined. The proof of the theorem is thereby complete. □

Remark 24. Concerning Theorem 23, the case in which $m - 1 \geq i$ is a little bit complicated: it seems that the assumption $A_j = c_j$ for some complex numbers $c_j \in \mathbb{C}$ $(1 \leq j \leq n-1)$ has to be imposed for establishing of any relevant result. Details can be left to the interested reader.

6. Examples and Applications

We have already noted that the method established in the proof of Theorem 18 will be further employed for reconsideration and improving some known results recently established by Keyantuo et al. [16] and Luong [17]. The main aim of the following example is to explain how we can do this.

Example 1. In [16], the authors have considered the abstract two-term fractional differential equation

$$\mathbf{D}_t^{\alpha'+1} u(t) + c_1 \mathbf{D}_t^{\beta'} u(t) = Au(t) + f(t), \quad t \geq 0,$$
$$u^{(k)}(0) = u_k, \quad k = 0, 1, \tag{73}$$

where $c_1 > 0$, A is a densely defined linear operator satisfying the condition (H) with $\beta = 1$, $C = I$ and $c < 0$, $0 < \alpha' \leq \beta' \leq 1$, and $f(t)$ is a given X-valued function; here, we have been forced to slightly change the notation used in [16]. For this, the notion of an $(\alpha', \beta')_{c_1}$-regularized family generated by A, which is a special case of the notion introduced in Definition 9 with $a(t) = \mathscr{L}^{-1}(1/\lambda^{\alpha'+1} + c_1\lambda^{\beta'})(\lambda)$, $t \geq 0$, and $k(t) = \mathscr{L}^{-1}(\lambda^{\alpha'}/\lambda^{\alpha'+1} + c_1\lambda^{\beta'})(\lambda)$, $t \geq 0$, has been introduced; compare [16, Lemma 2.5]. Our main contributions will be given in the case that $0 < \alpha' < \beta' \leq 1$, which is used in the formulations and proofs of [16, Theorems 4.3 and 5.3], the main results of afore-mentioned paper (although possible applications can be given in the study of two-term fractional Poisson heat equations on the space $H^{-1}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary [18], we will pay our attention to the case that $\mathscr{A} = A$ is a single-valued linear operator).

Let us define an exponentially bounded, analytic $(\alpha', \beta', C)_{c_1}$-regularized family $(S_{\alpha',\beta'}(t))_{t\geq 0}$ of angle $\nu \in (0, \pi/2]$, subgenerated by A, as the exponentially bounded, analytic (a,k)-regularized C-resolvent family of angle ν, subgenerated by the same operator, with $a(t)$ and $k(t)$ being defined as above. Then we have

$$\int_0^\infty e^{-\lambda t} S_{\alpha',\beta'}(t)\, x\, dt = \lambda^{\alpha'} \left(\lambda^{\alpha'+1} + c_1\lambda^{\beta'} - A\right)^{-1} Cx, \quad x \in X, \ \lambda > 0 \text{ suff. large} \tag{74}$$

and the assertion of [16, Theorem 3.1] holds with the initial values $x, y \in X$ and the inhomogeneity $f(t)$ replaced therein with the initial values $C^{-1}x, C^{-1}y \in X$ and the inhomogeneity $C^{-1}f(t)$, respectively, with the meaning clear.

Assume that the condition (H) holds with $\beta = 1$, $c < 0$, and $\theta = \pi/2 - \gamma'\pi/2$, where $\alpha' < \gamma' < \beta'$. Then we can argue as follows. Similarly as in the proof of Theorem 18, we have that the operator A is a subgenerator of an exponentially bounded, analytic $(\alpha', \beta', C)_{c_1}$-regularized family $(S_{\alpha',\beta'}(t))_{t\geq 0}$ of angle $\nu := (\pi/2(\gamma'+1))/(\alpha'+1) - \pi/2$; compare also [12, Theorem 2.2.5]. Estimating the term $(\lambda^{\alpha'+1} + c_1\lambda^{\beta'} - A)^{-1}C$ uniformly, as in the proof of Theorem 18, and using the integral computation given in the final part of this theorem and [2, Theorem 2.6.1], we get

