Hindawi
Abstract and Applied Analysis
Volume 2018, Article ID 2590158, 6 pages
http://dx.doi.org/10.1155/2018/2590158

Research Article

Some Oscillation Results for Even Order Delay Difference Equations with a Sublinear Neutral Term

Govindasamy Ayyappan and Gunasekaran Nithyakala

Department of Mathematics, Periyar University College of Arts and Science, Pappireddipatti 636 905, Tamilnadu, India

Correspondence should be addressed to Govindasamy Ayyappan; ayyapmath@gmail.com

Received 5 September 2018; Accepted 18 October 2018; Published 8 November 2018

Academic Editor: Agacik Zafer

In this paper, some new results are obtained for the even order neutral delay difference equation $\Delta(a_n\Delta^{m-1}(x_n+p_nx^{\alpha}_{n-k}))+q_nx^{\beta}_{n-\ell}=0$, where $m \geq 2$ is an even integer, which ensure that all solutions of the studied equation are oscillatory. Our results extend, include, and correct some of the existing results. Examples are provided to illustrate the importance of the main results.

1. Introduction

The aim of this paper is to investigate the oscillatory behavior of even order nonlinear neutral difference equation

$$\Delta\left(a_n\Delta^{m-1}\left(x_n+p_nx^{\alpha}_{n-k}\right)\right)+q_nx^{\beta}_{n-\ell}=0, \quad n\in\mathbb{N}\left(n_0\right) \tag{1}$$

where $\mathbb{N}(n_0)=\{n_0,n_0+1,\ldots\}$, n_0 is a positive integer, subject to the following conditions:

(C_1) $m \geq 2$ is an even integer, and α and β are ratio of odd positive integers with $0<\alpha\leq 1$ and $\beta\in(0,\infty)$;

(C_2) $\{a_n\}$ is a positive increasing sequence of real number for all $n\in\mathbb{N}(n_0)$;

(C_3) $\{p_n\}$ and $\{q_n\}$ are positive real sequences for all $n\in\mathbb{N}(n_0)$ with $0\leq p_n\leq p<1$;

(C_4) ℓ and k are positive integers.

Let $\theta=\max\{k,\ell\}$. Under a solution of (1), we mean a real sequence $\{x_n\}$ defined for $n\geq n_0-\theta$ and satisfying (1) for all $n\in\mathbb{N}(n_0)$. As usual a solution of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; else it is nonoscillatory.

In the past few years, there is a great interest in studying the oscillatory and asymptotic behavior of solutions of higher order neutral type difference equations, since such type of equations naturally arises in the applications including problems in population dynamics or in cobweb models in economics and so on. The problem of finding sufficient conditions which ensure that all solutions of the neutral type difference equations are oscillatory has been investigated by many authors; see, for example, [1–12] and the references cited therein. In all the results the neutral term is linear and few results are available when the neutral term is nonlinear; see [13–21].

In [20], the authors considered (1) with $\alpha\geq 1$ and $a_n\equiv 1$ and established sufficient conditions for the oscillation of all solutions. In view of these facts, in this paper our purpose is to obtain sufficient conditions for the oscillation of solution of (1) when

$$\sum_{n=n_0}^{\infty}\frac{1}{a_n}=\infty, \tag{2}$$

or

$$\sum_{n=n_0}^{\infty}\frac{1}{a_n}<\infty. \tag{3}$$

Thus the results presented here extend and generalize some of the results in [13, 14, 16, 18, 19, 21], complement the results in [20], and correct some of the results in [8].

2. Some Preliminary Lemmas

In this section, we provide some lemmas which will be useful in proving our main results. We begin with the following lemma that can be found in [22, Theorem 41, page 39].

Lemma 1. *If* $0 < \alpha \le 1$ *and* $a > 0$, *then* $a^\alpha \le \alpha a + (1-\alpha)$.

