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Research Article

Estimates on the Bergman Kernels in a Tangential Direction on Pseudoconvex Domains in $\mathbb{C}^3$

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Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$ and assume that $T_\Omega^{reg}(z_0) < \infty$ where $z_0 \in b\Omega$, the boundary of Ω. Then we get optimal estimates of the Bergman kernel function along some "almost tangential curve" $C_b(z_0, \delta_0) \subset \Omega \cup \{z_0\}$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. A natural operator on Ω is the orthogonal projection

$$P : L^2(\Omega) \longrightarrow H(\Omega) \cap L^2(\Omega), \tag{1}$$

where $H(\Omega)$ denotes the holomorphic functions on Ω. There is a corresponding kernel function $K_\Omega(z, w)$, called the Bergman kernel function on Ω. The nature of the singularity of $K_\Omega(z, w)$ tells us much about the holomorphic function theory of the domain in question and has been studied extensively since Bergman's original inquiries [1].

One of the methods for the estimates of the Bergman kernel is to construct maximal size of polydiscs in Ω where we have a plurisubharmonic function with maximal Hessian. For strongly pseudoconvex domains in $\mathbb{C}^n$, these polydiscs are of size $\delta > 0$ in normal direction and of size $\delta^{1/2}$ in tangential directions. For weakly pseudoconvex domains, the size of the polydisc in tangential directions depends on the boundary geometry of Ω near $z_0 \in b\Omega$, and hence we need complete analysis of the boundary geometry near z_0.

However these analyses and hence the optimal estimates on the Bergman kernels are done only for special type of pseudoconvex domains of finite type in $\mathbb{C}^n$. These domains are, for example, pseudoconvex domains of finite type in $\mathbb{C}^2$ [2–4], decoupled, convex, or uniformly extendable domains of finite type in $\mathbb{C}^n$ [5–7], or pseudoconvex domains in $\mathbb{C}^n$ with $(n-2)$ positive eigenvalues [8, 9]. For the estimates for weighted Bergman projections, one can also refer to [10–12]. Nevertheless, the optimal estimates for general pseudoconvex domains of finite type in $\mathbb{C}^n$, $n > 2$, are not known, even for $n = 3$ case.

Assume that Ω is a smoothly bounded domain in $\mathbb{C}^n$ with smooth defining function r with smooth boundary, $b\Omega$. Regular finite 1-type at $z_0 \in b\Omega$, denoted by $T_\Omega^{reg}(z_0)$, is the maximum order of vanishing of $r \circ \gamma$ for all one complex dimensional regular curve γ, $\gamma(0) = z_0$, and $\gamma'(0) \neq 0$. Thus $T_\Omega^{reg}(z_0)$ satisfies

$$\Delta_{n-1}(z_0) \leq T_\Omega^{reg}(z_0) \leq \Delta_1(z_0), \tag{2}$$

where $\Delta_q(z_0)$, $1 \leq q \leq n-1$, denotes finite q-type in the sense of D'Angelo [13]. Note that $\Delta_{n-1}(z_0) = T_{BG}(z_0)$ where $T_{BG}(z_0)$ is the type in the sense of Bloom-Graham.

Remark 1. Consider the domain Ω [13] in $\mathbb{C}^3$ defined by

$$r(z) = \operatorname{Re} z_3 + \left| z_1^2 - z_2^3 \right|^2. \tag{3}$$

Then $T_\Omega^{reg}(0) = 6$ and $\Delta_2(0) = 4$ while $\Delta_1(0) = \infty$ as the complex analytic curve $\gamma(t) = (t^3, t^2, 0)$ lies in the boundary. Note that $\gamma(t)$ is not regular curve.

In the sequel, we let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$, and assume that $T_\Omega^{reg}(z_0) = \eta < \infty$ where $z_0 \in b\Omega$. Let $C_b(z_0, \delta_0) \subset \Omega \cup \{z_0\}$ be the "almost tangential curve" connecting a point $z^{\delta_0} \in \Omega$ and $z_0 \in b\Omega$ as defined in (20). Note that $\operatorname{dist}(z^\delta, b\Omega) \approx \delta$ for each $z^\delta \in$

$C_b(z_0,\delta_0)$. Set $\tau_1 = \delta^{1/\eta}$, $\tau_2 = \tau(z^\delta,\delta)$ where $\tau(z^\delta,\delta)$ is defined in (51).

Theorem 2. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$ and assume that $T^{reg}_\Omega(z_0) < \infty$ where $z_0 \in b\Omega$. Then $K_\Omega(z^\delta,z^\delta)$, the Bergman kernel function of Ω at $z^\delta \in C_b(z_0,\delta_0)$, satisfies*

$$K_\Omega\left(z^\delta,z^\delta\right) \approx \delta^{-2}\tau_1^{-2}\tau_2^{-2}. \quad (4)$$

Theorem 3. *Let Ω and $z_0 \in b\Omega$ be as in Theorem 2. For each $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, there is a constant $C_\alpha > 0$, independent of $\delta > 0$, such that*

$$\left|D_z^\alpha K_\Omega\left(z,z^\delta\right)\right| \le C_\alpha \delta^{-2-\alpha_3}\tau_1^{-2-\alpha_1}\tau_2^{-2-\alpha_2}, \quad (5)$$

for $z \in \Omega$ and $z^\delta \in C_b(z_0,\delta_0)$.

Remark 4. (1) In Theorems 2 and 3, we do not assume that $\Delta_1(z_0) < \infty$, but we assume only that $T^{reg}_\Omega(z_0) < \infty$ (see Remark 1). With this weaker condition, we get optimal estimates for Bergman kernel function along special "almost tangential" direction, $C_b(z_0,\delta_0)$, but not normal or arbitrary direction.

(2) In [14], Herbort gives an example of a domain $\Omega_H \subset \mathbb{C}^3$ where the Bergman kernel grows logarithmically when $z \in \Omega$ approaches to $z_0 \in b\Omega$ in normal direction. Set

$$\Omega_H = \left\{z \in \mathbb{C}^3 \mid \operatorname{Re} z_3 + |z_1|^6 + |z_1|^2|z_2|^2 + |z_2|^6 < 0\right\}, \quad (6)$$

and for each small $\delta > 0$, set $z^\delta = (0,0,-\delta)$. Thus z^δ approaches to $0 \in b\Omega_H$ in normal direction as $\delta \longrightarrow 0$. In this case, Herbort shows that $K_\Omega(z^\delta,z^\delta) \approx \delta^{-3}(-\log\delta)^{-1}$; that is, the kernel grows logarithmically. For the same domain Ω_H in (6), we note that $\eta = T^{reg}_\Omega(z_0) = 6$ and hence $\tau_1 = \delta^{1/6}$, $\tau_2 = \delta^{1/3}$ in (4). Set $C_b(z_0,\delta_0) := \{(\delta^{1/6}/2,0,-\delta) : 0 \le \delta \le \delta_0\}$. Then $z^\delta := (\delta^{1/6}/2,0,-\delta) \in C_b(z_0,\delta_0)$ approaches to $0 \in b\Omega$ in "almost tangential direction". In the Appendix of this paper, we will show that

$$K_{\Omega_H}\left(z^\delta,z^\delta\right) \approx \delta^{-2}\tau_1^{-2}\tau_2^{-2} = \delta^{-3}. \quad (7)$$

In Section 2, we will construct special coordinates which reflect the regular finite type condition, $\Delta_2(z_0) \le T^{reg}_\Omega(z_0) = \eta < \infty$, and then show that $r(z)$ vanishes to order η in z_1-direction. We then consider the slices of Ω by fixing z_1. Then the domains become domains in $\mathbb{C}^2$, and hence we can handle them. Also, the condition $\Delta_2(z_0) < \infty$ acts like the condition $\Delta_1(z_0) < \infty$ on these slices.

For the estimates of $K(z,w)$, Catlin [2, 15] constructed plurisubharmonic functions with maximal Hessian near each thin δ-strip of $b\Omega$ (Section 3 of [2]). In this paper, however, we will construct these functions only on nonisotropic polydiscs $Q_{a\delta}(z^\delta) \subset\subset \Omega$ for each $z^\delta \in C_b(z_0,\delta_0)$ (Proposition 23). This avoids complicated technical parts in Section 3 of [2]. To get estimates of $K_\Omega(z,z^\delta)$, $z \in \Omega$, $z^\delta \in C_b(z_0,\delta_0)$, we consider dilated domains D_δ for each $\delta > 0$. Then the polydisc $Q_{a\delta}(z^\delta) \subset\subset \Omega$ becomes $P(0,1) \subset\subset D_\delta$, independent of $\delta > 0$, where $P(0,1)$ is a polydisc of radius one with center at the origin. Therefore the uniform 1/2-subelliptic estimates for $\bar\partial$-equation hold on $P(0,1)$, and the estimates for $K_\Omega(z,z^\delta)$ follow.

Remark 5. Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$, and assume that $\Delta_1(z_0) < \infty$, where $z_0 \in b\Omega$. Then the conditions of Theorems 2 and 3 are satisfied. Near future, using the results of Theorems 2 and 3, we hope we can prove some function theories on Ω, for example, the existence of peak function for Ω that peaks at $z_0 \in b\Omega$ or necessary conditions for the Hölder estimates for $\bar\partial$-equation.

2. Special Coordinates

In the sequel, we let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$ and assume that $m = \Delta_2(z_0) \le \eta = T^{reg}_\Omega(z_0) < \infty$, $z_0 \in b\Omega$. Note that m and η are positive integers. Without loss of generality, we may assume that $z_0 = 0$. In the sequel, we let $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and $\beta = (\beta_1,\beta_2,\beta_3)$ be multi-indices and set $\alpha' = (\alpha_1,\alpha_2)$ and $z' = (z_1,z_2)$, etc. In Theorem 3.1 in [16], You constructed special coordinates which represent the local geometry of $b\Omega$ near z_0.

Theorem 6. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$ with smooth defining function r and assume $T^{reg}_\Omega(0) = \eta < \infty$, $0 \in b\Omega$. Then there is a holomorphic coordinate system $z = (z_1,z_2,z_3)$ about 0 such that*

$$(1)\ \ r(z) = \operatorname{Re} z_3 + \sum_{\substack{|\alpha'|+|\beta'|=m \\ |\alpha'|,|\beta'|>0}}^{\eta} a_{\alpha',\beta'} z^{\alpha'}\bar z^{\beta'} + \mathcal{O}\left(|z_3||z| + |z'|^{\eta+1}\right), \quad (8)$$

$$(2)\ \ |r(t,0,0)| \approx |t|^\eta,$$

where

$$a_{\alpha',\beta'} \ne 0 \quad \text{with } \alpha_2 + \beta_2 = m \text{ for some } \alpha_2 > 0,\ \beta_2 > 0. \quad (9)$$

(Idea of the proof) by the standard holomorphic coordinate changes, $r(w)$ has the Taylor series expansion as in (8). Since $T^{reg}_\Omega(0) = \eta$, there is a regular curve which we may assume that $\gamma(t) = (t,\gamma_2(t),\mathcal{O}(t^\eta))$ satisfying $|r(\gamma(t))| \approx |t|^\eta$ for all sufficiently small $t \in \mathbb{C}$. Set $z = (w_1, w_2 + \gamma_2(w_1), w_3)$. Then, in z coordinates, $r(z)$ has representation satisfying (8). Also (9) follows from the condition that $m = \Delta_2(0)$.

Remark 7. (1) The second condition in (8) and property (9) say that $r(z)$ vanishes to order η along z_1 axis and order m along z_2 axis.

(2) There are much more terms (mixed with z_1, z_2 and their conjugates), compared to the h-extensible domain cases, in the summation part of (8).

