

Research Article

Stability for Linear Volterra Difference Equations in Banach Spaces

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This paper is devoted to studying the existence and stability of implicit Volterra difference equations in Banach spaces. The proofs of our results are carried out by using an appropriate extension of the freezing method to Volterra difference equations in Banach spaces. Besides, sharp explicit stability conditions are derived.

1. Introduction

In this paper, we study the existence and stability of solutions for a class of abstract functional difference equations described in the form

$$x(k) = \sum_{j=0}^k K(k, k-j)x(j) + f(k), \quad k \geq 1, \quad (1)$$

in a Banach space $(X, \|\cdot\|)$, $f \in l_{\infty}(Z^+, X)$, the space of bounded sequences equipped with the norm $\|\cdot\|$ on l_{∞} , and $K(k, j)$ is a function defined on $0 \leq j \leq k < \infty$, whose values are bounded operators in X . In addition, for any fixed integer $\tau \geq 0$, $K(\tau, \cdot)$ is summable and bounded on Z^+ , the set of nonnegative integers.

A solution of (1) is a sequence defined on Z^+ and satisfying (1) for all finite $k > 0$. The study of existence and stability of solutions for implicit Volterra difference equations of nonconvolution type, defined in abstract spaces, is a complicated problem. However, with appropriate conditions on $f(\cdot)$ and $K(k, \cdot)$, one can use the freezing method for abstract Volterra difference equations, so the difficulty is overcome.

Existence and uniqueness problems for the Volterra difference equations were discussed by some authors (e.g., see [1, 2]). Existence and stability of Volterra difference equations have been studied by many authors (Federson et al. [3], Murakami and Nagabuchi [4], Györi and Horvath [5],

Mingarelli [1], Gonzalez et al. [6], Kolmanovskii et al. [7], and Song and Baker [8]).

The main technique in the theory of stability and boundedness of Volterra difference equations is the direct Lyapunov method and its variants. In contrast, many alternative methods to Lyapunov's function have been successfully applied to the stability analysis of Volterra difference equations; for example, in Federson et al. [3], the Kurzweil-Henstok integral formalism is applied to establish the existence of solutions to integral equations of Volterra type. In Murakami and Nagabuchi [4], sufficient stability properties and the asymptotic almost periodicity for linear Volterra difference equations in Banach spaces are derived. Gonzalez et al. [6] considered an implicit Volterra difference equation in a Hilbert space and obtained sufficient conditions so that the solutions exist and have a bounded behavior. The coefficients of the considered equations are sequences of real numbers. In Mingarelli [1], Volterra-Stieltjes integral equations are studied, which can be considered as generalized Volterra difference equations. In Banás and Sadarangani [2], a class of operator-integral equations of Volterra-Stieltjes type which create a generalization of numerous integral equations appearing in mathematical literature is studied. In Györi and Horvath [5], sufficient conditions are presented under which the solutions to a linear nonconvolution Volterra difference equation converge to limits, which are given by a limit formula. In Kolmanovskii et al. [7], stability and boundedness problems of some classes of scalar Volterra

nonlinear difference equations are investigated. Their stability conditions are formulated in terms of the characteristic equations. In Song and Baker [8], the fixed point theory is used to establish sufficient conditions to ensure the stability of the zero solution of an implicit nonlinear Volterra difference equation. However, in the above-mentioned articles, Volterra equations with convolution kernels are mainly considered.

In this paper, formulating the Volterra discrete equations in the phase space $L_p(Z^+, X)$, where X is an appropriate Hilbert space, and assuming that the kernel operator is completely continuous, we obtain sufficient conditions for the existence and uniqueness problem. The suggested approach is based on the “freezing” method to abstract difference equations (Medina and Gil’ [9]), as well as on the concept of analytical pencils (analytic operator-valued functions of a complex argument). See, for example, [10–13]. In Medina [14], a class of nonlinear discrete-time Volterra equations in Banach spaces is considered. Using a linearization method, sufficient conditions of existence and boundedness are established. In fact, assuming that the kernels are Causal Operators, the existence and boundedness of solutions are derived. Consequently, the methodology and the corresponding results obtained in [14] are absolutely different compared with the results of this article.

Consider an X -valued Volterra-Stieltjes equation of the form

$$x(t) = \int_0^t K(t, s, x(s)) d\mu(s) + f(t), \quad t \geq 0, \quad (2)$$

where $K : [0, \infty) \times [0, \infty) \times X \rightarrow X$ and $f : L_{\mu, \infty} \rightarrow L_{\mu, \infty}$. A solution of this equation is a function $x \in L_{\mu, \infty}$, which is locally μ -integrable in the Riemann-Stieltjes sense.