$$\left\| S_{\alpha',\beta'}(t) \right\| = O\left(\frac{1}{1+t^{\alpha'+1}}\right) = O\left(\frac{1}{1+t^{\alpha'+1}+t^{\beta'}}\right), \quad t \geq 0; \tag{75}$$

here we would like to note that the arguments used in proof of [16, Theorem 4.1] are much more complicated than ours and that our estimate is better even in the case that $C = I$ because then, with the notion introduced in [16] accepted, we only require that the operator A is of sectorial angle $\gamma'\pi/2$, not of $\beta'\pi/2$, as the stronger estimate in [16] requires. Using the estimate (75), we can repeat literally the proofs of [16, Theorems 4.3 and 5.3] in order to see that their validity hold with C-sectorial operators of smaller angle $\gamma'\pi/2$, with the meaning clear. In the case that $C \neq I$, we can make applications to generators of analytic C-regularized semigroups generated by nonelliptic differential operators in L^p-spaces [44].

In [17], Luong has investigated the following abstract two-term fractional differential equation with nonlocal conditions in a Banach space X:

$$\mathbf{D}_t^{\alpha'+1} u(t) + c_1 \mathbf{D}_t^{\beta'} u(t) = Au(t) + f(t), \quad t \geq 0,$$
$$u(0) + g(u) = u_0, \tag{76}$$
$$u_t(0) + h(u) = y_0,$$

where $0 < \alpha' \leq \beta' \leq 1$, the functions $g(\cdot)$ and $h(\cdot)$ satisfy certain conditions, and A is of sectorial angle $\beta'\pi/2$. The improvements of main results of this paper, Theorem 13, to C-sectorial operators of angle $\gamma'\pi/2$, with γ' being clarified above, can be proved straightforwardly $(0 < \alpha' < \beta' \leq 1)$.

As mentioned before, Corollary 20 can be applied to the almost sectorial operators and multivalued linear operators used in the analysis of Poisson heat equations.

Example 2. (i) ([45]) Assume that $\eta \in (0,1)$, $q \in \mathbb{N}$, Ω is a bounded domain in $\mathbb{R}^n$ with boundary of class C^{4q}, and $E := C^\eta(\overline{\Omega})$. Consider the operator $A : D(A) \subseteq C^\zeta(\overline{\Omega}) \to C^\eta(\overline{\Omega})$ given by

$$Au(x) := \sum_{|\beta| \leq 2q} a_\beta(x) D^\beta u(x) \quad \forall x \in \overline{\Omega} \tag{77}$$

with domain $D(A) := \{u \in C^{2q+\eta}(\overline{\Omega}) : D^\beta u_{|\partial\Omega} = 0 \text{ for all } |\beta| \leq q-1\}$. In this place, $\beta \in \mathbb{N}_0^n$, $|\beta| = \sum_{i=1}^n \beta_j$, and $D^\beta = \prod_{i=1}^n ((1/i)(\partial/\partial x_i))^{\beta_i}$. Let $a_\beta : \overline{\Omega} \to \mathbb{C}$ satisfy the following:

(i) $a_\beta(x) \in \mathbb{R}$ for all $x \in \overline{\Omega}$ and $|\beta| = 2q$.

(ii) $a_\beta \in C^\eta(\overline{\Omega})$ for all $|\beta| \leq 2q$, and

(iii) there exists a constant $M > 0$ such that

$$M^{-1}|\xi|^{2q} \leq \sum_{|\beta|=2q} a_\beta(x)\xi^\beta \leq M|\xi|^{2q} \quad \forall \xi \in \mathbb{R}^n,\ x \in \overline{\Omega}. \tag{78}$$

Then there exists a sufficiently large number $\sigma > 0$ such that the operator $-A_\sigma \equiv -(A+\sigma)$ satisfies the condition (H) with $\beta = 1-\eta/2q$, $C = I$, $c < 0$, and some $\theta \in (0,\pi/2)$. Let us remind ourselves that A is not densely defined and that the value of exponent $\beta = 1-\eta/2q$ is sharp. Applications of Corollary 20 are clear and here we would like to illustrate just one of them: $c_j = 0$ for $j < n-1$, $\alpha_{n-1} = \alpha_n -$, $\alpha = 0+$, $i = 0$, and $\alpha_n(\eta/2q) < \zeta < \alpha_n - 1$.