Lemma 2 (Discrete Kneser's Theorem). *Let* $\{u_n\}$ *be a sequence of real number and* $u_n > 0$ *with* $\Delta^m u_n$ *being of constant sign eventually and not identically zero eventually. Then there exists an integer* j, $0 \le j \le m$, *with* $(m+j)$ *odd for* $\Delta^m u_n \le 0$ *and* $(m+j)$ *even for* $\Delta^m u_n \ge 0$ *and* $N \in \mathbb{N}(n_0)$ *such that*

$$\Delta^i u_n > 0 \quad \text{for all } i = 1, 2, \dots, j \tag{4}$$

and

$$(-1)^{i+j}\Delta^i u_n > 0 \quad \text{for all } i = j+1, j+2, \dots, m-1 \tag{5}$$

for all $n \ge N$.

Lemma 3. *Let* $\{u_n\}$ *be defined for* $n \in \mathbb{N}(n_0)$ *and* $u_n > 0$ *with* $\Delta^m u_n \le 0$ *for all* $n \in \mathbb{N}(n_0)$ *and not identically zero. Then there exists a large* $n_1 \in \mathbb{N}(n_0)$ *such that*

$$u_n \ge \frac{(n-n_1)^{m-1}}{(m-1)!}\Delta^{m-1}u_{2^{m-j-1}n}, \quad n \ge n_1 \tag{6}$$

where j *is defined in Lemma 2. Further, if* $\{u_n\}$ *is increasing, then*

$$u_n \ge \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1}u_n, \quad n \ge 2^{m-1}n_1. \tag{7}$$

The proof of the last two lemmas can be found in [1].

Next we define the sequence $\{z_n\}$ by

$$z_n = x_n + p_n x_{n-k}^\alpha. \tag{8}$$

Lemma 4. *Assume condition (2) holds. Let* $\{x_n\}$ *be a positive solution of (1). Then there is an integer* $n_1 \in \mathbb{N}(n_0)$ *such that*

$$\begin{gathered} z_n > 0, \\ \Delta z_n > 0, \\ \Delta^{m-1} z_n > 0, \\ \Delta^m z_n \le 0 \end{gathered} \tag{9}$$

for all $n \ge n_1$.

Proof. The proof is similar to that of Lemma 3 of [8], and hence the details are omitted. □

3. Oscillation Theorems

In this section, we present some sufficient conditions for the oscillation of all solutions of (1). To simplify our notation, for any positive real sequence $\{\rho_n\}$ which is decreasing to zero, we set

$$\begin{aligned} P(n) &= \left(1 - \alpha p_n - (1-\alpha)\frac{p_n}{\rho_n}\right), \\ Q(n) &= q_n P^\beta (n-\ell), \end{aligned} \tag{10}$$

and

$$A_n = \sum_{s=n}^{\infty}\frac{1}{a_s}. \tag{11}$$

Theorem 5. *Let condition (2) hold. Assume that there is a positive decreasing real sequence* $\{\rho_n\}$ *tending to zero such that* $P(n)$ *is positive for all* $n \ge N \in \mathbb{N}(n_0)$. *If*

$$\sum_{n=N}^{\infty} Q_n = \infty, \tag{12}$$

then every solution of (1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that $\{x_n\}$ is a positive solution of (1). Then there exists an integer $n_1 \in \mathbb{N}(n_0)$ such that $x_n > 0, x_{n-k} > 0$ and $x_{n-\ell} > 0$ for all $n \ge n_1$. From Lemma 4, we have $z_n > 0$, $\Delta z_n > 0$, $\Delta^{m-1}z_n > 0$ and $\Delta^m z_n \le 0$ for all $n \ge n_1$.

From the definition of z_n, we have

$$\begin{aligned} x_n = z_n - p_n x_{n-k}^\alpha &\ge z_n - p_n z_n^\alpha \\ &\ge z_n - \alpha p_n z_n - (1-\alpha) p_n \\ &= (1-\alpha p_n) z_n - (1-\alpha) p_n, \end{aligned} \tag{13}$$

where we have used Lemma 1. Since z_n is positive and increasing and ρ_n is positive and decreasing to zero, there is an integer $n_2 \ge n_1$ such that

$$z_n \ge \rho_n \quad \text{for all } n \ge n_2. \tag{14}$$

Using (14) in (13), one obtains

$$x_n \ge \left(1 - \alpha p_n - \frac{1}{\rho_n}(1-\alpha) p_n\right) z_n = P(n) z_n \tag{15}$$

and substituting this in (1) yields

$$\Delta\left(a_n \Delta^{m-1} z_n\right) + q_n P^\beta (n-\ell) z_{n-\ell}^\beta \le 0, \quad n \ge n_2. \tag{16}$$