If $\mu(t) = k - 1, k - 1 < t \leq k, \text{ for } k = 1, 2, \dots$, then we can restrict our attention to functions $x \in L_{\mu, \infty}$ which are piecewise constants with $x(t) = x_k$ for $k - 1 < t \leq k, k = 1, 2, \dots$. We can identify such a function x with a sequence (x_1, x_2, \dots) in the space ℓ_∞ . In this case, the Volterra-Stieltjes original equation is equivalent to a Volterra difference equation

$$x(j) = \sum_{k=1}^j K(j, k, x(k)) + f(j), \quad j = 2, 3, \dots \quad (3)$$

Consequently, [1, 2, 15, 16] are highly appropriate to our research.

Remark 1. We want to point out that the freezing method was introduced by V. M. Alekseev for linear ordinary differential equations (see Bylov et al. [17]) and extended to difference systems by Gil’ and Medina [18].

Our aim in this paper is to make new contributions to the development of the theory of existence and qualitative properties of solutions for the nonconvolution Volterra difference equations described by Volterra operators in Banach spaces.

The remainder of this article is organized as follows: In Section 2, we establish a preliminary result to a class of convolution Volterra difference equations which will be fundamental to formulating the corresponding nonconvolution

problem in Banach spaces. In Section 3, sufficient conditions on the existence and stability of solutions of nonconvolution Volterra difference equations are established. In Section 4, we illustrate the main result studying an interesting problem. Finally, Section 5 is devoted to the discussion of our results.

2. Preliminary Results

To prove our main results, it is convenient to establish some known qualitative results for a class of convolution Volterra difference equations (see [7, 8, 11, 19–25]).

Let $K(i), i = 0, 1, 2, \dots$, be bounded linear operators in a Banach space X with norm $\| \cdot \|$.

Consider the convolution Volterra difference equation

$$x(k) = \sum_{j=1}^k K(k-j)x(j) + h(k); \quad k = 1, 2, \dots \quad (4)$$

where $K(0) = 0, h = \{h(i)\}_{i=1}^\infty$, and $h(i) \in X$ is a given sequence.

Assume that $\overline{\lim_{k \rightarrow \infty} \sqrt[k]{\|K(k)\|}} < \infty$ and $\overline{\lim_{k \rightarrow \infty} \sqrt[k]{\|h(k)\|}} < \infty$.

To solve (4), put

$$T(z) = \sum_{j=1}^\infty K(j)z^j, \quad (5)$$

$$f(z) = \sum_{j=1}^\infty h(j)z^j, \quad (z \in \mathbb{C}).$$

Consider the equation

$$y(z) = T(z)y(z) + f(z). \quad (6)$$

In a neighborhood ω of zero, let $I - T(z)$ be boundedly invertible. Then

$$y(z) = (I - T(z))^{-1} f(z), \quad (z \in \omega). \quad (7)$$

Hence it follows that $y(z)$ is infinitely many times differentiable at zero.

Differentiating (6) j times, we get

$$y^{(j)}(z) = \sum_{i=0}^j C_j^i T^{(j-i)}(z) y^{(i)}(z) + f^{(j)}(z). \quad (8)$$

Since $K(i) = T^{(i)}(0)/i!$, substituting $z = 0$ into the later equality, we obtain the following relations:

$$b(j) = \sum_{i=0}^j K(j-i)b(i) + h(j); \quad (9)$$

$$(K(0) = 0; j = 1, 2, \dots),$$

where $b(j) = y^{(j)}(0)/j!$. Therefore, we arrive at (7). Hence, the sequence $x(k) = b(k)$ is a solution to (7). According to (7), we obtain

$$\begin{aligned} x(j) &= \frac{1}{j!} \left. \frac{d^j y(z)}{dz^j} \right|_{z=0} \\ &= \frac{1}{j!} \left. \frac{d^j}{dz^j} (1 - T(z))^{-1} f(z) \right|_{z=0}. \end{aligned} \tag{10}$$

Thanks to the Cauchy formula

$$\begin{aligned} x(j) &= \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z^{j+1}} (1 - T(z))^{-1} f(z) dz, \\ &\quad (j = 1, 2, \dots), \end{aligned} \tag{11}$$

where γ is a smooth contour surrounding zero, provided that $I - T(z)$ is boundedly invertible and f is regular inside γ and on γ . Thus, the next result can be established.