(ii) ([18]) Let $X := L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, and $1 < p < \infty$. Suppose that the operator $A := \Delta - b$ acts on X with the Dirichlet boundary conditions and that B is the multiplication operator by the function $m(x)$. By the analysis contained in [18, Example 3.6], the condition (H) is satisfied for the multivalued linear operator $\mathscr{A} := AB^{-1}$ with $\beta = 1/p$, $C = I$, and some numbers $c < 0$ and $\theta \in (0,\pi/2)$; here it is worth noting that the validity of additional condition [18, (3.42)] on the function $m(x)$ enables us to get the better exponent β in (H), provided that $p > 2$. Applications in the study of the existence and uniqueness of asymptotically almost periodic and asymptotically almost automorphic solutions of multiterm fractional integrodifferential Poisson heat equation

$$\begin{aligned}
&\left[u(t,x) - \left(g_{\zeta+1+i} * f\right)(t,x)\varphi(x)\right] + \sum_{j=1}^{n-1} c_j g_{\alpha_n-\alpha_j} \\
&\quad * \left[u(t,x) - \left(g_{\zeta+1+i} * f\right)(t,x)\varphi(x)\right] \\
&\quad + \sum_{j\in\mathbb{N}_{n-1}\setminus D_i} c_j\left[g_{\alpha_n-\alpha_j+i+\zeta+1} * f\right](t,x) \\
&= (\Delta - b)v(t,x); \\
&\qquad v(t,x) = 0,\ (t,x) \in [0,\infty)\times\partial\Omega,
\end{aligned} \tag{79}$$

where $m(x)v(t,x) = \int_0^t g_{\alpha_n-\alpha}(t-s)u(s,x)ds$ for all $t \geq 0$ and $x \in \Omega$, are immediate ($\varphi \in X$).

Example 3. In [12, Subsection 3.3.2], we have analyzed hypercyclic and topologically mixing properties of solutions of abstract multiterm fractional Cauchy problem (1) with $\mathscr{A} = A$ being single-valued and $A_j = c_j I$ for some $c_j \in \mathbb{C}$ $(1 \leq i \leq n-1)$. Let it be the case. With the help of Proposition 21, we can reconsider a great number of examples given in the above-mentioned part of [12] and provide several interesting applications in the investigation of problem about the existence of a dense linear subspace X' of X such that the mapping $t \mapsto R_i(t)x$, $t \geq 0$, is asymptotically almost periodic for all $x \in X'$; here, $(R_i(t))_{t\geq 0}$ is a given k_i-regularized C-propagation family for (1) with a subgenerator A (cf. [7] for further information in this direction). For the sake of illustration, we will consider only the situation of [12, Example 3.3.12(ii)]; see also Ji and Weber [31]. Suppose that X is a symmetric space of noncompact type and rank one, $p > 2$, and the parabolic domain P_p and the positive real number c_p possess the same meaning as in [31]. Suppose, further, that $P(z) = \sum_{j=0}^k a_j z^j$, $z \in \mathbb{C}$ is a nonconstant complex polynomial with $a_k > 0$, $n = 2$, $0 < a < 2$, $\alpha_2 = 2a$, $\alpha_1 = 0$, $\alpha = a$, $c_1 > 0$, $i = 0$, and $|\theta| < \min(\pi/2 - n\arctan(|p-2|/2\sqrt{p-1}), \pi/2 - n\arctan(|p-2|/2\sqrt{p-1}) - (\pi/2)a)$. Then we already know that the operator $-e^{i\theta}P(\Delta^\natural_{X,p})$ is the integral generator of an exponentially bounded, analytic resolvent propagation family $((R_{\theta,P,0}(t))_{t\geq0}, \ldots, (R_{\theta,P,\lceil 2a\rceil-1}(t))_{t\geq 0})$ of angle $\min((\pi - n\arctan(|p-2|/2\sqrt{p-1}) - |\theta|)/a - \pi/2, \pi/2)$, and that $(R_{\theta,P,0}(t))_{t\geq0}$ is topologically mixing provided the condition