In conjunction with multitype $\mathscr{M}(0) = (1, m, m_3)$, we need to consider the dominating terms (in size) among the mixed terms in z_1 and z_2 variables in the summation part of (8). Using the notations of Section 3.2 in [16], set

$$\begin{aligned} \Gamma &= \{(\alpha', \beta'); a_{\alpha',\beta'} \neq 0,\ m \leq |\alpha'| + |\beta'| \\ &\leq \eta,\ \text{and}\ |\alpha'|, |\beta'| > 0\} \\ S &= \{(p,q); \alpha_1 + \beta_1 = p,\ \alpha_2 + \beta_2 \\ &= q \text{ for some } (\alpha', \beta') \in \Gamma\} \cup \{(\eta, 0)\}. \end{aligned} \tag{10}$$

Then there are $(p_\nu, q_\nu) \in S$ for $\nu = 0, 1, \ldots, N$ and $\eta_\nu, \lambda_\nu > 0$ for $\nu = 1, \ldots, N$, such that

$$\begin{aligned} &(1)\ (p_0, q_0) = (\eta, 0),\ (p_N, q_N) = (0, m),\ \lambda_N = m,\ \eta_1 = \eta \\ &(2)\ p_0 > p_1 > \ldots > p_N \text{ and } q_0 < q_1 < \ldots < q_N \\ &(3)\ \lambda_1 < \lambda_2 < \ldots < \lambda_N \text{ and } \eta_1 > \eta_2 > \ldots > \eta_N \\ &(4)\ \frac{p_{\nu-1}}{\eta_\nu} + \frac{q_{\nu-1}}{\lambda_\nu} = 1 \text{ and } \frac{p_\nu}{\eta_\nu} + \frac{q_\nu}{\lambda_\nu} = 1 \\ &(5)\ a_{\alpha',\beta'} = 0 \text{ if } \frac{\alpha_1 + \beta_1}{\eta_\nu} + \frac{\alpha_2 + \beta_2}{\lambda_\nu} < 1 \text{ for each } \nu = 1, \ldots, N. \end{aligned} \tag{11}$$

Remark 8. (1) Here, p_ν's and q_ν's are the exponents of z_1 and z_2, respectively, in the dominating terms in the summation part of (8).

(2) If $\Delta_1(z_0) < \infty$, then the expression in (8) will be similar to that of $\mathbb{C}^2$ case in [2], and hence we need not consider the above complicated pairs.

Set $t_0 = \eta$. If $1 \leq k \leq m$, then $q_{\nu-1} < k \leq q_\nu$ for some $\nu = 1, \ldots, N$. In this case, set

$$t_k = \eta_\nu \left(1 - \frac{k}{\lambda_\nu}\right), \quad \text{i.e., } \frac{t_k}{\eta_\nu} + \frac{k}{\lambda_\nu} = 1. \tag{12}$$

Then $(p_{\nu-1}, q_{\nu-1})$, (t_l, l), and (p_ν, q_ν) are colinear points in the first quadrant of the plane, and λ_ν (resp., η_ν) is the intercept of q-axis (resp., p-axis) of this line. Let L_ν be the line segment from $(p_{\nu-1}, q_{\nu-1})$ to (p_ν, q_ν) for $\nu = 1, \ldots, N$, set $L = L_1 \cup L_2 \cup \ldots \cup L_N$, $\Gamma_L = \{(\alpha', \beta') \in \Gamma; (\alpha_1 + \beta_1, \alpha_2 + \beta_2) \in L\}$, and set

$$\begin{aligned} \Lambda = \{&(\alpha', \beta') \in \Gamma_L;\ \alpha' + \beta' = (p_\nu, q_\nu),\ \alpha_2 > 0,\ \beta_2 \\ &> 0,\ \nu = 1, \ldots, N\}. \end{aligned} \tag{13}$$

As in Corollary 3.8 and Remark 3.9 in [16], we can rewrite (8) so that

$$\begin{aligned} r(z) &= \operatorname{Re} z_3 + \sum_{\Gamma_L - \Lambda} a_{\alpha,\beta} z'^{\alpha} \bar{z}'^{\beta} \\ &+ \sum_{\nu=1}^{N} \sum_{\substack{\alpha_2+\beta_2=q_\nu \\ \alpha_2>0,\beta_2>0}} M^\nu_{\alpha_2,\beta_2}(z_1) z_2^{\alpha_2} \bar{z}_2^{\beta_2} \\ &+ \mathcal{O}\left(|z_3||z| + \sum_{\nu=1}^{N} \sum_{l=q_{\nu-1}}^{q_\nu} |z_1|^{[t_l]+1} |z_2|^l + |z_2|^{m+1}\right), \end{aligned} \tag{14}$$

where $M^\nu_{\alpha_2,\beta_2}(z_1)$ is a nontrivial homogeneous polynomial of degree p_ν given by

$$M^\nu_{\alpha_2,\beta_2}(z_1) = \sum_{\alpha_1+\beta_1=p_\nu} a_{\alpha_1,\beta_1} z_1^{\alpha_1} \bar{z}_1^{\beta_1}, \tag{15}$$

and there are a small constant $a_0 > 0$, and $d \in \{z_1 \in \mathbb{C}; |z_1| = 1\}$ such that

$$\begin{aligned} &M^\nu_{\alpha_2,\beta_2}(z_1) \neq 0 \quad \text{for } |z_1 - d| < a_0, \\ &\text{and } \left|M^\nu_{\alpha_2,\beta_2}(d\delta^{1/\eta})\right| \approx \delta^{p_\nu/\eta}, \end{aligned} \tag{16}$$

for all $\alpha_2 + \beta_2 = q_\nu$ with all $\nu = 1, \ldots, N-1$. Property (16) means that there is (α', β') with terms mixed in z_2 and $\bar{z}_2$ variables for $|z_1 - d| < a_0$. Let $|d| = 1$ be the constant (direction) in (16) and we will fix d in the rest of this paper. In the sequel, we set $\hat{z}_l$ equal to z_l or $\bar{z}_l$, $l = 1, 2, 3$.

Remark 9. (a) $\{t_k\}$ defined in (12) is strictly decreasing on k.

(b) Each of the summation parts of (14) contains the terms of the form $\hat{z}_1^{t_k} \hat{z}_2^k$ where (t_k, k)'s are the pairs, defined in (12), on the polyline L.

(c) Each term of the first summation part in (14) is pure in z_2 or $\bar{z}_2$ variables.

(d) Each term of the second summation part in (14) has terms mixed in z_2 and $\bar{z}_2$, and it corresponds to the pair of integers (p_ν, q_ν), the vertices of the polyline L.

Lemma 10. *Let $d_0(z_1) := r(z_1, 0, 0)$ be the term containing only z_1 or $\bar{z}_1$ variables in the first sum of (14). Then*

$$|d_0(z_1)| \approx |z_1|^\eta. \tag{17}$$

Proof. From (8) and (14), we see that $|r(z_1, 0, 0)| \lesssim |z_1|^\eta$. On the other hand, since the regular 1-type at $0 \in b\Omega$ is equal to $\eta = T^{reg}_\Omega(0)$, there is $\tilde{c}_1 > 0$ such that $|r(z_1, 0, 0)| \geq \tilde{c}_1 |z_1|^\eta$. □

In the sequel, we let V be a small neighborhood of $z_0 = 0 \in b\Omega$ where $r(z)$ has expression as in (14). Since $(\partial r/\partial z_3)(0) \neq 0$, we may assume that $|(\partial r/\partial z_3)(z)| \geq c_0$ for all $z \in V$ for a uniform constant $c_0 > 0$ by shrinking V if necessary. For each fixed $\delta > 0$ and for each $z = (z_1, z_2, z_3) \in V$ satisfying $|z_1 - d\delta^{1/\eta}| < \gamma\delta^{1/\eta}$, for a sufficiently small $\gamma > 0$ to be chosen, we set $\pi(z) = (z_1, 0, e_\delta) := \tilde{z} \in b\Omega$, where $\pi(z)$

is the composition of the projection onto $z_1 z_3$ plane and then the projection onto $b\Omega$ along the $\operatorname{Re} z_3$ direction. Using the Taylor series method in z_3 variable about e_δ, we see that

$$r(z_1, 0, 0) = 2\operatorname{Re}\left[\frac{\partial r(\tilde{z})}{\partial z_3}(-e_\delta)\right] + \mathcal{O}\left(e_\delta^2\right). \tag{18}$$

Since $|e_\delta| \ll 1$ and $2\operatorname{Re}(\partial r/\partial z_3) = 1 + \mathcal{O}(|z|) \geq 1/2$ on V, it follows from (17) that

$$|e_\delta| \approx |z_1|^\eta, \tag{19}$$

for z_1 near 0.

Now for each small $\delta > 0$, set $z(\delta) := (d\delta^{1/\eta}, 0, 0)$ and set $\tilde{z}^\delta = \pi(z(\delta)) = (d\delta^{1/\eta}, 0, e_\delta) \in b\Omega$. For a small constant $b > 0$ to be chosen, set $z^\delta = (d\delta^{1/\eta}, 0, e_\delta - b\delta) \in \Omega$, and for a fixed small $\delta_0 > 0$ satisfying $r(z^{\delta_0}) < 0$, set

$$\begin{aligned} &C_b(z_0, \delta_0) \\ &:= \left\{z^\delta : z^\delta = \left(d\delta^{1/\eta}, 0, e_\delta - b\delta\right),\ 0 \leq \delta \leq \delta_0\right\} \\ &\subset \Omega \cup \{z_0\}, \end{aligned} \tag{20}$$

connecting $z^{\delta_0} \in V \cap \Omega$ and $z_0 = 0 \in b\Omega$.

Following the same arguments as in the proof of Proposition 1.2 in [2], for each fixed $\tilde{z} \in V$, we can construct special coordinates about $\tilde{z}$ so that, in terms of new coordinates, there is no pure terms in z_2 or $\overline{z}_2$ variables in the first summation part of $r(z)$ in (14). We will fix z_1 variable and consider the coordinate changes only on $z'' = (z_2, z_3)$ variables.

Proposition 11. *For each fixed $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) \in V$, there is a holomorphic coordinate system $z'' = \Phi_{\tilde{z}}(\zeta'') = (\zeta_2, \Phi_3(\zeta''))$ such that in the new coordinates ζ'' defined by*

$$\Phi_{\tilde{z}}(\zeta'') = \left(\tilde{z}_2 + \zeta_2, \Phi_3(\zeta'')\right), \tag{21}$$

where

$$\Phi_3(\zeta'') = \tilde{z}_3 + \left(\frac{\partial r}{\partial z_3}(\tilde{z})\right)^{-1}\left(\frac{\zeta_3}{2} - \sum_{l=1}^{m} c_l(\tilde{z})\zeta_2^l\right), \tag{22}$$

and where $c_l(\tilde{z})$, $l = 2, 3, \ldots, m$, depends smoothly on $\tilde{z}$, the function given by $\rho(\tilde{z}_1, \zeta'') := r(\tilde{z}_1, \Phi_{\tilde{z}}(\zeta''))$ satisfies

$$\begin{aligned} \rho(\tilde{z}_1, \zeta'') = {} & r(\tilde{z}) + \operatorname{Re}\zeta_3 + \sum_{\substack{j+k=2 \\ j,k>0}}^{m} a_{j,k}(\tilde{z}_1)\zeta_2^j\overline{\zeta}_2^k \\ & + \mathcal{O}\left(|\zeta_3||\zeta| + |\zeta_2|^{m+1}\right). \end{aligned} \tag{23}$$

Proof. For $\tilde{z} \in V$, define

$$\begin{aligned} \Phi^1(w'') = \Bigg(& \tilde{z}_2 + w_2, \tilde{z}_3 \\ & + \left(\frac{\partial r}{\partial z_3}(\tilde{z})\right)^{-1}\left(\frac{w_3}{2} - \frac{\partial r}{\partial z_2}(\tilde{z})w_2\right)\Bigg), \end{aligned} \tag{24}$$

where $w'' = (w_2, w_3)$. Then we have

$$\begin{aligned} \rho_2(\tilde{z}_1, w'') &:= r\left(\tilde{z}_1, \Phi^1(w'')\right) \\ &= r(\tilde{z}) + \operatorname{Re} w_3 + \mathcal{O}\left(|w''|^2\right). \end{aligned} \tag{25}$$

Assume that (22) and (23) hold for $l \geq 2$. That is, we have defined $\Phi^{l-1} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ so that $\rho_l(\tilde{z}_1, w'') := r(\tilde{z}_1, \Phi^{l-1}(w''))$ can be written as

$$\begin{aligned} \rho_l(\tilde{z}_1, w'') = {} & \operatorname{Re} w_3 + \sum_{\substack{j+k=2 \\ j,k>0}}^{l-1} a_{j,k}^{l-1}(\tilde{z})w_2^j\overline{w}_2^k \\ & + \mathcal{O}\left(|w_3||w''| + |w_2|^l\right). \end{aligned} \tag{26}$$