Theorem 2 (see [19, 25]). *Inside γ and on γ , let $I - T(z)$ be boundedly invertible and f be regular. Then a solution of (4) is given by formula (11).*

Remarks 3. Theorem 2 will play a fundamental role to establish the existence and stability of the solution of non-convolution equations of kind (1). In doing so, we will use the freezing method.

Definition 4 (see [7, 8, 22–24]). We will say that (1) is stable if, for any $f \in l_{\infty}(Z^+, X)$, a solution x of (1) satisfies the inequality

$$\|x\|_{l_{\infty}} \leq c_0 \|f\|_{l_{\infty}}, \tag{12}$$

where the constant c_0 does not depend on f .

Let H be a separable Hilbert space and A a linear compact operator in H . If $\{e_k\}_{k=1}^{\infty}$ is an orthogonal basis in H and the series $\sum_{k=1}^{\infty} (Ae_k, e_k)$ converges, then the sum of the series is called the trace of the operator A and is denoted by

$$\text{Trace}(A) = \text{Tr}(A) = \sum_{k=1}^{\infty} (Ae_k, e_k). \tag{13}$$

Definition 5 (see [20, 26]). An operator A satisfying the relation $\text{Tr}(A^*A) < \infty$ is said to be a *Hilbert-Schmidt operator*, where A^* is the adjoint operator of A .

The norm

$$N_2(A) = N(A) = \sqrt{\text{Tr}(A^*A)} \tag{14}$$

is called the *Hilbert-Schmidt norm* of A .

Definition 6 (see [20, 26]). A bounded linear operator A is said to be *quasi-Hermitian* if its imaginary component

$$A_I = \frac{A - A^*}{2i} \tag{15}$$

is a Hilbert-Schmidt operator, where A^* is the adjoint operator of A .

Theorem 7 (see [17, 19, 26]). *Let V be a Hilbert-Schmidt completely continuous quasinilpotent (Volterra) operator acting in a separable Hilbert space H . Then the inequality*

$$\|V^k\| \leq \frac{N_p^k(V)}{\sqrt{k!}}, \quad \text{for any natural } k \tag{16}$$

is true.

3. Main Results

Now, we are in a position to establish sufficient conditions on the existence and stability of solutions of (1).

Assume that, for any fixed integer $\tau \geq 0$, $K(\tau, \cdot)$ is l_1 -summable and bounded on Z^+ . In addition, assume that there exists a nonnegative constant q such that

$$\begin{aligned} \sum_{j=0}^k \|K(k, j) - K(\tau, j)\| &\leq q|k - \tau|, \\ (q = \text{const}; k, \tau \geq 0). \end{aligned} \tag{17}$$

Under (17), the function $K(\tau, j)$, for a fixed integer τ , admits the Z -transform $\tilde{K}_{\tau}(z) = \sum_{j=0}^{\infty} z^{-j} K(\tau, j)$, $|z| \geq \rho$, where ρ is the radius of convergence of $\tilde{K}_{\tau}(z)$. Besides, it is assumed that the operator $W_{\tau}(z) = 1 - \tilde{K}_{\tau}(z)$ is boundedly invertible for all z in a neighborhood ω of zero.

Introduce the Green function

$$G_{\tau}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-iyk} W_{\tau}^{-1}(e^{iy}) dy. \tag{18}$$

Theorem 8. *Under assumption (17), let*

$$q \sum_{k=0}^{\infty} k \sup_{\tau \geq 0} \|G_{\tau}(k)\| < 1. \tag{19}$$

Then (1) is stable. Moreover, constant c_0 in (12) is explicitly pointed below.

Proof. Consider the convolution equation

$$x(k) = \sum_{j=0}^k K(\tau, k-j) x(j) + f(k), \quad k \geq 1, \tag{20}$$

with a fixed integer $\tau \geq 0$.

The solution of (20) may be written as

$$x(k) = \sum_{j=1}^k G_{\tau}(k-j) f(j), \tag{21}$$

whereby the discrete Green's function $G(k, j)$ is defined.