$$-e^{i\theta}P\left(\operatorname{int}\left(P_p\right)\right) \cap \left\{(it)^a + c_1(it)^{-a} : t \in \mathbb{R}\setminus\{0\}\right\} \neq \emptyset; \tag{80}$$

compare [12] for the notion. Our new assumption will be, instead of (80), that there exist a number $\lambda_0 \in \mathbb{C}\setminus(-\infty,0]$ and a sufficiently small number $\epsilon > 0$ such that $\lambda^a + c_1\lambda^{-a} \in -e^{i\theta}P(\operatorname{int}(P_p))$ and that $\lambda^a \notin \Sigma_{a\pi/2}$ for $|\lambda - \lambda_0| \leq \epsilon$. Let $|\lambda-\lambda_0| \leq \epsilon$. Since there exists an $x \neq 0$ such that $-e^{i\theta}P(\Delta^\natural_{X,p})x = (\lambda^a + c_1\lambda^{-a})x$ due to our assumption $\lambda^a + c_1\lambda^{-a} \in -e^{i\theta}P(\operatorname{int}(P_p)) \subseteq \sigma_p(-e^{i\theta}P(\Delta^\natural_{X,p}))$, we can employ [12, Lemma 3.3.7] in order to see that

$$\begin{aligned}
R_{\theta,P,0}(t)x &= \frac{\lambda^a t^{-a}}{\lambda^{2a} - c_1}\left(E_{a,2-a}\left(\lambda^a t^a\right)\right. \\
&\quad \left.- E_{a,2-a}\left(c_1\lambda^{-a}t^a\right)\right)x + \frac{\lambda^a}{\lambda^{2a}-c_1}\left[\lambda^a E_a\left(\lambda^a t^a\right)\right. \\
&\quad + (a-1)\lambda^a E_{a,2}\left(\lambda^a t^a\right) - c_1\lambda^{-a}E_a\left(c_1\lambda^{-a}t^a\right) \\
&\quad \left.- (a-1)c_1\lambda^{-a}E_{a,2}\left(c_1\lambda^{-a}t^a\right)\right]x + \left(\lambda^a + c_1\lambda^{-a}\right) \\
&\quad \cdot \frac{\lambda^a}{\lambda^{2a}-c_1}\left[E_a\left(\lambda^a t^a\right) - E_a\left(c_1\lambda^{-a}t^a\right)\right]x, \quad t > 0;
\end{aligned} \tag{81}$$

see also [12, p. 418]. Due to our assumption $\lambda^a \notin \Sigma_{a\pi/2}$, the asymptotic expansion formulae (8)-(9), and the fact that the first term in the above expression can be continuously extended to the nonnegative real axis, we get the mapping $t \mapsto R_{\theta,P,0}(t)x$, $t \geq 0$, is asymptotically almost periodic. Since the set $\{x \in X : \exists\lambda \in \mathbb{C}\setminus(-\infty,0]$ s.t. $|\lambda-\lambda_0| \leq \epsilon$ and $-e^{i\theta}P(\Delta^\natural_{X,p})x = (\lambda^a + c_1\lambda^{-a})x\}$ is total in X, Proposition 21 implies that there exists a dense linear subspace X' of X such that the mapping $t \mapsto R_{\theta,P,0}(t)x$, $t \geq 0$, is asymptotically almost periodic for all $x \in X'$.