Now summing the last inequality from n_2 to $n-1$, we obtain

$$a_n \Delta^{m-1} z_n - a_{n_2}\Delta^{m-1} z_{n_2} + \sum_{s=n_2}^{n-1} q_s P^\beta (s-\ell) z_{s-\ell}^\beta \le 0 \tag{17}$$

for all $n \ge n_2$. That is

$$\sum_{s=n_2}^{n-1} Q_s z_{s-\ell}^\beta \le a_{n_2}\Delta^{m-1} z_{n_2} - a_n \Delta^{m-1} z_n, \quad n \ge n_2. \tag{18}$$

Since $\Delta z_n > 0$ and $z_n > 0$ eventually, there exists a positive constant M such that $z_{n-\ell} \geq M$ for all $n \geq n_2$. Using this and the positivity of $a_n\Delta^{m-1}z_n$ in (18) and letting $n \longrightarrow \infty$, we obtain

$$\sum_{n=n_1}^{\infty} Q_n < \infty \tag{19}$$

which is a contradiction to (12). This completes the proof. □

Remark 6. In the above theorem, we did not impose any condition on β and hence our result is more general than some of the existing results in the literature.

In the following, we present other oscillation criteria using Lemma 3.

Theorem 7. *Let condition (2) hold. Assume that there is a positive decreasing real sequence $\{\rho_n\}$ tending to zero such that $P(n)$ is positive for all $n \geq N \in \mathbb{N}(n_0)$. If*

(i)

$$\lim_{n\longrightarrow\infty} \inf \sum_{s=n-\ell}^{n-1} \frac{Q_s}{a_{s-\ell}} (s-\ell)^{m-1} > \frac{1}{\lambda}\left(\frac{\ell}{\ell+1}\right)^{\ell+1} \tag{20}$$

for $\beta = 1$,

(ii)

$$\sum_{n=N}^{\infty} \frac{Q_n}{a_{n-\ell}^{\beta}} (n-\ell)^{\beta(m-1)} = \infty \tag{21}$$

for $0 < \beta < 1$,

(iii) *there exists a $\delta > (1/\ell)\log\beta$ such that*

$$\lim_{n\longrightarrow\infty} \inf \left[\frac{Q_n}{a_{n-\ell}^{\beta}} (n-\ell)^{\beta(m-1)} \exp\left(e^{-\delta n}\right)\right] > 0 \tag{22}$$

for $\beta > 1$,

then every solution of (1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that there is an integer $n_1 \in \mathbb{N}(n_0)$ such that $x_n > 0, x_{n-k} > 0$ and $x_{n-\ell} > 0$ for all $n \geq n_1$. Now proceeding as in the proof of the previous theorem, we obtain (16). That is,

$$\Delta\left(a_n\Delta^{m-1}z_n\right) + Q_n z_{n-\ell}^{\beta} \leq 0, \quad n \geq n_2. \tag{23}$$

Since $\Delta^{m-1}z_n > 0$, $\Delta^m z_n \leq 0$ and using Lemma 3, we have from (23) that

$$\begin{aligned}&\Delta\left(a_n\Delta^{m-1}z_n\right)\\ &\quad + Q_n\left(\frac{1}{(m-1)!}\left(\frac{n-\ell}{2^{m-1}}\right)^{m-1}\right)^{\beta}\left(\Delta^{m-1}z_{n-\ell}\right)^{\beta}\\ &\quad \leq 0, \quad n \geq n_2.\end{aligned} \tag{24}$$

Set $w_n = a_n\Delta^{m-1}z_n$. Then $w_n > 0$ and the last inequality becomes

$$\Delta w_n + \frac{\lambda Q_n}{a_{n-\ell}^{\beta}} (n-\ell)^{\beta(m-1)} w_{n-\ell}^{\beta} \leq 0, \quad n \geq n_2, \tag{25}$$

where $\lambda = ((1/(m-1)!)(1/2^{m-1})^{m-1})^{\beta} > 0$. Now, using Lemma 1.1 of [20], we see that the equation

$$\Delta w_n + \frac{\lambda Q_n}{a_{n-\ell}^{\beta}} (n-\ell)^{\beta(m-1)} w_{n-\ell}^{\beta} = 0, \quad n \geq n_2 \tag{26}$$

has an eventually positive solution.