If we define $\Phi^l = \Phi^{l-1} \circ \phi^l$, where

$$\phi^l(\zeta'') = \left(\zeta_2, \zeta_3 - \frac{2}{l!}\frac{\partial^l \rho_l}{\partial w_2^l}(\tilde{z}_1, 0, 0)\zeta_2^l\right), \tag{27}$$

then $\rho_{l+1}(\tilde{z}_1, \zeta'') := r(\tilde{z}_1, \Phi^l(\zeta''))$ satisfies (26) for l replaced by $l + 1$. If we proceed up to $l = m$ and set $\Phi_{\tilde{z}} = \Phi^m = \Phi^1 \circ \phi^2 \circ \cdots \circ \phi^m$, then by setting $\rho = \rho_{m+1} = r(\tilde{z}, \Phi_{\tilde{z}}(\cdot))$, we see that (22) and (23) hold. ☐

In the sequel, we will use the coordinate changes in Proposition 11 only at $\tilde{z} = (\tilde{z}_1, 0, \tilde{z}_3) \in V$, (in particular at $\tilde{z}^\delta = (d\delta^{1/\eta}, 0, e_\delta) \in b\Omega$ in Section 3). We want to study the dependence of $\Phi_{\tilde{z}}$ about $\tilde{z}$. For each $\tilde{z} = (\tilde{z}_1, 0, \tilde{z}_3) \in V$, set $c_0(\tilde{z}) := (\partial r/\partial z_3)(\tilde{z}) = 1 + \mathcal{O}(|\tilde{z}|)$, and we note that

$$c_l(\tilde{z}) = \frac{1}{l!}\frac{\partial^l \rho_l}{\partial w_2^l}(\tilde{z}_1, 0, 0), \quad l = 1, 2, \ldots, m, \tag{28}$$

where ρ_l is defined in the inductive step of the proof of Proposition 11. Set

$$\begin{aligned} e_0(\tilde{z}) &= \frac{1}{2}c_0(\tilde{z})^{-1}, \\ e_1(\tilde{z}) &= -c_0(\tilde{z})^{-1}\frac{\partial r}{\partial z_2}(\tilde{z}) \text{ and} \\ e_l(\tilde{z}) &= -c_0(\tilde{z})^{-1}c_l(\tilde{z}) \quad l = 2, \ldots, m, \end{aligned} \tag{29}$$

and set $\rho_0 = r$. Then $\rho_1(\tilde{z}_1, \zeta'') = r(\tilde{z}_1, \zeta_2, \tilde{z}_3 + e_0\zeta_3)$ and

$$\rho_{i+1} = \rho_i\left(\tilde{z}_1, \zeta_2, \zeta_3 + e_i(\tilde{z})\zeta_2^i\right), \quad i = 1, 2, \ldots, m. \tag{30}$$

To study the dependence of $\Phi_{\tilde{z}}$ and hence dependence of $a_{j,k}(\tilde{z}_1)$ about $\tilde{z}_1$ in (23), we thus need to study the dependence of $e_l(\tilde{z})$ on $\tilde{z}$ variable. For a convenience, set $\tilde{z} = (z_1, 0, z_3)$, i.e., remove tilde's, and assume that $\tilde{z}$ satisfies

$$\begin{aligned} &\left|z_1 - d\delta^{1/\eta}\right| < \gamma\delta^{1/\eta}, \\ &\text{and } |z_3| \lesssim |z_1|^\eta, \end{aligned} \tag{31}$$

for a sufficiently small $\gamma > 0$ to be chosen. In view of (19), we see that $\tilde{z}^{\delta} = (d\delta^{1/\eta}, 0, e_{\delta})$ satisfies (31). In following we let z be the given coordinates, and we let ζ be the coordinates obtained from holomorphic coordinates changes of z, as in l-th step of coordinate changes in the proof of Proposition 11. Also we let D_k^s (resp., $\widetilde{D}_k^s$), $k = 1, 2, 3$, denote any partial derivative operator of order s with respect to z_k and $\overline{z}_k$ (resp., ζ_k and $\overline{\zeta}_k$) variables. According to the coordinate changes in Proposition 11, we note that $D_1 = \widetilde{D}_1$.

Proposition 12. *Assume that $\tilde{z} = (z_1, 0, z_3) \in V$ satisfies (31). Then for each $i = 0, 1, \ldots, m+1$, we have*

$$\left|D_1^{l_1}\widetilde{D}_2^k \rho_i(\tilde{z})\right| \lesssim |z_1|^{t_k - l_1}, \quad 0 \le k \le m, \ 0 \le l_1 \le \eta, \tag{32}$$

and for each $\alpha_2, \beta_2 > 0$ with $\alpha_2 + \beta_2 = q_\nu$, for some q_ν in (14), we have

$$\left|D_1^{l_1}\widetilde{D}_2^{q_\nu} \rho_i(\tilde{z})\right| \approx \left|D_1^{l_1} D_2^{q_\nu} r(\tilde{z})\right| \approx |z_1|^{p_\nu - l_1}, \qquad l_1 \le p_\nu, \ 2 \le i \le m+1. \tag{33}$$

Proof. We will prove by induction on i. From (14), (17), and (31) one obtains

$$\left|D_1^{l_1} D_2^k r(\tilde{z})\right| \lesssim |z_1|^{\eta - l_1} + |z_1|^{t_k - l_1} + |z_3| + |z_1|^{[t_k]+1-l_1} \lesssim |z_1|^{t_k - l_1}, \tag{34}$$

and hence (32) follows for $i = 0$. Since $\rho_1(z_1, \zeta'') = r(z_1, \zeta_2, z_3 + e_0\zeta_3)$, it follows, from (31) and chain rule, that

$$\left|D_1^{l_1}\widetilde{D}_2^k \rho_1(\tilde{z})\right| \lesssim \left|D_1^{l_1}\widetilde{D}_2^k r(\tilde{z})\right| + |z_3| \lesssim |z_1|^{t_k - l_1}, \tag{35}$$

because we are evaluating at $\tilde{\zeta} = (z_1, 0, 0)$. This proves (32) for $i = 1$.

By induction, assume that (32) holds for $i = 0, 1, \ldots, s$. For the $e_i(\tilde{z})$ defined in (29), it follows, from (34) and induction hypothesis, that

$$\left|D_1^{l_1} e_i(\tilde{z})\right| \lesssim \sum_{j=0}^{l_1} \left|D_1^j D_2^i \rho_i(\tilde{z})\right| \lesssim \sum_{j=0}^{l_1} |z_1|^{t_i - j} \lesssim |z_1|^{t_i - l_1}, \tag{36}$$

for $i = 1, \ldots, s$. Since we are evaluating at $\zeta_2 = 0$, it follows, for $s \ge 1$, that

$$\begin{aligned} D_1^{l_1}\widetilde{D}_2^k \rho_{s+1}(\tilde{z}) &= D_1^{l_1}\widetilde{D}_2^k \rho_s(\tilde{z}), \quad \text{if } k < s, \text{ and} \\ &= D_1^{l_1}\widetilde{D}_2^k \rho_s(\tilde{z}) + \mathcal{O}\left(\sum_{j=0}^{l_1} D_1^j e_s(\tilde{z})\right), \\ &\qquad \text{if } k \ge s. \end{aligned} \tag{37}$$

By (30), (36), and (37) and by induction, (32) holds for $i = s+1$ because $t_k \le t_s$ if $k \ge s$.

Now we prove (33). Assume $\alpha_2 + \beta_2 = q_\nu$ with $\alpha_2 > 0$, $\beta_2 > 0$ where (p_ν, q_ν) are the pairs corresponding to the second summation part of (14). Note that the first summation part of (14) will be annihilated by $D_2^{q_\nu}$ because it contains the pure terms of z_2 or $\overline{z}_2$ mixed with $\tilde{z}_1^k$. Thus it follows from (14), (16), and (31) that

$$\left|D_1^{l_1} D_2^{q_\nu} r(\tilde{z})\right| \approx \left|D_1^{l_1} M_{\alpha_2, \beta_2}^{\nu}(z_1)\right| + \mathcal{O}\left(|z_3| + |z_1|^{p_\nu + 1 - l_1}\right) \approx |z_1|^{p_\nu - l_1}. \tag{38}$$

Since $\rho_1(z_1, \zeta'') = r(z_1, \zeta_2, z_3 + e_0\zeta_3)$, it follows from (31) and (38) that

$$\left|D_1^{l_1}\widetilde{D}_2^{q_\nu} \rho_1(\tilde{z})\right| \approx \left|D_1^{l_1}\widetilde{D}_2^{q_\nu} r(\tilde{z})\right| + \mathcal{O}(|z_3|) \approx |z_1|^{p_\nu - l_1}. \tag{39}$$

Similarly, since $\rho_2(z_1, \zeta'') = \rho_1(z_1, \zeta_2, \zeta_3 + e_1\zeta_2)$, it follows from (36) that

$$\left|D_1^{l_1}\widetilde{D}_2^{q_\nu} \rho_2(\tilde{z})\right| \approx \left|D_1^{l_1}\widetilde{D}_2^{q_\nu} \rho_1(\tilde{z})\right| + \sum_{j=0}^{l_1} \left|D_1^j e_1\right| \approx |z_1|^{p_\nu - l_1}, \tag{40}$$

because $p_\nu = t_{q_\nu} < t_1$ for $q_\nu \ge 2$. This proves (33) for $i = 2$.

By induction assume that (33) holds for $i = 2, \ldots, s$. If $k = q_\nu = \alpha_\nu + \beta_\nu$ with $\alpha_\nu > 0$ and $\beta_\nu > 0$, that is, if $\widetilde{D}_2^k$ has mixed derivatives of $\partial/\partial\zeta_2$ and $\partial/\partial\overline{\zeta}_2$, we note that (37) becomes

$$\begin{aligned} D_1^{l_1}\widetilde{D}_2^k \rho_{s+1}(\tilde{z}) &= D_1^{l_1}\widetilde{D}_2^k \rho_s(\tilde{z}), \quad \text{if } k \le s, \text{ and} \\ &= D_1^{l_1}\widetilde{D}_2^k \rho_s(\tilde{z}) + \mathcal{O}\left(\sum_{j=0}^{l_1} D_1^j e_s(\tilde{z})\right), \\ &\qquad \text{if } k > s. \end{aligned} \tag{41}$$

If $k = q_\nu \le s$, (33) follows from (41) and induction hypothesis of (33). If $q_\nu > s$, it follows, from (36), (41), and induction hypothesis of (33), that

$$\left|D_1^{l_1}\widetilde{D}_2^{q_\nu} \rho_{s+1}(\tilde{z})\right| = \left|D_1^{l_1}\widetilde{D}_2^{q_\nu} \rho_s(\tilde{z})\right| + \mathcal{O}\left(\sum_{j=0}^{l_1} \left|D_1^j e_s(\tilde{z})\right|\right) \approx |z_1|^{p_\nu - l_1}, \tag{42}$$

because $t_s > p_\nu = t_{q_\nu}$ for $q_\nu > s$. Therefore (33) is proved for $i = s+1$. □

Recall the expression of $\rho = \rho_{m+1}$ and coefficient functions $a_{j,k}(\overline{z}_1)$ in (23).