Now, rewrite (20) in the form

$$\begin{aligned} x(k) - \sum_{j=1}^k K(\tau, k-j) x(j) &= f_0(k, \tau) + f(k), \\ (k \geq 0), \end{aligned} \tag{22}$$

where

$$f_0(k, \tau) = \sum_{j=1}^k [K(k, k-j) - K(\tau, k-j)] x(j). \quad (23)$$

Hence, according to (21),

$$x(k) = \sum_{j=1}^k G_\tau(k-j) [f(j) + f_0(j, k)], \quad (24)$$

$$= F(k) + \sum_{j=1}^k G_\tau(k-j) f_0(j, k), \quad (25)$$

where $F(k) = \sum_{j=1}^k G_\tau(k-j) f(j)$.

Denote

$$M(k) = \sup_{\tau \geq 0} \|G_\tau(k)\|. \quad (26)$$

Hence, we get

$$\|F\|_{l_\infty} \leq \|f\|_{l_\infty} \sup_k \left(\sum_{j=1}^k M(k-j) \right) \leq \|M\|_{l_\infty} \|f\|_{l_\infty}. \quad (27)$$

Due to (22),

$$\begin{aligned} \|f_0(k, \tau)\| &\leq \sum_{j=1}^k \|K(k, k-j) - K(\tau, k-j)\| \|x(j)\| \\ &\leq q \|x\|_{l_\infty} |k - \tau|. \end{aligned} \quad (28)$$

By (24), we have

$$\|x(k)\| \leq \|M\|_{l_\infty} \|f\|_{l_\infty} + q \|x\|_{l_\infty} \sum_{j=1}^k M(k-j) |j - \tau|. \quad (29)$$

Take $k = \tau$; then

$$\begin{aligned} \|x(\tau)\| &\leq \|M\|_{l_\infty} \|f\|_{l_\infty} \\ &\quad + q \sup_{1 \leq j \leq \tau} \|x(j)\| \sum_{j=1}^{\tau} (\tau - j) M(\tau - j). \end{aligned} \quad (30)$$

Hence

$$\|x(\tau)\| \leq \|M\|_{l_\infty} \|f\|_{l_\infty} + q \sup_{1 \leq j \leq \tau} \|x(j)\| \sum_{j=1}^{\infty} jM(j). \quad (31)$$

Therefore, for any $k_0 > 0$,

$$\sup_{\tau \leq k_0} \|x(\tau)\| \leq \|M\|_{l_\infty} \|f\|_{l_\infty} + \lambda \sup_{\tau \leq k_0} \|x(\tau)\|, \quad (32)$$

where $\lambda = q \sum_{j=1}^{\infty} jM(j)$.

On the other hand, condition $q \sum_{j=1}^{\infty} jM(j) < 1$ implies

$$\sup_{1 \leq k \leq k_0} \|x(k)\| \leq \frac{\|M\|_{l_\infty} \|f\|_{l_\infty}}{1 - \lambda}. \quad (33)$$

Since the right hand side does not depend on k_0 , condition (12) follows with $c_0 = \|M\|_{l_\infty} / (1 - \lambda)$.

The existence of solutions is due to the convergence of the Neumann series

$$x = \sum_{j=0}^{\infty} V^j f, \quad (34)$$

where

$$(Vx)(k) = \sum_{j=1}^k K(k, j) x(j), \quad (35)$$

provided that $\sum_{k=1}^{\infty} \|K(k, \tau)\| < \infty$ for any fixed integer $\tau \geq 0$. In fact, (1) is rewritten in the operator form

$$x = Vx + f \quad (36)$$

Hence

$$x = (I - V)^{-1} f = \sum_{j=0}^{\infty} V^j f. \quad (37)$$

This yields

$$\|x\|_{l_\infty} \leq \sum_{j=0}^{\infty} \|V^j\|_{l_\infty} \|f\|_{l_\infty}. \quad (38)$$

Since V is a quasinilpotent Hilbert-Schmidt operator, it follows by [19, 25] that

$$\|V^j\|_{l_\infty} \leq \frac{N_\infty^p(V)}{\sqrt[j]{k!}} < \infty. \quad (39)$$

Consequently, the Neumann series $\sum_{j=0}^{\infty} V^j f$ is convergent. \square

Remark 9. The stability theory of Volterra difference equations has been considered, for example, by Song and Baker [8], Mingarelli [1], and Gonzalez et al. [6]. However, the “freezing” method has not been used previously to study qualitative properties of Volterra difference systems in Banach spaces. Consequently, the theoretical contributions of this paper are significantly new.