(i) If (20) holds, then by Theorem 7.6.1 of [23], (26) with $\beta = 1$ has no positive solution, which is a contradiction.

(ii) If (21) holds, then by Theorem 1 of [24], (26) with $0 < \beta < 1$ has no positive solution, which is a contradiction.

(iii) If (22) holds, then by Theorem 2 of [24], (26) with $\beta > 1$ has no positive solution which is a contradiction. This completes the proof of the theorem. □

Theorem 8. *Assume that (3) and $\beta = 1$ hold. Assume that there is a positive decreasing real sequence $\{\rho_n\}$ tending to zero such that $P(n)$ is positive for all $n \geq N \in \mathbb{N}(n_0)$. If (20) holds and*

$$\begin{aligned}&\lim_{n\longrightarrow\infty} \sup \sum_{s=N}^{n}\left(MQ_sA_{s+1}(s-\ell)^{m-2} - \frac{1}{4a_sA_{s+1}}\right)\\ &\quad = \infty\end{aligned} \tag{27}$$

where $M = (1/(m-2)!)(1/2^{m-2})^{m-2}$, then every solution of (1) either is oscillatory or tends to zero as $n \longrightarrow \infty$.

Proof. Assume that (1) has a nonoscillatory solution $\{x_n\}$ which is eventually positive such that $\lim_{n\longrightarrow\infty} x_n \neq 0$. From the definition of z_n, we have $z_n > 0$ for all $n \geq n_1 \in \mathbb{N}(n_0)$. By virtue of (1) and Lemma 2 there are two possibilities, either

$$\begin{aligned}z_n &> 0,\\ \Delta z_n &> 0,\\ \Delta^{m-1}z_n &> 0,\\ \Delta^m z_n &\leq 0,\\ \Delta\left(a_n\Delta^{m-1}z_n\right) &\leq 0\end{aligned} \tag{28}$$

or

$$\begin{aligned}z_n &> 0,\\ \Delta z_n &> 0,\\ \Delta^{m-2}z_n &> 0,\\ \Delta^{m-1}z_n &< 0,\\ \Delta\left(a_n\Delta^{m-1}z_n\right) &\leq 0\end{aligned} \tag{29}$$

for all $n \geq n_1 \geq n_0$.

Case (i). Suppose conditions (28) hold for all $n \geq n_1$; then the proof for this case is similar to that of Case (i) of Theorem 7 and hence the details are omitted.

Case (ii). Assume now that conditions (29) hold for all $n \geq n_1$. Since $a_n\Delta^{m-1}z_n$ is decreasing, then we have

$$a_j\Delta^{m-1}z_j \leq a_n\Delta^{m-1}z_n \quad \text{for } j \geq n \geq n_1. \tag{30}$$

Dividing the last inequality by a_j and summing the resulting inequality from n to $j-1$, we obtain

$$\Delta^{m-2}z_j - \Delta^{m-2}z_n \leq a_n\Delta^{m-1}z_n\sum_{s=n}^{j-1}\frac{1}{a_s}. \tag{31}$$

Letting $j \longrightarrow \infty$, we obtain

$$0 \leq \Delta^{m-2}z_n + A_n a_n\Delta^{m-1}z_n \quad \text{for } n \geq n_1. \tag{32}$$

Define

$$w_n = A_n\left(\frac{1}{A_n} + \frac{a_n\Delta^{m-1}z_n}{\Delta^{m-2}z_n}\right), \quad n \geq n_1 \tag{33}$$