Corollary 13. *Assume that $\tilde{z} = (z_1, 0, z_3)$ satisfies (31). Then*

$$\left|D_1^{l_1} a_{j,k}(z_1)\right| \lesssim |z_1|^{t_{j+k} - l_1}, \tag{43}$$

and if $j + k = q_\nu$ for some q_ν in (14), then

$$\left|D_1^{l_1} a_{j,k}(z_1)\right| \approx |z_1|^{p_\nu - l_1}. \tag{44}$$

Proof. From (23) we see that

$$D_1^{l_1} a_{j,k}(z_1) = D_1^{l_1} \widetilde{D}_2^{j+k} \rho(\widetilde{\zeta}), \tag{45}$$

where $\widetilde{\zeta} = (z_1, 0, 0)$ and $j, k > 0$. Hence it follows from (32) that

$$\left|D_1^{l_1} a_{j,k}(z_1)\right| = \left|D_1^{l_1} \widetilde{D}_2^{j+k} \rho(\widetilde{\zeta})\right| \lesssim |z_1|^{t_{j+k}-l_1}. \tag{46}$$

Assume $q_\nu = j + k \le m$ for some q_ν. Thus $j, k > 0$ and it follows from (23), (33), and (41) that

$$\begin{aligned} \left|D_1^{l_1} a_{j,k}(z_1)\right| &= \left|D_1^{l_1} \widetilde{D}_2^{q_\nu} \rho(\widetilde{\zeta})\right| = \left|D_1^{l_1} \widetilde{D}_2^{q_\nu} \rho_{q_\nu}(\widetilde{\zeta})\right| \\ &\approx |z_1|^{t_{q_\nu}-l_1} = |z_1|^{p_\nu - l_1}, \end{aligned} \tag{47}$$

because $t_{q_\nu} = p_\nu$. □

Remark 14. Suppose that $q_{\nu-1} < l \le q_\nu$ and $p_\nu \le t_l < p_{\nu-1}$. Then (p_ν, q_ν), (t_l, l), and $(p_{\nu-1}, q_{\nu-1})$ are colinear points. From the standard interpolation method, we have

$$a^{t_l} b^l \le a^{p_\nu} b^{q_\nu} + a^{p_{\nu-1}} b^{q_{\nu-1}}, \tag{48}$$

for all sufficiently small $a, b \ge 0$. Assume that $j, k > 0$ and $l = j + k \ne q_\nu$ for any of $\nu = 1, 2, \ldots, N$. Therefore it follows from (43) and (48) that

$$\begin{aligned} \left|a_{j,k}(z_1) \zeta_2^j \overline{\zeta}_2^k\right| &\lesssim |z_1|^{t_{j+k}} |\zeta_2|^{j+k} \\ &\le |z_1|^{p_\nu} |\zeta_2|^{q_\nu} + |z_1|^{p_{\nu-1}} |\zeta_2|^{q_{\nu-1}}. \end{aligned} \tag{49}$$

Therefore the terms of the form $a_{j,k}(z_1)\zeta_2^j\overline{\zeta}_2^k$, with $j+k = q_\nu$ for some q_ν, in the summation part in (23), are the major terms which bounds the other summation terms from above.

In the sequel, we assume that $\widetilde{z} = (z_1, 0, z_3)$ satisfies (31). As in Section 1 in [2], for each $\widetilde{z} = (z_1, 0, z_3)$, set

$$A_l(\widetilde{z}) = A_l(z_1) = \max\left\{\left|a_{j,k}(z_1)\right|; j + k = l\right\}, \quad l = 2, \ldots, m. \tag{50}$$

In view of Remark 14, we will consider $A_l(z_1)$ only for $l = q_\nu$, $0 \le \nu \le N - 1$. From (9) and (44) we note that

$$\begin{aligned} &\left|A_{q_0}(z_1)\right| = \left|A_m(z_1)\right| \approx 1, \text{ and} \\ &\left|A_{q_\nu}(z_1)\right| \approx |z_1|^{p_\nu}, \quad 1 \le \nu \le N - 1, \end{aligned} \tag{51}$$

because $q_0 = m$. For each sufficiently small $\delta > 0$, set

$$\begin{aligned} \tau(\widetilde{z}, \delta) &= \tau(z_1, \delta) \\ &= \min\left\{\left(\frac{\delta}{A_{q_\nu}(z_1)}\right)^{1/q_\nu}; 0 \le \nu \le N - 1\right\}, \end{aligned} \tag{52}$$

and set

$$\begin{aligned} T(\widetilde{z}, \delta) &= T(z_1, \delta) \\ &= \min\left\{q_\nu; \left(\frac{\delta}{A_{q_\nu}(z_1)}\right)^{1/q_\nu} = \tau(z_1, \delta)\right\}. \end{aligned} \tag{53}$$

From (51) and (52), we see that if $\delta' < \delta$, then

$$\left(\frac{\delta'}{\delta}\right)^{1/2} \tau(\widetilde{z}, \delta) \le \tau(\widetilde{z}, \delta') \le \left(\frac{\delta'}{\delta}\right)^{1/m} \tau(\widetilde{z}, \delta). \tag{54}$$

Lemma 15. *For each* $0 < \epsilon \le 1$, $T(\widetilde{z}, \epsilon\delta) \le T(\widetilde{z}, \delta)$.

Proof. Set $q_\nu^\epsilon = T(\widetilde{z}, \epsilon\delta)$ and $q_\nu = T(\widetilde{z}, \delta)$. Then

$$\begin{aligned} \tau(\widetilde{z}, \epsilon\delta) &= \left(\frac{\epsilon\delta}{A_{q_\nu^\epsilon}(\widetilde{z})}\right)^{1/q_\nu^\epsilon} = \epsilon^{1/q_\nu^\epsilon} \left(\frac{\delta}{A_{q_\nu^\epsilon}(\widetilde{z})}\right)^{1/q_\nu^\epsilon} \\ &\ge \epsilon^{1/q_\nu^\epsilon} \tau(\widetilde{z}, \delta) = \epsilon^{1/q_\nu^\epsilon} \left(\frac{\delta}{A_{q_\nu}(\widetilde{z})}\right)^{1/q_\nu} \\ &= \epsilon^{1/q_\nu^\epsilon - 1/q_\nu} \left(\frac{\epsilon\delta}{A_{q_\nu}(\widetilde{z})}\right)^{1/q_\nu} \\ &\ge \epsilon^{1/q_\nu^\epsilon - 1/q_\nu} \tau(\widetilde{z}, \epsilon\delta). \end{aligned} \tag{55}$$

Therefore $q_\nu^\epsilon \le q_\nu$ because $0 < \epsilon \le 1$. □

Proposition 16. *Assume* $\widetilde{z} = (z_1, 0, z_3)$ *satisfies (31). Then*

$$\tau(\widetilde{z}, \delta) \approx \tau(\widetilde{z}^\delta, \delta), \tag{56}$$

where $\widetilde{z}^\delta = (d\delta^{1/\eta}, 0, e_\delta)$.

Proof. By (31), we note that $|z_1| \approx \delta^{1/\eta}$. Assume that $T(\widetilde{z}^\delta, \delta) = q_\nu$. Then $A_{q_\nu}(\widetilde{z}^\delta) \approx \delta^{p_\nu/\eta}$ by (51). Therefore it follows, from (50) and (52), that

$$A_{q_\nu}(\widetilde{z}) \approx |z_1|^{p_\nu} \approx \delta^{p_\nu/\eta} \approx A_{q_\nu}(\widetilde{z}^\delta), \tag{57}$$

and hence it follows from (52) and (53) that

$$\tau(\widetilde{z}^\delta, \delta)^{q_\nu} = \frac{\delta}{A_{q_\nu}(\widetilde{z}^\delta)} \approx \frac{\delta}{A_{q_\nu}(\widetilde{z})} \ge \tau(\widetilde{z}, \delta)^{q_\nu}. \tag{58}$$

Thus $\tau(\widetilde{z}^\delta, \delta) \gtrsim \tau(\widetilde{z}, \delta)$ follows. Similarly, one can show that $\tau(\widetilde{z}^\delta, \delta) \lesssim \tau(\widetilde{z}, \delta)$. □

Let $0 < \sigma < 1$ be a small constant to be determined (in Remark 22). By Lemma 15, $T(\widetilde{z}^\delta, \sigma\delta) \le T(\widetilde{z}^\delta, \delta)$ for each $0 < \sigma < 1$, independent of $\delta > 0$. Therefore there is a smallest integer $s = s(\widetilde{z}^\delta)$, $0 \le s \le m - 1$, such that

$$T(\widetilde{z}^\delta, \sigma^{s+1}\delta) = T(\widetilde{z}^\delta, \sigma^s\delta) := t_s. \tag{59}$$

Then $t_s = q_{\nu(s)}$ for some $q_{\nu(s)}$ by (53). In following, for the fixed integer $s = s(\widetilde{z}^\delta)$ in (59), set $\delta_s = \sigma^s\delta$, $\tau_s := \tau(\widetilde{z}^\delta, \delta_s)$, and $\tau_1 = \delta^{1/\eta}$ as usual. If we define $\Phi_{\widetilde{z}}(\zeta) = (\zeta_1, \zeta_2, \Phi_3(\zeta''))$, where $\Phi_3(\zeta'')$ is defined in (22), we may regard that $\Phi_{\widetilde{z}}(\zeta) : \mathbb{C}^3 \longrightarrow$

$\mathbb{C}^3$ is a biholomorphism. For each $\tilde{z}^\delta = (d\delta^{1/\eta}, 0, e_\delta) \in b\Omega$, set $\tilde{\zeta}^\delta = (d\delta^{1/\eta}, 0, 0) = \Phi_{\tilde{z}^\delta}^{-1}(\tilde{z}^\delta)$. For each small $\gamma > 0$, define

$$R^s_{\gamma\delta}(\tilde{\zeta}^\delta) := \{\zeta \,|\zeta_1 - d\delta^{1/\eta}| < \gamma\tau_1,\ |\zeta_2| < \gamma\tau_s,\ |\zeta_3| < \gamma\delta_s\}, \quad \text{and,}$$
$$Q^s_{\gamma\delta}(\tilde{z}^\delta) := \{\Phi_{\tilde{z}^\delta}(\zeta); \zeta \in R^s_{\gamma\delta}(\tilde{\zeta}^\delta)\}, \tag{60}$$

and set $R^0_{\gamma\delta}(\tilde{\zeta}^\delta) = R_{\gamma\delta}(\tilde{\zeta}^\delta)$ and $Q^0_{\gamma\delta}(\tilde{z}^\delta) = Q_{\gamma\delta}(\tilde{z}^\delta)$ when $s = 0$.

Proposition 17. *The function $\rho = r \circ \Phi_{\tilde{z}^\delta}$ satisfies*

$$\left|D_1^{l_1}\rho(\tilde{\zeta}^\delta)\right| \lesssim \delta\tau_1^{-l_1}, \quad \text{and,}$$
$$\left|D_1^{l_1}\tilde{D}_2^k\rho(\tilde{\zeta}^\delta)\right| \lesssim \delta_s\tau_1^{-l_1}\tau_s^{-k}, \quad 1 \le k \le m. \tag{61}$$

Proof. Recall that $\rho = \rho_{m+1}$, and $|z_1| = \delta^{1/\eta}$ in (32). When $k = 0$, it follows from (12) ($t_0 = \eta$) and (32) that

$$\left|D^{l_1}\rho(\tilde{\zeta}^\delta)\right| \lesssim (\delta^{1/\eta})^{\eta - l_1} = \delta\tau_1^{-l_1}. \tag{62}$$

Assume $1 \le k \le m$. Then by (12), $q_{\nu-1} < k \le q_\nu$ for some ν, and hence it follows that $p_\nu \le t_k \le p_{\nu-1}$. Therefore one obtains, from (48)–(52), that

$$|z_1|^{t_k}\tau_s^k \lesssim |z_1|^{p_{\nu-1}}\tau_s^{q_{\nu-1}} + |z_1|^{p_\nu}\tau_s^{q_\nu} \lesssim A_{q_{\nu-1}}\tau_s^{q_{\nu-1}} + A_{q_\nu}\tau_s^{q_\nu} \lesssim \delta_s. \tag{63}$$

From (32) and (63), it follows that

$$\left|D_1^{l_1}\tilde{D}_2^k\rho(\tilde{\zeta}^\delta)\right| \lesssim |z_1|^{t_k - l_1} = (|z_1|^{t_k}\tau_s^k)|z_1|^{-l_1}\tau_s^{-k} \lesssim \delta_s\tau_1^{-l_1}\tau_s^{-k}. \tag{64}$$

□

Using the z coordinates defined in (14), set

$$L_3 = \frac{\partial}{\partial z_3} \quad \text{and}$$
$$L_k = \frac{\partial}{\partial z_k} - \left(\frac{\partial r}{\partial z_3}\right)^{-1}\frac{\partial r}{\partial z_k}\frac{\partial}{\partial z_3} := \frac{\partial}{\partial z_k} + b_k(z)\frac{\partial}{\partial z_3}, \quad k = 1, 2. \tag{65}$$

Then L_k, $k = 1, 2$, are tangential holomorphic vector fields and $|L_3 r| \ge c_0 > 0$ on $V \cap \Omega$ for a uniform constant $c_0 > 0$. For any j, k with $j, k > 0$, define

$$\mathscr{L}_{j,k}\partial\bar{\partial}r(z) = \underbrace{L_2 \ldots L_2}_{(j-1)\text{times}}\underbrace{\bar{L}_2 \ldots \bar{L}_2}_{(k-1)\text{times}}\partial\bar{\partial}r(L_2, \bar{L}_2)(z). \tag{66}$$

In ζ-coordinates defined by $z = (z_1, \Phi_{\tilde{z}}(\zeta'')) := \Phi_{\tilde{z}}(\zeta)$, set $L'_k = (d\Phi_{\tilde{z}}^{-1})L_k$, $k = 1, 2, 3$ and set $\tilde{b}_k(\zeta) = b_k(\Phi_{\tilde{z}}(\zeta))$, $k = 1, 2$. If we define

$$\mathscr{L}'_{j,k}\partial\bar{\partial}\rho(\zeta) = \underbrace{L'_2 \ldots L'_2}_{(j-1)\text{times}}\underbrace{\bar{L}'_2 \ldots \bar{L}'_2}_{(k-1)\text{times}}\partial\bar{\partial}\rho(L'_2, \bar{L}'_2)(\zeta), \tag{67}$$

then by functoriality,

$$\mathscr{L}_{j,k}\partial\bar{\partial}r(z) = \mathscr{L}'_{j,k}\partial\bar{\partial}\rho(\zeta). \tag{68}$$

Lemma 18. *There is a small constant $c_2 > 0$ such that*

$$\partial\bar{\partial}r(z)(L_1, \bar{L}_1) \ge c_2\delta\tau_1^{-2}, \quad z \in Q_{\gamma\delta}(\tilde{z}^\delta), \tag{69}$$

provided $\gamma > 0$ is sufficiently small.