4. Example

To illustrate the main result, consider in X the equation

$$x(k) - A(k) \sum_{j=0}^k e^{-(k-j)h} x(j) = f(k); \quad (40)$$

$$h = \text{const.} > 0,$$

where $A(k)$ is a variable bounded operator in X satisfying

$$\|A(k) - A(\tau)\| \leq q_0 |k - \tau|; \quad (k, \tau \geq 0). \quad (41)$$

Take $K(k, j) = A(k)e^{-jh}$. Then, by (41),

$$\begin{aligned} \sum_{j=0}^k \|K(k, j) - K(\tau, j)\| &\leq \|A(k) - A(\tau)\| \sum_{j=0}^k e^{-jh} \\ &\leq \frac{q_0}{e^h} |k - \tau|. \end{aligned} \tag{42}$$

We also have

$$\begin{aligned} \tilde{K}_\tau(z) &= \sum_{j=0}^{\infty} \frac{K(\tau, j)}{z^j} = A(\tau) \sum_{j=0}^{\infty} \frac{(e^{-h})^j}{z^j} \\ &= A(\tau) \frac{z}{z - e^{-h}}, \quad |z| > \frac{1}{e^h}. \end{aligned} \tag{43}$$

This relation yields

$$W_\tau(z) = I - \frac{A(\tau)z}{z - e^{-h}}. \tag{44}$$

On the other hand,

$$\begin{aligned} kG_\tau(k) &= k \left(\frac{1}{2\pi i} \int_\gamma z^{k-1} W_\tau^{-1}(z) dz \right) \\ &= \frac{1}{2\pi i} \int_\gamma z^{k-1} T_\tau(z) dz, \end{aligned} \tag{45}$$

where

$$T_\tau(z) = -\frac{d}{dz} (W_\tau^{-1}(z)). \tag{46}$$

If $|z| < b < 1$ and letting $T_\tau(z)$ be regular and

$$\lambda_b = \sup_{\tau \geq 0} \frac{1}{2\pi} \int_0^{2\pi} \|T_\tau(e^{iy-b})\| dy < \infty, \tag{47}$$

then

$$\|kG_\tau(k)\| \leq b^{k-1} \frac{1}{2\pi} \int_0^{2\pi} \|T_\tau(e^{iy-b})\| dy \leq b^{k-1} \lambda_b. \tag{48}$$

Consequently,

$$\sum_{k=0}^{\infty} k \sup_{\tau \geq 0} \|G_\tau(k)\| \leq \frac{\lambda_b}{b(1-b)}. \tag{49}$$

Theorem 10. Under conditions (42), for a positive b and all z , with $|z| < b < 1$, let $T_\tau(z)$ be regular and $\lambda_b/b(1-b) < 1$. Then (40) is stable.

For example, if $A(t) = \sin(t)B$, where B is a constant operator, then condition (41) holds with

$$q(t) = 2 \|B\| \sin\left(\frac{t}{2}\right), \tag{50}$$

since

$$\begin{aligned} \sin \alpha - \sin \beta &= 2 \sin\left(\frac{\alpha - \beta}{2}\right) \cos\left(\frac{\alpha + \beta}{2}\right), \\ &(\alpha, \beta \in \mathbb{R}). \end{aligned} \tag{51}$$

Taking $K(k, j) = \sin(t)Be^{-jh}$. Then, by (41),

$$\begin{aligned} \sum_{j=0}^k \|K(k, j) - K(\tau, j)\| &\leq \|A(k) - A(\tau)\| \sum_{j=0}^k e^{-jh} \\ &\leq \frac{2 \|B\|}{e^h} |k - \tau|. \end{aligned} \tag{52}$$

Thus, every statement of Theorem 10 can be easily verified.

5. Concluding Remarks

The stability problem for Volterra difference equations of nonconvolution type in an infinite dimensional Hilbert space is more complicated than that for equations in R^n (a finite dimensional Euclidean space). However, with appropriate conditions on $f(\cdot)$ and $K(j, \cdot)$, one can use the freezing method for abstract difference equations, so the difficulty is overcome. In fact, considering the time j as a parameter, we obtain an infinite family of convolution Volterra difference equations. Thus, using the freezing method, we deduce the qualitative properties corresponding to the nonconvolution Volterra difference equations and to the convolution original equation. On the other hand, the study of existence of solutions of this kind of implicit Volterra difference equations is a complicated problem. Our proof of the existence of solutions is carried out using the convergence of Neumann series of quasinilpotent Hilbert-Schmidt operators [27].

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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