and then $w_n > 0$, and using (16), we have

$$\begin{aligned}\Delta w_n &= -\frac{1}{a_nA_n}w_n + A_{n+1}\left(\frac{1}{a_nA_nA_{n+1}}\right.\\ &\quad\left.+\frac{\Delta\left(a_n\Delta^{m-1}z_n\right)}{\Delta^{m-2}z_{n+1}} - \frac{a_n\Delta^{m-1}z_n}{\Delta^{m-2}z_n\Delta^{m-2}z_{n+1}}\Delta^{m-1}z_n\right)\\ &\leq \frac{1-w_n}{a_nA_n} - A_{n+1}Q_n\frac{z_{n-\ell}}{\Delta^{m-2}z_{n+1}}\\ &\quad - \frac{A_{n+1}}{a_n}\left(\frac{a_n\Delta^{m-1}z_n}{\Delta^{m-2}z_n}\right)^2\end{aligned} \tag{34}$$

where we have used $\Delta^{m-2}z_n$ as positive and decreasing. Now using $(w_n-1)/A_n = a_n\Delta^{m-1}z_n/\Delta^{m-2}z_n$ in the above inequality, it follows that

$$\begin{aligned}\Delta w_n &\leq \frac{1-w_n}{a_nA_n} - A_{n+1}Q_n\frac{z_{n-\ell}}{\Delta^{m-2}z_{n+1}}\\ &\quad - \frac{A_{n+1}}{a_nA_n^2}\left(1-w_n\right)^2, \quad n \geq n_1.\end{aligned} \tag{35}$$

Now from Lemma 3, we obtain

$$z_{n-\ell} \geq \frac{1}{(m-2)!}\left(\frac{n-\ell}{2^{m-2}}\right)^{m-2}\Delta^{m-2}z_{n-\ell}. \tag{36}$$

Since $\Delta^{m-1}z_n < 0$ and $n-\ell < n+1$, we have

$$\Delta^{m-2}z_{n+1} < \Delta^{m-2}z_{n-\ell}. \tag{37}$$

Combining the inequalities (35) and (37), we have

$$\begin{aligned}\Delta w_n &\leq -MA_{n+1}Q_n\left(n-\ell\right)^{m-2} + \frac{\left(1-w_n\right)}{a_nA_n}\\ &\quad - \frac{A_{n+1}}{a_nA_n^2}\left(1-w_n\right)^2, \quad n \geq n_1,\end{aligned} \tag{38}$$

where $M = (1/(m-2)!)(1/2^{m-2})^{m-2}$. Completing the square in the above inequality, we have

$$\begin{aligned}\Delta w_n &\leq -MA_{n+1}Q_n\left(n-\ell\right)^{m-2}\\ &\quad - \frac{A_{n+1}}{a_nA_n^2}\left(\left(1-w_n\right) - \frac{1}{2}\frac{A_n}{A_{n+1}}\right)^2 + \frac{1}{4a_nA_{n+1}}\end{aligned} \tag{39}$$

or

$$\Delta w_n \leq -MA_{n+1}Q_n\left(n-\ell\right)^{m-2} + \frac{1}{4a_nA_{n+1}}, \quad n \geq n_1. \tag{40}$$

By summing the last inequality from n_1 to n, we obtain

$$\sum_{s=n_1}^{n}\left[MA_{s+1}Q_s\left(s-\ell\right)^{m-2} - \frac{1}{4a_sA_{s+1}}\right] \leq w_{n_1}. \tag{41}$$

Taking lim sup as $n \longrightarrow \infty$, in the above inequality we obtain a contradiction with (27). This completes the proof. □

Theorem 9. *Assume that (3) and $0 < \beta < 1$ hold. Further assume that there is a positive decreasing real sequence $\{\rho_n\}$ tending to zero such that $P(n)$ is positive for all $n \geq N \in \mathbb{N}(n_0)$. If (21) holds and*

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\sup\sum_{s=N}^{n}\left(M^\beta A_{s+1}Q_s\left(s-\ell\right)^{\beta(m-2)} - \frac{M_1^{1-\beta}}{4a_sA_{s+1}}\right)\\ &\quad = \infty\end{aligned} \tag{42}$$

for some constant $M_1 > 0$, then every solution of (1) either is oscillatory or tends to zero as $n \longrightarrow \infty$.

Proof. Assume that $\{x_n\}$ is an eventually positive solution of (1) such that $\lim_{n\longrightarrow\infty} x_n \neq 0$. Proceeding as in the proof of Theorem 8, we see that $\{z_n\}$ satisfies two possible cases (28) and (29) for all $n \geq n_1$.