Proof. Since the level sets of ρ are pseudoconvex, it follows from (61) that

$$\partial\bar{\partial}\rho(\zeta)(L'_1, \bar{L}'_1) = \left|\frac{\partial^2\rho}{\partial\zeta_1\partial\bar{\zeta}_1}(\zeta) + \mathcal{O}(\tilde{b}_1)\right| \ge \left|\frac{\partial^2\rho}{\partial\zeta_1\partial\bar{\zeta}_1}(\zeta)\right| - \tilde{C}_1\delta\tau_1^{-1}. \tag{70}$$

Recall that $d_0(z_1) = \sum_{\alpha_1+\beta_1=\eta} a_{\alpha_1,\beta_1} z_1^{\alpha_1}\bar{z}_1^{\beta_1}$ is the term which contains only z_1 or $\bar{z}_1$ variables in the first summation part of (14). Therefore it follows, from (17), (19), and (23), that

$$\left|\frac{\partial^2\rho}{\partial\zeta_1\partial\bar{\zeta}_1}(\tilde{\zeta}^\delta)\right| = \left|\frac{\partial^2 r(\tilde{z}^\delta)}{\partial z_1\partial\bar{z}_1}\right| = \left|\frac{\partial^2 d_0(z_1)}{\partial z_1\partial\bar{z}_1}\right| + \mathcal{O}(|e_\delta| + |z_1|^{\eta-1}) \approx |z_1|^{\eta-2} = \delta\tau_1^{-2}, \tag{71}$$

because $|z_1| = |d\delta^{1/\eta}| = \tau_1 = \delta^{1/\eta}$. If $\zeta \in R_{\gamma\delta}(\tilde{\zeta}^\delta)$, it follows from (61) and (71) and by using the Taylor series method that

$$\left|\frac{\partial^2\rho}{\partial\zeta_1\partial\bar{\zeta}_1}(\zeta)\right| \ge \left|\frac{\partial^2\rho}{\partial\zeta_1\partial\bar{\zeta}_1}(\tilde{\zeta}^\delta)\right| - \gamma\tilde{C}_1\delta\tau_1^{-2} \ge 2c_2\delta\tau_1^{-2}, \quad \zeta \in R_{\gamma\delta}(\tilde{\zeta}^\delta), \tag{72}$$

provided $\gamma > 0$ is sufficiently small. Thus (69) follows from (68), (70), and (72). □

In the sequel, we let c_2 and C_2 be the constants which may different from time to time but depend only on the derivatives of r or ρ up to order η. Recall that $\tilde{b}_k(\zeta) = b_k(\Phi_{\tilde{z}}(\zeta))$, $k = 1, 2$. By using (61), and by using Taylor series method, one obtains that

$$\left|D_1^{l_1}D_2^{l_2}\tilde{b}_k(\zeta)\right| \le C_2\delta_s\tau_1^{-l_1}\tau_s^{-l_2}\tilde{\tau}_k^{-1}, \quad \zeta \in R^s_{\gamma\delta}(\tilde{\zeta}^\delta), \tag{73}$$

provided $\gamma > 0$ is sufficiently small, where $\tilde{\tau}_1 = \tau_1$ and $\tilde{\tau}_2 = \tau_s$. Note that we can write

$$\partial\bar{\partial}\rho(\zeta)(L'_2, \bar{L}'_2) = \frac{\partial^2\rho}{\partial\zeta_2\partial\bar{\zeta}_2} + R_1, \tag{74}$$

where $R_1 = \mathcal{O}(\tilde{b}_2)$. By applying L'_2 or $\bar{L}'_2$ successively to $\partial\bar{\partial}\rho(\zeta)(L'_2, \bar{L}'_2)$, we obtain that

$$D_1^{l_1}\tilde{D}_2^{l_2}\mathscr{L}'_{j,k}\partial\bar{\partial}\rho(\zeta) = D_1^{l_1}\tilde{D}_2^{l_2}\frac{\partial^{j+k}\rho}{\partial\zeta_2^j\partial\bar{\zeta}_2^k} + D_1^{l_1}\tilde{D}_2^{l_2}R_{j+k-1}, \tag{75}$$

where, by (73) and by using induction method, R_{j+k-1} satisfies

$$\left|D_1^{l_1}\widetilde{D}_2^{l_2}R_{j+k-1}(\zeta)\right| \le C_2\delta_s\tau_1^{-l_1}\tau_s^{-l_2-j-k+1}, \quad \zeta \in R^s_{\gamma\delta}\left(\widetilde{\zeta}^\delta\right). \tag{76}$$

Combining the estimate in (61), (75), and (76), one obtains that

$$\left|\mathscr{L}'_{j,k}\partial\overline{\partial}\rho(\zeta)\right| \le C_2\delta_s\tau_s^{-j-k}, \quad \zeta \in R^s_{\gamma\delta}\left(\widetilde{\zeta}^\delta\right). \tag{77}$$

Assume that (59) holds. Thus $t_s = q_{\nu(s)}$ for some $q_{\nu(s)}$, and hence it follows from (53) that $A_{q_{\nu(s)}}(z_1) = \delta_s\tau_s^{-q_{\nu(s)}}$. Therefore it follows from (23) and (50) that there exist integers $j, k > 0$ with $j + k = t_s = q_{\nu(s)}$, such that

$$\left|\frac{\partial^{j+k}\rho}{\partial\zeta_2^j\partial\overline{\zeta}_2^k}\left(\widetilde{\zeta}^\delta\right)\right| = \left|a_{j,k}\left(\widetilde{z}^\delta\right)\right| = A_{q_{\nu(s)}}(z_1) = \delta_s\tau_s^{-q_{\nu(s)}} = \delta_s\tau_s^{-j-k}. \tag{78}$$

For these $j, k > 0$, it follows from (61), (75), (76), and (78) and by using the Taylor series method that there are constants $c_2, C_2 > 0$ such that

$$c_2\delta_s\tau_s^{-j-k} \le \left|\mathscr{L}_{j,k}\partial\overline{\partial}r(z)\right| \le C_2\delta_s\tau_s^{-j-k}, \quad z \in Q^s_{\gamma\delta}\left(\widetilde{z}^\delta\right), \tag{79}$$

provided $\gamma > 0$ is sufficiently small.

Lemma 19. *There is $C_2 > 0$ such that*

$$\left|\partial\overline{\partial}r(z)\left(L_1, \overline{L}_2\right)\right| \le C_2\gamma\delta_s\tau_1^{-1}\tau_s^{-1}, \quad z \in Q^s_{\gamma\delta}\left(\widetilde{z}^\delta\right). \tag{80}$$

Proof. By functoriality, we have

$$\partial\overline{\partial}r(z)\left(L_1, \overline{L}_2\right) = \partial\overline{\partial}\rho(\zeta)\left(L'_1, \overline{L}'_2\right) = \frac{\partial^2\rho}{\partial\zeta_1\partial\overline{\zeta}_2}(\zeta) + \mathscr{O}\left(\widetilde{b}_1(\zeta) + \widetilde{b}_2(\zeta)\right). \tag{81}$$

From (23), we see that

$$D_1\left(\frac{\partial^2\rho}{\partial\zeta_1\partial\overline{\zeta}_2}\right)\left(\widetilde{\zeta}^\delta\right) = \mathscr{O}\left(\left|e_\delta\right|\right) = \mathscr{O}(\delta) = \frac{\partial^2\rho}{\partial\zeta_1\partial\overline{\zeta}_2}\left(\widetilde{\zeta}^\delta\right), \tag{82}$$

and it follows from (61) that

$$\left|\widetilde{D}_2\frac{\partial^2\rho}{\partial\zeta_1\partial\overline{\zeta}_2}\left(\widetilde{\zeta}^\delta\right)\right| \lesssim \delta_s\tau_1^{-1}\tau_s^{-2}. \tag{83}$$

Therefore (80) follows from (73), (81), and (83) and by using Taylor series method. □

Note that $T(\widetilde{z}^\delta, \sigma^s\delta) := t_s = q_{\nu(s)}$, for some $q_{\nu(s)}$, and hence there exist $j > 0$, $k > 0$ with $j + k = t_s$. In view of (79), we may assume that

$$\left|L_2\left(\mathrm{Re}\,\mathscr{L}_{j-1,k}\partial\overline{\partial}r(z)\right)\right| \approx \delta_s\tau_s^{-t_s}, \quad z \in Q^s_{\gamma\delta}\left(\widetilde{z}^\delta\right), \tag{84}$$

is valid (when $j = 1$, we replace $\mathscr{L}_{j-1,k}$ by $\mathscr{L}_{j,k-1}$). Set

$$G(z) = \mathrm{Re}\mathscr{L}_{j-1,k}\partial\overline{\partial}r(z). \tag{85}$$

By using the estimates (73)–(76), one obtains that

$$\left|L_1G\right| \le \left|D_1\frac{\partial^{t_s-1}\rho}{\partial\zeta_2^{j-1}\partial\overline{\zeta}_2^k}\right| + \left|D_1R_{j+k-1}\right| + \left|b_1G\right| \le C_2\delta_s\tau_1^{-1}\tau_s^{-t_s+1}, \tag{86}$$

and similarly,

$$\left|\partial\overline{\partial}G\left(L_j, \overline{L}_k\right)(z)\right| \le C_2\delta_s\widetilde{\tau}_j^{-1}\widetilde{\tau}_k^{-1}\tau_s^{-t_s+1}, \quad j, k = 1, 2, \tag{87}$$

for $z \in Q^s_{\gamma\delta}(\widetilde{z}^\delta)$, where $\widetilde{\tau}_1 = \tau_1$ and $\widetilde{\tau}_2 = \tau_s$.