Conflicts of Interest

The authors have no conflicts of interest with this publication.

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References

[1] L. Amerio and G. Prouse, *Almost-Periodic Functions and Functional Equations*, Van Nostrand-Reinhold, New York, NY, USA, 1971.

[2] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, vol. 96 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 2001.

[3] D. N. Cheban, *Asymptotically Almost Periodic Solutions of Differential Equations*, Hindawi Publishing Corporation, New York, NY, USA, 2009.

[4] T. Diagana, *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*, Springer, New York, NY, USA, 2013.

[5] G. M. N'Guérékata, *Almost Automorphic and Almost Periodicity Functions in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht, Netherlands, 2001.

[6] Y. Hino, T. Naito, N. V. Minh, and J. S. Shin, *Almost Periodic Solutions of Differential Equations in Banach Spaces. Stability and Control: Theory, Methods and Applications*, vol. 15, Taylor and Francis Group, London, UK, 2002.

[7] M. Kostić, *Abstract Volterra Integro-Differential Equations: Almost Periodicity and Asymptotically Almost Periodic Properties of Solutions*, Book Manuscript, 2017.

[8] M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, London, UK, 1982.

[9] T.-J. Xiao and J. Liang, *The Cauchy Problem for Higher–Order Abstract Differential Equations*, Springer-Verlag, Berlin, Germany, 1998.

[10] S. Zaidman, *Almost-Periodic Functions in Abstract Spaces*, vol. 126, Pitman, Boston, Massachusetts, USA, 1985.

[11] E. Bazhlekova, *Fractional evolution equations in Banach spaces [Ph.D. thesis]*, Eindhoven University of Technology, Eindhoven, Netherlands, 2001.

[12] M. Kostić, *Abstract Volterra Integro-Differential Equations*, Taylor and Francis Group/CRC Press/Science Publishers, Boca Raton, New York, NY, USA, 2015.

[13] M. Kostić, C.-G. Li, and M. Li, "On a class of abstract time-fractional equations on locally convex spaces," *Abstract and Applied Analysis*, p. 41, 2012.

[14] M. Sova, "The Laplace transform of analytic vector-valued functions," *Casopis pro Pestovani Matematiky*, vol. 104, pp. 267–280, 1979.

[15] E. Cuesta, "Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations," *Discrete and Continuous Dynamical Systems - Series A*, pp. 277–285, 2007.

[16] V. Keyantuo, C. Lizama, and M. Warma, "Asymptotic behavior of fractional-order semilinear evolution equations," *Differential and Integral Equations. An International Journal for Theory and Applications*, vol. 26, no. 7-8, pp. 757–780, 2013.

[17] V. T. Luong, "Decay mild solutions for two-term time fractional differential equations in Banach spaces," *Journal of Fixed Point Theory and Applications*, vol. 18, no. 2, pp. 417–432, 2016.

[18] A. Favini and A. Yagi, *Degenerate Differential Equations in Banach Spaces*, Chapman and Hall/CRC Pure and Applied Mathematics, New York, NY, USA, 1998.

[19] M. Kostić, *Abstract Degenerate Volterra Integro-Differential Equations: Linear Theory and Applications*, Book Manuscript, 2016.

[20] K. Diethelm, *The Analysis of Fractional Differential Equations*, vol. 2004, Springer-Verlag, Berlin, Germany, 2010.

[21] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations, Elsevier Science B.V*, Elsevier Science, Amsterdam, Netherlands, 2006.

[22] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.

[23] J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser-Verlag, Basel, Switzerland, 2012.

[24] W. M. Ruess and W. H. Summers, "Asymptotic almost periodicity and motions of semigroups of operators," *Linear Algebra and its Applications*, vol. 84, no. C, pp. 335–351, 1986.

[25] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, New York, NY, USA, 1993.

[26] J. Andres, A. M. Bersani, and R. F. Grande, "Hierarchy of almost-periodic function spaces," *Rendiconti di Matematica*, vol. 26, pp. 121–188, 2006.