Case (i). Suppose conditions (28) hold for all $n \geq n_1$; then the proof for this case is similar to that of Case (ii) of Theorem 7 and hence the details are omitted.

Case (ii). Assume now that conditions (29) hold for all $n \geq n_1$; proceeding as in Case (ii) of Theorem 8 we have

$$\begin{aligned}\Delta w_n &\leq \frac{\left(1-w_n\right)}{a_nA_n} - A_{n+1}\frac{Q_nz_{n-\ell}^\beta}{\Delta^{m-2}z_{n+1}}\\ &\quad - \frac{A_{n+1}}{a_nA_n^2}\left(1-w_n\right)^2, \quad n \geq n_1.\end{aligned} \tag{43}$$

Now using (36) and (37) in (43), we obtain

$$\begin{aligned}\Delta w_n &\leq -M^\beta A_{n+1}Q_n\left(n-\ell\right)^{\beta(m-2)}\left(\Delta^{m-2}z_{n-\ell}\right)^{\beta-1}\\ &\quad + \frac{1}{4a_nA_{n+1}}.\end{aligned} \tag{44}$$

Since $\{\Delta^{m-2}z_n\}$ is positive and decreasing and $\beta < 1$, there is a constant $M_1 > 0$ such that $(\Delta^{m-2}z_{n-\ell})^{\beta-1} \geq M_1^{\beta-1}$ for all $n \geq n_2 \geq n_1$. Using this in (44) and then summing the resulting inequality from n_2 to n, we obtain

$$\sum_{s=n_2}^{n}\left(M^{\beta}A_{s+1}Q_s(s-\ell)^{\beta(m-2)}-\frac{M_1^{1-\beta}}{4a_sA_{s+1}}\right) \leq M_1^{1-\beta}w_{n_2}<\infty. \quad (45)$$

Taking lim sup as $n \longrightarrow \infty$, in the above inequality, we obtain a contradiction with (42). This completes the proof. □

Theorem 10. *Assume that (3) and $\beta > 1$ hold. Further assume that there is a positive decreasing and sequence $\{\rho_n\}$ tending to zero such that $P(n)$ is positive for all $n \geq N \in \mathbb{N}(n_0)$. If (22) holds and*

$$\lim_{n\longrightarrow\infty}\sup\sum_{s=N}^{n}\left(M^{\beta}A_{s+1}^{\beta}Q_s(s-\ell)^{\beta(m-2)}-\frac{1}{4M_2^{\beta-1}a_sA_{s+1}}\right) = \infty \quad (46)$$

for some constant $M_2 > 0$, then every solution of (1) either is oscillatory or tends to zero as $n \longrightarrow \infty$.

Proof. Let us assume that $\{x_n\}$ is an eventually positive solution of (1) such that $\lim_{n\longrightarrow\infty} x_n \neq 0$. Proceeding as in the proof of Theorem 8, we see that $\{z_n\}$ satisfies two possible cases (28) and (29) for all $n \geq n_1$.

Case (i). If conditions (28) hold for all $n \geq n_1$, then the proof is similar to that of Case (iii) of Theorem 7 and hence the details are omitted.

Case (ii). Assume now that conditions (29) hold for all $n \geq n_1$. Proceeding as in Case (ii) of Theorem 9, we have

$$\Delta w_n \leq -M^{\beta}A_{n+1}Q_n(n-\ell)^{\beta(m-2)}\left(\Delta^{m-2}z_{n-\ell}\right)^{\beta-1} + \frac{1}{4a_nA_{n+1}}, \quad n \geq n_1. \quad (47)$$

Now from (32), one can see that $\Delta^{m-2}z_n/A_n$ is nondecreasing and hence there is a constant $M_2 > 0$ such that $\Delta^{m-2}z_n/A_n \geq M_2$ for all $n \geq n_1$. Using this in (47) and since $\beta > 1$, we have

$$\Delta w_n \leq -M^{\beta}M_2^{\beta-1}A_{n+1}^{\beta}Q_n(n-\ell)^{\beta(m-2)}+\frac{1}{4a_nA_{n+1}}, \quad n \geq n_1. \quad (48)$$

Summing the last inequality from n_1 to n, we obtain

$$\sum_{s=n_1}^{n}\left(M^{\beta}A_{s+1}^{\beta}Q_s(s-\ell)^{\beta(m-2)}-\frac{1}{4M_2^{\beta-1}a_sA_{s+1}}\right) \leq \frac{w_{n_1}}{M_2^{\beta-1}}. \quad (49)$$

Taking lim sup as $n \longrightarrow \infty$ in the above inequality, we get a contradiction with (46). This completes the proof. □

4. Examples

In this section, we present two examples to illustrate the importance of the main results.