Lemma 20. *Assume that (59) holds. Then*

$$|G(z)| \le C_2\sigma^{1/t_s}\delta_s\tau_s^{-t_s+1}, \quad z \in Q^s_{\gamma\delta}\left(\widetilde{z}^\delta\right). \tag{88}$$

Proof. Suppose $z \in Q^s_{\gamma\delta}(\widetilde{z}^\delta)$. In view of (51)–(53), (56), and (59), we see that

$$\left(\frac{\sigma^{s+1}\delta}{A_{t_s-1}(z)}\right)^{1/(t_s-1)} \ge \tau\left(z, \sigma^{s+1}\delta\right) = \left(\frac{\sigma^{s+1}\delta}{A_{t_s}(z)}\right)^{1/t_s} = \sigma^{1/t_s}\tau\left(z, \sigma^s\delta\right) \approx \sigma^{1/t_s}\tau_s, \tag{89}$$

and hence it follows that

$$A_{t_s-1}(z) \lesssim \sigma^{1/t_s}\delta_s\tau_s^{-t_s+1}. \tag{90}$$

This together with (73)–(78) implies the estimate (88). □

In the sequel, we write

$$L = a_1L_1 + a_2L_2 + a_3L_3. \tag{91}$$

Lemma 21. *There is a positive number $\sigma > 0$, independent of $\widetilde{z}^\delta$ and δ, such that if $z \in Q^s_{\gamma\delta}(\widetilde{z}^\delta)$ and if (59) holds, then there are constants $c_2 > 0$ and $C_2 > 0$, independent of $\widetilde{z}$, δ and $\sigma > 0$, such that*

$$\partial\overline{\partial}G^2\left(L, \overline{L}\right)(z) \ge c_2\delta_s^2\tau_s^{-2t_s}\left|a_2\right|^2 - C_2\delta_s^2\tau_1^{-2}\tau_s^{-2t_s+2}\left|a_1\right|^2 - C_2\left|a_3\right|^2. \tag{92}$$

Proof. Suppose $z \in Q^s_{\gamma\delta}(\widetilde{z}^\delta)$. From (87) and (88), we note that

$$\left|G(z)\,\partial\overline{\partial}G(z)\left(L_j, \overline{L}_k\right)\right| \lesssim \sigma^{1/t_s}\delta_s^2\tau_s^{-2t_s+2}\widetilde{\tau}_j^{-1}\widetilde{\tau}_k^{-1}, \tag{93}$$

for $j,k=1,2$ where $\widetilde{\tau}_1=\tau_1$ and $\widetilde{\tau}_2=\tau_s$. Using (84)–(88) and (93) and by using small (large) constant method, one obtains that

$$\begin{aligned}\partial\bar{\partial}G(z)^2(L,\overline{L}) &= 2|LG(z)|^2+2G(z)\partial\bar{\partial}G(L,\overline{L})(z)\\ &\geq 2c_2\delta_s^2\tau_s^{-2t_s}|a_2|^2\\ &\quad+4\,\mathrm{Re}\left(\sum_{1\leq j<k\leq 3}(L_jG)(\overline{L}_kG)a_j\bar{a}_k\right)\\ &\quad+2\sum_{1\leq j\leq k\leq 3}G(z)\partial\bar{\partial}G(z)(L_j,\overline{L}_k)a_j\bar{a}_k\\ &\geq c_2\delta_s^2\tau_s^{-2t_s}|a_2|^2-C_2\delta_s^2\tau_1^{-2}\tau_s^{-2t_s+2}|a_1|^2-C_2|a_3|^2,\end{aligned}\tag{94}$$

for some $c_2>0$ and $C_2>0$ provided $\sigma>0$ is sufficiently small. □

Remark 22. From now on, we fix constants $c_2>0$ and $C_2>0$, which depend only on the derivatives of r or ρ of order up to η on V, satisfying (69), (73), (80), and (86)–(92), and set $C_2=c_2^{-1}$ for a convenience. Now we choose and fix $\gamma>0$ and then fix $\sigma>0$ so that

$$\begin{aligned}40C_2^2\gamma^{1/2}&\leq 1,\ \text{and}\\ 420C_2^4\gamma^{-7/2}\sigma^{2/m}&\leq\frac{1}{16}.\end{aligned}\tag{95}$$

3. Estimates on the Bergman Kernels

Recall that $\widetilde{z}^\delta=\pi(z(\delta))=(d\delta^{1/\eta},0,e_\delta)\in b\Omega$ where $z(\delta)=(d\delta^{1/\eta},0,-\delta)$ and where π is the projection defined before (19). Also note that $\Phi_{\widetilde{z}^\delta}(\widetilde{\zeta}^\delta)=\widetilde{z}^\delta$ where $\widetilde{\zeta}^\delta=(d\delta^{1/\eta},0,0)$ and where $\Phi_{\widetilde{z}^\delta}$ is the holomorphic coordinate function defined in Proposition 11 about $\widetilde{z}=\widetilde{z}^\delta$. Also recall $C_b(z_0,\delta_0)$ defined in (20). In this section we estimate the Bergman kernel function $K_\Omega(z,z^\delta)$, for $z\in\Omega$ and $z^\delta\in C_b(z_0,\delta_0)$.

To get optimal estimates of the Bergman kernel, we need to construct a plurisubharmonic function which has maximal Hessian near each thin neighborhood of $b\Omega$ as in [2, 15]. It contains complicated estimates depending on the type conditions of each boundary points. In this paper, however, we will construct such functions only at $\widetilde{z}^\delta\in b\Omega$. This will make the estimates much simpler than those in [2, 15] but still contain many complicated estimates.

Note that $\sigma>0$ and $\gamma>0$ are fixed in Remark 22 and hence the type t_s and the integer s defined in (59) depend only on $\widetilde{z}^\delta\in b\Omega$. Recall that $\delta_s=\sigma^s\delta$, $\tau_1=\delta^{1/\eta}$, $\tau_2=\tau(\widetilde{z}^\delta,\delta)$, and $\tau_s=\tau(\widetilde{z}^\delta,\delta_s)$. From (54) we have

$$\sigma^{s/2}\tau_2\leq\tau_s\leq\sigma^{s/m}\tau_2.\tag{96}$$

Let us write $L=a_1L_1+a_2L_2+a_3L_3$.

Proposition 23. *There exist a smooth plurisubharmonic function $g_{\widetilde{z}^\delta}$ on $\overline{\Omega}$ that satisfies the following:*

(i) $|g_{\widetilde{z}^\delta}(z)|\leq 1$, for $z\in\overline{\Omega}$, and $g_{\widetilde{z}^\delta}$ is supported in $Q^s_{\gamma\delta}(\widetilde{z}^\delta)\cap\overline{\Omega}$.

(ii) There exist a small constant $b>0$ such that if $z\in Q_{2b\delta}(\widetilde{z}^\delta)\cap\overline{\Omega}$, then

$$\partial\bar{\partial}g_{\widetilde{z}^\delta}(L,\overline{L})(z)\approx\tau_1^{-2}|a_1|^2+\tau_2^{-2}|a_2|^2+\delta^{-2}|a_3|^2.\tag{97}$$

(iii) If $\Phi_{\widetilde{z}^\delta}(\zeta)=(z_1,z_2,\Phi_3(\zeta))$ where Φ_3 is defined in (22), then

$$\left|\widetilde{D}^\alpha\left(g_{\widetilde{z}^\delta}\circ\Phi(\zeta)\right)\right|\leq C_\alpha\tau_1^{-\alpha_1}\tau_2^{-\alpha_2}\delta^{-\alpha_3}.\tag{98}$$

holds for all $\zeta\in R^s_{\gamma\delta}(\widetilde{\zeta}^\delta)$ where $\widetilde{D}^\alpha=\widetilde{D}_1^{\alpha_1}\widetilde{D}_2^{\alpha_2}\widetilde{D}_3^{\alpha_3}$.

Proof. For each fixed $\widetilde{z}^\delta$, we note that the integers $s=s(\widetilde{z}^\delta)$ and t_s, defined in (59), will be fixed. Set $\widetilde{\tau}_1=\tau_1$ and $\widetilde{\tau}_2=\tau_s$. Note that $\gamma^{-2}\sigma^{-4s}\widetilde{\tau}_i^2\leq 1$ provided $\delta>0$ is sufficiently small. Since $\delta_s=\sigma^s\delta$, it follows from (80) that

$$\begin{aligned}&\left|\partial\bar{\partial}r(z)(L_1,\overline{L}_2)a_1\bar{a}_2\right|\\ &\quad\leq C_2\gamma\delta\left(\tau_1^{-2}|a_1|^2+\sigma^{2s}\tau_s^{-2}|a_2|^2\right),\ \text{and}\\ &\left|\partial\bar{\partial}r(z)(L_i,\overline{L}_3)a_i\bar{a}_3\right|\\ &\quad\leq C_2\gamma\sigma^{2s}\delta\widetilde{\tau}_i^{-2}|a_i|^2+C_2\gamma^{-1}\sigma^{-2s}\delta^{-1}\widetilde{\tau}_i^2|a_3|^2\\ &\quad\leq C_2\gamma\sigma^{2s}\left(\delta\widetilde{\tau}_i^{-2}|a_i|^2+\delta^{-1}|a_3|^2\right),\quad i=1,2,\end{aligned}\tag{99}$$

for $z\in Q^s_{\gamma\delta}(\widetilde{z}^\delta)$. From now on, we fix $\lambda=420C_2^2\gamma^{-9/2}$ and set $\lambda_s=\sigma^{-2s}\lambda$.

We may assume that the level sets of r are pseudoconvex on V and $|L_3r|^2\geq c_0^2>0$ on $V\cap\Omega$, where we may assume that $c_0^2\geq 4c_2$. Also $4C_2\gamma^{1/2}\leq c_2/10$ by (95). Therefore it follows from (69) and (99) that

$$\begin{aligned}&\lambda_s\delta^{-1}\partial\bar{\partial}r(L,\overline{L})+(\lambda_s\delta^{-1})^2|Lr|^2\\ &\quad=\lambda_s\delta^{-1}\sum_{k=1}^{3}\partial\bar{\partial}r(L_k,\overline{L}_k)|a_k|^2+2\lambda_s\delta^{-1}\\ &\qquad\cdot\mathrm{Re}\sum_{1\leq j<k\leq 3}\partial\bar{\partial}r(L_j,\overline{L}_k)a_j\bar{a}_k+\lambda_s^2\delta^{-2}|a_3|^2|L_3r|^2\\ &\quad\geq\lambda_s\delta^{-1}\left[\frac{4c_2}{5}\delta\tau_1^{-2}|a_1|^2\right.\\ &\qquad\left.+\left(\partial\bar{\partial}r(z)(L_2,\overline{L}_2)-\frac{c_2}{10}\gamma^{1/2}\sigma^{2s}\delta\tau_s^{-2}\right)|a_2|^2\right]\\ &\qquad+3c_2\lambda_s^2\delta^{-2}|a_3|^2,\end{aligned}\tag{100}$$

for $z\in Q^s_{\gamma\delta}(\widetilde{z}^\delta)$.

Let $\psi(\zeta)$ be defined by

$$\psi(\zeta)=\chi\left(\tau_1^{-2}\left|\zeta_1-d\delta^{1/\eta}\right|^2+\tau_s^{-2}|\zeta_2|^2+\delta_s^{-2}|\zeta_3|^2\right),\tag{101}$$

where χ is a smooth function such that $\chi(t)=1$ for $t<\gamma^2/9$ and $\chi(t)=0$ for $t\geq\gamma^2$, satisfying $|D^k\chi|\leq C_k\gamma^{-2k}$. Set $\Psi(z)=$

$\psi((\Phi_{\tilde{z}^\delta})^{-1}(z))$. Note that $\Phi_{\tilde{z}^\delta}^{-1}(z)$ has similar expression as in (22). Thus it follows, from (22), (29), (30), and chain rule, that

$$\left|D^\alpha \Psi(z)\right| \le C_{|\alpha|}\gamma^{-2|\alpha|}\tau_1^{-\alpha_1}\tau_s^{-\alpha_2}\delta_s^{-\alpha_3} \quad z \in Q_{\gamma\delta}^s\left(\tilde{z}^\delta\right). \tag{102}$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. Since $C_2 = c_2^{-1}$, one obtains

$$\begin{aligned}
&\left|\partial\bar{\partial}\Psi(z)\left(L, \bar{L}\right)\right| \\
&\quad \le C_2\gamma^{-4}\left(\tau_1^{-2}\left|a_1\right|^2 + \tau_s^{-2}\left|a_2\right|^2 + \delta_s^{-2}\left|a_3\right|^2\right), \\
&\lambda_s\delta^{-1}\left|L_i\Psi(z)\right|\left|a_i\right|\left|a_3\right| \\
&\quad \le 10C_2\gamma^{-4}\tilde{\tau}_i^{-2}\left|a_i\right|^2 + \frac{c_2}{10}\lambda_s^2\delta^{-2}\left|a_3\right|^2, \quad i = 1, 2,
\end{aligned} \tag{103}$$

where $\tilde{\tau}_1 = \tau_1$ and $\tilde{\tau}_2 = \tau_s$.

Suppose that z satisfies $\Psi(z) \ge 1/4$. Using the fact that $L_k r = 0$, $k = 1, 2$, and the fact that $84C_2\gamma^{-9/2} = (c_2/5)\lambda \le (c_2/5)\lambda_s$, it follows from (100)–(103) that

$$\begin{aligned}
&\partial\bar{\partial}\left(\Psi e^{\lambda_s\delta^{-1}r}\right)\left(L, \bar{L}\right) = e^{\lambda_s\delta^{-1}r}\Bigg[\partial\bar{\partial}\Psi\left(L, \bar{L}\right) \\
&\quad + 2\lambda_s\delta^{-1}\sum_{i=1}^{3}\operatorname{Re}\left(\left(L_i\Psi\right)\left(\bar{L}_3 r\right)\right)a_i\bar{a}_3\Bigg] \\
&\quad + e^{\lambda_s\delta^{-1}r}\left[\lambda_s\delta^{-1}\Psi\partial\bar{\partial}r\left(L, \bar{L}\right) + \lambda_s^2\delta^{-2}\Psi\left|Lr\right|^2\right] \ge \frac{1}{4} \\
&\quad \cdot e^{\lambda_s\delta^{-1}r}\left[\frac{3c_2}{5}\lambda_s\tau_1^{-2}\left|a_1\right|^2 + c_2\lambda_s^2\delta^{-2}\left|a_3\right|^2\right] + \frac{1}{4} \\
&\quad \cdot e^{\lambda_s\delta^{-1}r}\left[\lambda_s\delta^{-1}\partial\bar{\partial}r(z)\left(L_2, \bar{L}_2\right) - \frac{2c_2}{5}\gamma^{1/2}\lambda\tau_s^{-2}\right] \\
&\quad \cdot \left|a_2\right|^2.
\end{aligned} \tag{104}$$

We note that the negative part in (104) contains $\gamma^{1/2}\lambda$ instead of $\gamma^{1/2}\lambda_s$.