[27] A. S. Besicovitch, *Almost Periodic Functions*, Dover Publications Inc., New York, NY, USA, 1954.

[28] G. M. N'Guérékata and A. Pankov, "Stepanov-like almost automorphic functions and monotone evolution equations," *Nonlinear Analysis. Theory, Methods and Applications. An International Multidisciplinary Journal*, vol. 68, no. 9, pp. 2658–2667, 2008.

[29] G. M. N'Guérékata, *Spectral Theory for Bounded Functions and Applications to Evolution Equations*, Nova Science Publishers, New York, NY, USA, 2017.

[30] H. R. Henríquez, "On Stepanov-almost periodic semigroups and cosine functions of operators," *Journal of Mathematical Analysis and Applications*, vol. 146, no. 2, pp. 420–433, 1990.

[31] L. Ji and A. Weber, "Dynamics of the heat semigroup on symmetric spaces," *Ergodic Theory Dynam. Systems*, vol. 3, pp. 457–468, 2010.

[32] M. Kostić, "Some contributions to the theory of abstract degenerate volterra integro-differential equations," *Journal of Mathematics and Statistics*, vol. 12, no. 2, pp. 65–76, 2016.

[33] M. Kostić, "Existence of generalized almost periodic and asymptotic almost periodic solutions to abstract Volterra integro-differential equations," *Electronic Journal of Differential Equations*, vol. 2017, no. 239, pp. 1–30, 2017.

[34] M. Kostić, "Degenerate k-regularized (C1,C2)-existence and uniqueness families," *CUBO: A Mathematical Journal*, vol. 17, no. 3, pp. 15–41, 2015.

[35] M. Kostić, "Weyl-almost periodic solutions and asymptotically Weyl-almost periodic solutions of abstract Volterra integro-differential equations," *Journal of Mathematical Analysis and Applications*, Paper No. 239, 30 pages, 2017.

[36] A. M. Fink, "Extensions of almost automorphic sequences," *Journal of Mathematical Analysis and Applications*, vol. 27, no. 3, pp. 519–523, 1969.

[37] S. Abbas, "A note on Weyl pseudo almost automorphic functions and their properties," *Mathematical Sciences*, vol. 6, no. 1, p. 29, 2012.

[38] F. Bedouhene, N. Challali, O. Mellah, P. Raynaud de Fitte, and M. Smaali, "Almost automorphy and various extensions for stochastic processes," *Journal of Mathematical Analysis and Applications*, vol. 429, no. 2, pp. 1113–1152, 2015.

[39] M. Kostić, "Generalized almost automorphic and generalized asymptotically almost automorphic solutions of abstract

Volterra integro-differential inclusions," *Fractional Differential Calculus*, 2017.

[40] R. Cross, *Multivalued Linear Operators*, Marcel Dekker, Inc., New York, NY, USA, 1998.

[41] R. W. Carroll and R. W. Showalter, *Singular and Degenerate Cauchy Problems*, Academic Press, New York, NY, USA, 1976.

[42] I. V. Melnikova and A. I. Filinkov, *Abstract Cauchy Problems: Three Approaches*, Chapman Hall/CRC Press, Boca Raton, New York, NY, USA, 2001.

[43] G. A. Sviridyuk and V. E. Fedorov, *Linear Sobolev Type Equations and Degenerate Semigroups of Operators*, Inverse and Ill-Posed Problems (Book 42), VSP, Utrecht, Boston, USA, 2003.

[44] Q. Zheng and M. Li, *Regularized Semigroups and Non-Elliptic Differential Operators*, Science Press, Beijing, China, 2014.

[45] W. Von Wahl, "Gebrochene Potenzen eines elliptischen Operators und parabolische Differentialgleichungen in Räumen hölderstetiger Funktionen," *Nachrichten von der Gesellschaft der wissenschaften zu Gottingen, Mathematisch-Physikalische Klasse*, vol. 11, pp. 231–258, 1972.