Example 1. Consider the neutral difference equation

$$\Delta\left(n\Delta^{m-1}\left(x_n+\frac{1}{n}x_{n-2}^{1/3}\right)\right)+\frac{1}{n}x_{n-1}^{3}=0, \quad n \geq 2, \quad (50)$$

where $m \geq 2$ is an even integer. Here $a_n = n$, $p_n = 1/n$, $q_n = 1/n$, $k = 2$, $\ell = 1$, $\alpha = 1/3$, and $\beta = 3$. By taking $\rho_n = 1/n$, we see that $P(n) = (1/3)((n-1)/n) > 0$ for all $n \geq 2$. Now it is easy to see that the hypotheses $(C_1) - (C_4)$ are satisfied. Also condition (12) holds and therefore, by Theorem 5, every solution of (50) is oscillatory.

Example 2. Consider the neutral difference equation

$$\Delta\left(n(n+1)\Delta^{m-1}\left(x_n+\frac{1}{n}x_{n-2}^{1/3}\right)\right)+nx_{n-1}^{1/3}=0, \quad n \geq 2, \quad (51)$$

where $m \geq 2$ is an even integer. Here $a_n = n(n+1)$, $p_n = 1/n$, $q_n = n$, $k = 2$, $\ell = 1$, $\alpha = \beta = 1/3$. By taking $\rho_n = 1/n$, we see that $P(n) = (1/3)((n-1)/n) > 0$ for all $n \geq 2$. Now condition (21) becomes

$$\sum_{n=2}^{\infty}\frac{n((n-2)/(n-1))^{1/3}}{3^{1/3}n^{1/3}(n-1)^{1/3}}(n-1)^{(1/3)(m-1)} = \sum_{n=2}^{\infty}\frac{n^{2/3}(n-2)^{1/3}}{3^{1/3}}(n-1)^{(1/3)(m-3)}=\infty \quad (52)$$

since $m \geq 2$. Also a simple calculation shows that $A_n = 1/n$ and, using this, condition (42) becomes

$$\lim_{n\longrightarrow\infty}\sup\sum_{s=2}^{n}\left(M^{1/3}\frac{s}{(s+1)}(s-2)^{1/3}(s-1)^{(1/3)(m-3)}-\frac{M_1^{2/3}}{4s}\right) = \infty. \quad (53)$$

Thus all conditions of Theorem 9 are satisfied and hence every solution of (51) either is oscillatory or tends to zero as $n \longrightarrow \infty$.

5. Conclusion

The results obtained in this paper extend and complement some of the results reported in the literature. Further, Theorem 8, where $\alpha = 1$, corrects the conclusion of Theorem 4 established in [8]. The results reported in the papers [3, 4, 6–12, 17, 20] cannot be applicable to (50) and (51) to yield this conclusion since these equations have sublinear neutral terms. It would be interesting to improve Theorems 8, 9, and 10 so that all solutions are oscillatory instead of either being oscillatory or tending to zero.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

References

[1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.

[2] R. P. Agarwal, M. Bohner, S. R. Grace, and D. O'Regan, *Discrete oscillation theory*, Hindawi Publishing Corporation, New York, NY, USA, 2005.

[3] R. P. Agarwal, E. Thandapani, and P. J. Wong, "Oscillations of higher-order neutral difference equations," *Applied Mathematics Letters*, vol. 10, no. 1, pp. 71–78, 1997.

[4] Y. Bolat, O. Akin, and H. Yildirim, "Oscillation criteria for a certain even order neutral difference equation with an oscillating coefficient," *Applied Mathematics Letters*, vol. 22, no. 4, pp. 590–594, 2009.