Let h be a smooth convex function such that $h(t) = 0$ for $t \le 1/2$ and $h(t) > 0$ for $t > 1/2$ and $h(9/8) \le 1$. Set $G_{\tilde{z}^\delta}(z) = \Psi(z)e^{\lambda_s\delta^{-1}r(z)}$ and set $g_{\tilde{z}^\delta}(z) = h(G_{\tilde{z}^\delta}(z))$. Suppose $T(\tilde{z}^\delta, \delta) = 2$. Then $s = 0$, and hence (79) holds for $\delta_s = \delta$ with $j = k = 1$; that is,

$$c_2\delta\tau_2^{-2} \le \partial\bar{\partial}r(z)\left(L_2, \bar{L}_2\right) \le C_2\delta\tau_2^{-2}, \quad z \in Q_{\gamma\delta}\left(\tilde{z}^\delta\right). \tag{105}$$

For those z with $\Psi(z) \ge 1/4$, it follows from (104) (with $\lambda_s = \lambda$) and (105) that

$$\begin{aligned}
&\partial\bar{\partial}G_{\tilde{z}^\delta}(z) \\
&\quad \ge \frac{3c_2\lambda}{20}e^{\lambda\delta^{-1}r}\left[\tau_1^{-2}\left|a_1\right|^2 + \tau_2^{-2}\left|a_2\right|^2 + \delta^{-2}\left|a_3\right|^2\right].
\end{aligned} \tag{106}$$

If $\Psi(z) \le 1/4$, then $G_{\tilde{z}^\delta}(z) \le 1/4$ and hence $g_{\tilde{z}^\delta}(z) = 0$. Hence $g_{\tilde{z}^\delta}$ is a smooth plurisubharmonic function supported on $Q_{\gamma\delta}(\tilde{z}^\delta)$, and $|g_{\tilde{z}^\delta}| \le 1$.

Now assume $T(\tilde{z}^\delta, \delta) > 2$ and assume that (59) holds. Then (79) holds for some positive integers j, k with $j + k = t_s$. Let $G(z)$ be the function defined in (85). From (88) and (95), we see that

$$\begin{aligned}
\lambda\gamma^{1/2}\delta_s^{-2}\tau_s^{2t_s-2}G(z)^2 &\le \lambda\gamma^{1/2}C_2^2\sigma^{2/t_s} \\
&\le 420C_2^4\gamma^{-7/2}\sigma^{2/m} \le \frac{1}{16},
\end{aligned} \tag{107}$$

$z \in Q_{\gamma\delta}^s(\tilde{z}^\delta)$, because $\lambda = 420C_2^2\gamma^{-4}$ and $t_s \le m$. Set

$$\begin{aligned}
&g_{\tilde{z}^\delta}(z) \\
&\quad = h\left(\Psi(z)e^{\lambda_s\delta^{-1}r(z)} + \phi\left(\lambda\gamma^{1/2}\delta_s^{-2}\tau_s^{2t_s-2}G(z)^2\right)\right),
\end{aligned} \tag{108}$$

where $\phi(t)$ is a smooth function that satisfies $\phi(t) = t$, for $t \le 1/16$, $\phi(t) = 0$ for $t \ge 1$, and $\phi(t) \le 1/8$ for all t. Thus $g_{\tilde{z}^\delta} \in C_0^\infty(Q_{\gamma\delta}^s(\tilde{z}^\delta))$ and $|g_{\tilde{z}^\delta}| \le 1$ because $h(9/8) \le 1$. By (107) we note that $\phi(z) = z$ on $Q_{\gamma\delta}^s(\tilde{z}^\delta)$, and we also note that $g_{\tilde{z}^\delta} = 0$ if $\Psi(z)e^{\lambda_s\delta^{-1}r(z)} \le 3/8$, in particular, $g_{\tilde{z}^\delta} = 0$ outside $Q_{\gamma\delta}^s(\tilde{z}^\delta)$. From (92), we obtain that

$$\begin{aligned}
&\lambda\gamma^{1/2}\delta_s^{-2}\tau_s^{2t_s-2}\partial\bar{\partial}G(z)^2\left(L, \bar{L}\right) \\
&\quad \ge \gamma^{1/2}\left(c_2\lambda\tau_s^{-2}\left|a_2\right|^2 - C_2\lambda\tau_1^{-2}\left|a_1\right|^2 - C_2\lambda\delta_s^{-2}\left|a_3\right|^2\right) \\
&\quad \ge c_2\gamma^{1/2}\lambda\tau_s^{-2}\left|a_2\right|^2 - \frac{c_2}{40}\lambda\tau_1^{-2}\left|a_1\right|^2 - \frac{c_2}{40}\lambda\delta_s^{-2}\left|a_3\right|^2,
\end{aligned} \tag{109}$$

for $z \in Q_{\gamma\delta}^s(\tilde{z}^\delta)$, because $\gamma^{1/2}C_2 \le c_2/40$.

Assuming that $\Psi(z)e^{\lambda_s\delta^{-1}r(z)} \ge 3/8$, we note that the negative coefficient part of $|a_2|^2$ of the Hessian of $\Psi(z)e^{\lambda_s\delta^{-1}r(z)}$ in (104) is controlled by the first term in the third line of (109), and the error terms of the coefficients of $|a_1|^2$ and $|a_3|^2$ in the third line of (109) are controlled by the corresponding coefficients of the Hessian of $\Psi(z)e^{\lambda_s\delta^{-1}r(z)}$ in (104). In either $T(\tilde{z}^\delta, \delta) = 2$ or $T(\tilde{z}^\delta, \delta) > 2$ cases, it follows from (104), (106), and (109) that

$$\begin{aligned}
\partial\bar{\partial}g_{\tilde{z}^\delta}\left(L, \bar{L}\right) \ge \frac{c_2}{32}e^{\lambda_s\delta^{-1}r(z)}&\left(\lambda_s\tau_1^{-2}\left|a_1\right|^2\right. \\
&\left. + \gamma^{1/2}\lambda\tau_s^{-2}\left|a_2\right|^2 + \lambda_s^{-2}\delta^{-2}\left|a_3\right|^2\right),
\end{aligned} \tag{110}$$

for $z \in Q_{\gamma\delta}^s(\tilde{z}^\delta)$.

Note that parameters, c_2, C_2, γ, σ, and λ, are fixed in Remark 22, independent of $\delta > 0$. Therefore the upper bound of $g_{\tilde{z}^\delta}$ follows from (84)–(88), (96), (99), (102), and (103). Note that $e^{\lambda_s\delta^{-1}r(z)} > e^{-1/4} > 3/4$, if $r(z) > -\delta/4\lambda_s = -\delta\sigma^{2s}/4\lambda$, and this property holds on $Q_{2b\delta}(\tilde{z}^\delta)$ if we take $b > 0$ sufficiently small; say, $0 < 2b < \sigma^{2m}/\lambda^2$. Also note that $\Psi = 1$ on $Q_{2b\delta}(\tilde{z}^\delta)$. This fact together with (96) and (110) proves properties (i) and (ii). Property (iii) follows from (22), (30), (32), and (96). □

For each $z^\delta = (d\delta^{1/\eta}, 0, e_\delta - b\delta) \in C_b(z_0, \delta_0)$, set $\zeta^\delta := \Phi_{\tilde{z}^\delta}^{-1}(z^\delta) = (d\delta^{1/\eta}, 0, -b\delta)$.

Proposition 24. *There is a small constant $a > 0$ such that $R_{2a\delta}(\zeta^\delta) \subset\subset \Omega$ for all sufficiently small $\delta > 0$.*

Proof. From (22)–(29), we obtain that

$$\rho\left(\zeta^\delta\right) = r\left(z^\delta\right) = -b\delta + \mathcal{O}\left(\delta^{1+1/\eta}\right) < -\frac{b\delta}{2}, \quad (111)$$

for all sufficiently small $\delta > 0$. Assume $\zeta \in R_{2a\delta}(\zeta^\delta)$ and write

$$\begin{aligned}\rho(\zeta) &= \left[\rho(\zeta) - \rho\left(d\delta^{1/\eta}, \zeta_2, \zeta_3\right)\right] \\ &\quad + \left[\rho\left(d\delta^{1/\eta}, \zeta_2, \zeta_3\right) - \rho\left(\zeta^\delta\right)\right] + \rho\left(\zeta^\delta\right) \\ &:= E_1 + E_2 + \rho\left(\zeta^\delta\right).\end{aligned} \quad (112)$$

From (61), and by using Taylor series method, one obtains that

$$|E_1| \le a \max_{|\tilde{\zeta}_1 - d\delta^{1/\eta}| < 2a\delta^{1/\eta}} \left|D_1\rho\left(\tilde{\zeta}_1, \zeta_2, \zeta_3\right)\right| \delta^{1/\eta} \le 2aC_2\delta, \quad (113)$$

for a uniform constant $C_2 > 0$. Similarly, we obtain $|E_2| \le 4aC_2\delta$. Combining these estimates and (111) and if we set $a = b/24C_2$, then we obtain that

$$\rho(\zeta) < 6aC_2\delta - \frac{b\delta}{2} = -\frac{b\delta}{4}, \quad \zeta \in R_{a\delta}\left(\zeta^\delta\right). \quad (114)$$

□

Remark 25. (1) Set $\tilde{g}_\delta(\zeta) := g_{\tilde{z}^\delta} \circ \Phi_{\tilde{z}^\delta}(\zeta)$. Then, by functoriality, Proposition 23 holds, where $g_{\tilde{z}^\delta}$ is replaced by $\tilde{g}_\delta$, and $Q_{\gamma\delta}(\tilde{z}^\delta)$ is replaced by $R_{\gamma\delta}(\tilde{\zeta}^\delta)$.

For each fixed $\delta > 0$, and for each fixed $\tilde{z}^\delta = (d\delta^{1/\eta}, 0, e_\delta) \in b\Omega$, set $\Omega_{\tilde{z}^\delta} = \Phi^{-1}_{\tilde{z}^\delta}(\Omega)$. Note that $|\det(J_{\mathbb{C}}\Phi^{-1}_{\tilde{z}^\delta}(z))| = 2|(\partial r/\partial z_3)(\tilde{z}^\delta)| \ge 2c_0 > 0$ on V. Thus it follows, from transformation formula, that

$$K_\Omega\left(z, z^\delta\right) = 4\left|\frac{\partial r}{\partial z_3}\left(\tilde{z}^\delta\right)\right|^2 K_{\Omega_{\tilde{z}^\delta}}\left(\zeta, \zeta^\delta\right). \quad (115)$$

In view of Propositions 23 and 24, there is a smooth plurisubharmonic weight function $g_{\tilde{z}^\delta}$ which has maximal Hessian on $Q_{a\delta}(z^\delta) \subset\subset \Omega$. We also note that $\tau(\tilde{z}^\delta, \delta) \approx \tau(z^\delta, \delta)$ by (56). If we use these properties and (115), we get the following estimates for the Bergman kernel function $K_\Omega(z^\delta, z^\delta)$ at $z^\delta \in C_b(z_0, \delta_0)$ as in Theorem 6.1 in [2]:

$$K_\Omega\left(z^\delta, z^\delta\right) \approx \delta^{-2}\delta^{-2/\eta}\tau\left(z^\delta, \delta\right)^{-2}, \quad z^\delta \in C_b\left(z_0, \delta_0\right). \quad (116)$$

This proves Theorem 2.