[5] Yaşar Bolat and Jehad Alzabut, "On the oscillation of even-order half-linear functional difference equations with damping term," *International Journal of Differential Equations*, vol. 2014, Article ID 791631, 6 pages, 2014.

[6] W.-T. Li, "Oscillation of higher-order neutral nonlinear difference equations," *Applied Mathematics Letters*, vol. 11, no. 4, pp. 1–8, 1998.

[7] N. Parhi and A. K. Tripathy, "Oscillation of a class of nonlinear neutral difference equations of higher order," *Journal of Mathematical Analysis and Applications*, vol. 284, no. 2, pp. 756–774, 2003.

[8] D. Seghar, S. Selvarangam, and E. Thandapani, "Oscillation criteria for even order nonlinear neutral difference equations," *Differential Equations & Applications*, vol. 6, no. 3, pp. 441–453, 2014.

[9] S. Selvarangam, S. A. Rupadevi, E. Thandapani, and S. Pinelas, "Oscillation criteria for even order neutral difference equations," *Opuscula Mathematica*, vol. 39, no. 1, pp. 91–108, 2019.

[10] M. K. Yildiz and O. Ocalan, "Oscillation results for higher order nonlinear neutral delay difference equations," *Applied Mathematics Letters*, vol. 20, no. 3, pp. 243–247, 2007.

[11] A. Zafer, "Oscillatory and asymptotic behavior of higher order difference equations," *Mathematical and Computer Modelling*, vol. 21, no. 4, pp. 43–50, 1995.

[12] Z. Zhou, J. Yu, and G. Lei, "Oscillations for even-order neutral difference equations," *The Korean Journal of Computational & Applied Mathematics*, vol. 7, no. 3, pp. 601–610, 2000.

[13] C. Dharuman and E. Thandapani, "Oscillation of solutions of nonlinear difference equation with a super-linear neutral term," *Nonautonomous Dynamical Systems*, vol. 5, pp. 52–58, 2018.

[14] B. Kamaraj and R. Vasuki, "Oscillation of Second Order Difference Equation with a Superlinear Neutral Term," *Journal of Advances in Mathematics and Computer Science*, vol. 23, no. 5, pp. 1–10, 2017.

[15] W.-T. Li and J.-R. Yau, "Oscillation of first order difference equations with a nonlinear neutral term," *Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 6, no. 3, pp. 337–343, 1999.

[16] X. Liu, "Oscillation of solutions of neutral difference equations with a nonlinear neutral term," *Computers & Mathematics with Applications*, vol. 52, no. 3-4, pp. 439–448, 2006.

[17] H. A. El-Morshedy and S. R. Grace, "Oscillation of some nonlinear difference equations," *Journal of Mathematical Analysis and Applications*, vol. 281, no. 1, pp. 10–21, 2003.

[18] E. Thandapani, S. Pandian, and R. K. Balasubramaniam, "Oscillation of solutions of nonlinear neutral difference equations with nonlinear neutral term," *Far East Journal of Applied Mathematics*, vol. 15, no. 1, pp. 47–62, 2004.

[19] C. Dharuman, J. R. Graef, E. Thandapani, and K. S. Vidhya, "Oscillation of second order difference equation with a sublinear neutral term," *Journal of Mathematics and Applications*, vol. 40, pp. 59–67, 2017.

[20] M. K. Yildiz and H. Ogunmez, "Oscillation results of higher order nonlinear neutral delay difference equations with a nonlinear neutral term," *Hacettepe Journal of Mathematics and Statistics*, vol. 43, no. 5, pp. 809–814, 2014.

[21] Z. Zhang, J. Chen, and C. Zhang, "Oscillation of solutions for second-order nonlinear difference equations with nonlinear neutral term," *Computers & Mathematics with Applications*, vol. 41, no. 12, pp. 1487–1494, 2001.

[22] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, 1934.

[23] I. Győri and G. Ladas, *Oscillation Theory of Delay Differnetial Equations with Applications*, Oxford Mathematical Monographs, Clarendon Press, New York, NY, USA, 1991.

[24] X. H. Tang and Y. Liu, "Oscillation for nonlinear delay difference equations," *Tamkang Journal of Mathematics*, vol. 32, no. 4, pp. 275–280, 2001.