Now we want to get derivative estimates of $K(z, z^\delta)$ for $z \in \Omega$ and $z^\delta \in C_b(z_0, \delta_0)$. In view of (115), we will estimate $K_{\Omega_{\tilde{z}^\delta}}(\zeta, \zeta^\delta)$ where $z = \Phi_{\tilde{z}^\delta}(\zeta)$ and $z^\delta = \Phi_{\tilde{z}^\delta}(\zeta^\delta)$. We will follow the methods in [3, 9] which use dilated coordinates. For each fixed $\delta > 0$, we recall that $\tau_1 = \delta^{1/\eta}$, $\tau_2 = \tau(z^\delta, \delta)$ and $\tau_3 = \delta$. Define a dilation map D_δ given by

$$\begin{aligned}D_\delta(\zeta) &= \left(\frac{\zeta_1 - d\delta^{1/\eta}}{a\tau_1}, \frac{\zeta_2}{a\tau_2}, \frac{\zeta_3 + b\delta}{a\tau_3}\right) := (w_1, w_2, w_3) \\ &= w,\end{aligned} \quad (117)$$

set

$$\begin{aligned}\rho_\delta(w) &:= \delta^{-1}\left(\rho \circ D_\delta^{-1}(w)\right), \\ \Omega_\delta &= \left\{w \in \mathbb{C}^3; \rho_\delta(w) < 0\right\},\end{aligned} \quad (118)$$

and set

$$\lambda_\delta(w) := \tilde{g}_\delta \circ D_\delta^{-1}(w), \quad (119)$$

where $\tilde{g}_\delta(\zeta) := g_{\tilde{z}^\delta} \circ \Phi_{\tilde{z}^\delta}(\zeta)$ and where $g_{\tilde{z}^\delta}$ is defined in Proposition 23 .

Set

$$\begin{aligned}L_3^\delta &= \frac{\partial}{\partial w_3}, \\ L_k^\delta &= \frac{\partial}{\partial w_k} - \left(\frac{\partial\rho_\delta}{\partial w_3}\right)^{-1}\frac{\partial\rho_\delta}{\partial w_k}\frac{\partial}{\partial w_3}, \\ &\qquad k = 1, 2,\end{aligned} \quad (120)$$

and write $L^\delta = b_1L_1^\delta + b_2L_2^\delta + b_3L_3^\delta$. The properties of $\lambda_\delta(w)$, which follow from Propositions 23 and 24 and Remark 25, are summarized in the following proposition.

Proposition 26. *For each $\delta > 0$ there is $\lambda_\delta(w)$, defined on Ω_δ, such that*

(1) $\lambda_\delta(w)$ is smooth plurisubharmonic in Ω_δ, and $|\lambda_\delta| \le 1$;
(2) supp *$\lambda_\delta(w) \subset P(0, \tilde{C})$, for some $\tilde{C} = 2a^{-1}\gamma > 1$;*
(3) $\partial\bar{\partial}\lambda_\delta(L^\delta, \overline{L}^\delta)(w) \approx |b_1|^2 + |b_2|^2 + |b_3|^2$ if $w \in P(0, 1)$;
(4) $|D_w^\alpha\lambda_\delta(w)| \le C_\alpha$.

The weight function with the properties in Proposition 26 is the key ingredient for the derivative estimates of the Bergman kernel function off the diagonal. Set $P = P(0, \tilde{C})$ and let N_δ be the Neumann operator on Ω_δ. Then we have the following L^2 estimates of N_δ (Proposition 3.14 in [3]).

Proposition 27. *Let $h \in L^2$ be a $(0, 1)$ form and* supp *$h \subset P$. Then there is $C > 0$, independent of $\delta > 0$, so that*

$$\int_{\Omega_\delta \cap P} \left|N_\delta h\right|^2 \le C\|h\|^2. \quad (121)$$

Note that $D_\delta(\zeta^\delta) = 0$. Set

$$P(0, r) := \left\{w = (w_1, w_2, w_3) : \left|w_k\right| \le r, \ k = 1, 2, 3\right\}. \quad (122)$$

From (117) and Proposition 24, we note that

$$D_\delta\left(R_{a\delta}\left(\zeta^\delta\right)\right) = P(0, 1) \subset\subset P(0, 2) \subset\subset \Omega_\delta, \quad (123)$$

independent of $\delta > 0$. Let $\xi_1, \xi_2 \in C_0^\infty(P(0,1))$ with $\xi_1 = 1$ in a neighborhood of 0 and $\xi_2 = 1$ on $\operatorname{supp}\xi_1$. From (123), we see that $\operatorname{supp}\xi_2 \subset P(0,1) \subset\subset P(0,2) \subset\subset \Omega_\delta$, independent of $\delta > 0$. Therefore we have the following elliptic estimates:

$$\|\xi_1 f\|_{s+2}^2 \le C_s\left(\|\xi_2 \square_\delta f\|_s^2 + \|f\|^2\right), \quad s \ge 0,\ f \in \operatorname{Dom}\left(\overline{\partial}\right) \cap \operatorname{Dom}\left(\overline{\partial}^*\right), \tag{124}$$

where $\square_\delta$ is the complex Laplacian on Ω_δ.

Remark 28. The estimates in (124) are on the polydisc $P(0,1) \subset\subset P(0,2) \subset\subset \Omega_\delta$, strictly inside of Ω_δ, independent of $\delta > 0$. Therefore we gain two derivatives in (124) and it is stable; that is, C_s is independent of $\delta > 0$. Also we note that we do not require that $\Delta_1(z_0) < \infty$. Since $P(0,2) \subset\subset P = P(0,\widetilde{C})$ where $\widetilde{C} = 2a^{-1}\gamma > 2$, we can also apply the estimate (121) on $P(0,2)$.

Let $\phi \in C_0^\infty(P(0,1))$, $\int \phi = 1$, and ϕ be polyradial. In terms of w-coordinates in (117), we have the following well known representation of Bergman kernel function on Ω_δ.

$$K_{\Omega_\delta}(w,0) = \phi(w) - \overline{\partial}_* N_\delta \overline{\partial}\phi(w). \tag{125}$$

Let $\chi \in C^\infty(\Omega_\delta)$ with $\chi = 1$ outside $P(0,1)$ and $\chi = 0$ on $\operatorname{supp}\phi$. Combining (121)–(125), we can prove the following lemma as in the proof of Theorem 4.2 in [3].

Lemma 29. *For each $s \ge 0$ there is $C_s > 0$ such that*

$$\left\|\chi N_\delta \overline{\partial}\phi\right\|_s \le C_s. \tag{126}$$

Now, if we use the estimate (126) with $s = |\alpha| + 3$, we can prove Theorem 3 as in the proof of Theorem 4.2 in [3].

Appendix

We recall Herbort's example Ω_H in (6). Therefore $\eta = 6 = \Delta_1(0)$ and hence $\tau_1 = \delta^{1/6}$, $\tau_2 = \delta^{1/3}$, and $\tau_3 = \delta$ in our notations. For each fixed $\delta > 0$, set $z^\delta = (\delta^{1/6}/2, 0, -\delta)$. Then $z^\delta \in \Omega_H$ and approaches to $0 \in b\Omega_H$ in "almost tangential direction" as the points do along $C_b(z_0, \delta_0)$. In this case, we will show that

$$K_{\Omega_H}\left(z^\delta, z^\delta\right) \approx \delta^{-3} = \delta^{-2}\tau_1^{-2}\tau_2^{-2}, \tag{A.1}$$

which is exactly same result as Theorem 2.

Proof of (A.1). Set

$$P_\delta\left(z^\delta\right) := \left\{z : \left|z_1 - \frac{\delta^{1/6}}{2}\right| < \frac{\delta^{1/6}}{10},\ |z_2| < \frac{\delta^{1/3}}{10},\ |z_3 + \delta| < \frac{\delta}{10}\right\}. \tag{A.2}$$

Then the polydisc $P_\delta(z^\delta)$ about z^δ is contained in Ω_H. Therefore the upper bound $K_{\Omega_H}(z^\delta, z^\delta) \lesssim \delta^{-3}$ follows. Let us show lower bounds.

Let Δ_3 be the unit polydisc in $\mathbb{C}^3$. Since the localization lemma is valid for Ω_H, we will estimate $K_{\Omega_H\cap\Delta_3}(z^\delta, z^\delta)$. Set

$$\begin{aligned} P_1(z_1,z_2) &= |z_1|^6 + |z_1|^2|z_2|^2 + |z_2|^6, \text{ and} \\ P_2(z_1,z_2) &= |z_1|^6 + |z_1|^2|z_2|^2, \end{aligned} \tag{A.3}$$

and set

$$\begin{aligned} G_1 &= \{\operatorname{Re} z_3 + P_1(z_1,z_2) < 0\}, \text{ and} \\ G_2 &= \{\operatorname{Re} z_3 + P_2(z_1,z_2) < 0\}. \end{aligned} \tag{A.4}$$

Then $\Omega_H = G_1 \subset G_2$ and $z^\delta \in G_1 \cap G_2$. Set $f_\delta = 8\delta^{11/6} z_1/(z_3 - \delta)^2$. Then $f_\delta(z^\delta) = 1$. Note that

$$\begin{aligned} \|f_\delta\|_{L^2(G_1\cap\Delta_3)}^2 &\le \|f_\delta\|_{L^2(G_2\cap\Delta_3)}^2 \\ &= 16\delta^{11/3}\int_{|z_1|,|z_2|<1} |z_1|^2 \cdot \left[\int_{\operatorname{Re} z_3 < -P_2} |z_3 - \delta|^{-4}\, dV_2(z_3)\right] dV_4 \\ &\lesssim \delta^{11/3}\int_{|z_1|,|z_2|<1} |z_1|^2 (\delta + P_2)^{-2}\, dV_4 := (*), \end{aligned} \tag{A.5}$$

where we have used $\int_{\mathbb{R}}(ds/(1+s^2)^2) = c_1 < \infty$. Set $z_1 = \delta^{1/6}z_1'$ and $z_2 = \delta^{1/3}z_2'$. Then

$$\begin{aligned} (*) &\lesssim \delta^3 \int_{|z_2'|<\delta^{-1/3}}\int_{|z_1'|<\delta^{-1/6}} |z_1'|^2 \left(1 + P_2(z_1', z_2')\right)^{-2} dV_4(z_1', z_2') \\ &\lesssim \delta^3 \int_0^{\delta^{-1/3}}\int_0^{\delta^{-1/6}} r_1^3 r_2 \left(1 + r_1^2 r_2^2 + r_1^6\right)^{-2} dr_1\, dr_2 := (**) \end{aligned} \tag{A.6}$$

where we have used the polar coordinates: $z_k' = r_k\sigma^{i\theta}, k = 1,2$ in the second line.

Set $r_1^4 = x$ and $r_2^2 = y$. Then

$$\begin{aligned} (**) &\lesssim \delta^3 \int_0^{\delta^{-2/3}}\int_0^{\delta^{-2/3}} \left(1 + x^{1/2}y + x^{3/2}\right)^{-2} dy\, dx \\ &= \delta^{7/3}\int_0^{\delta^{-2/3}} \frac{dx}{(1+x^{3/2})(1+\delta^{-2/3}x^{1/2} + x^{3/2})} \\ &:= \delta^{7/3} I(\delta). \end{aligned} \tag{A.7}$$

Write $I(\delta) = \int_0^1 + \int_1^{\delta^{-2/3}} := I_1(\delta) + I_2(\delta)$. Set $\delta^{-2/3}x^{1/2} = x'$. Then

$$I_1(\delta) \lesssim \int_0^{\delta^{-2/3}} \frac{\delta^{4/3}x'\,dx'}{1+x'} \lesssim \delta^{4/3}\int_0^{\delta^{-2/3}} dx' = \delta^{2/3}. \tag{A.8}$$

Also,

$$I_2(\delta) \lesssim \int_1^{\delta^{-2/3}} \frac{dx}{x^{3/2}\left(x^{3/2}+\delta^{-2/3}x^{1/2}\right)} \\ \le \delta^{2/3}\int_1^{\delta^{-2/3}} x^{-2}dx \le \delta^{2/3}. \tag{A.9}$$

Combining (A.5)–(A.9), we obtain that $\|f_\delta\|^2_{L^2(\Omega_H)} \lesssim \delta^3$. Therefore $K_{\Omega_H}(z^\delta, z^\delta) \gtrsim \delta^{-3}$. □

Remark 30. Set $f(z) = \exp(z_3/(1-z_3))$. Then f is a peak function that peaks at $0 \in b\Omega_H$ for the domain Ω_H.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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