Hindawi
Abstract and Applied Analysis
Volume 2018, Article ID 1327691, 6 pages
http://dx.doi.org/10.1155/2018/1327691

Research Article

Fixed Point Theorems for $\mathscr{L}$-Contractions in Generalized Metric Spaces

Seong-Hoon Cho

Department of Mathematics, Hanseo University, Chungnam 356-706, Republic of Korea

Correspondence should be addressed to Seong-Hoon Cho; shcho@hanseo.ac.kr

Received 21 August 2018; Revised 8 October 2018; Accepted 4 November 2018; Published 2 December 2018

Academic Editor: Aref Jeribi

In this paper, the notion of $\mathscr{L}$-contractions is introduced and a new fixed point theorem for such contractions is established.

1. Introduction and Preliminaries

Branciari [1] introduced the notion of generalized metric spaces and obtained a generalization of the Banach contraction principle, whereafter many authors proved various fixed point results in such spaces, for example, [2–8] and references therein. Also, Suzuki *et al.* [9] and Abtahi *et al.* [10] studied ν-generalized metric spaces and proved the Banach and Kannan contraction principles in such spaces, and Mitrović *et al.* [11] introduced the notion of $b_\nu(s)$-generalized metric spaces and Banach and Reich contraction principles in such spaces.

In particular, Jleli and Samet [12] introduced the notion of θ-contractions and gave a generalization of the Banach contraction principle in generalized metric spaces, where $\theta : (0, \infty) \longrightarrow (1, \infty)$ is a function satisfying the following conditions:

(θ1) θ is nondecreasing;

(θ2) $\forall \{t_n\} \subset (0, \infty)$,

$$\lim_{n\longrightarrow\infty} \theta(t_n) = 1 \Longleftrightarrow \lim_{n\longrightarrow\infty} t_n = 0^+; \tag{1}$$

(θ3) $\exists r \in (0,1) \wedge l \in (0,\infty)$:

$$\lim_{t\longrightarrow 0^+} \frac{\theta(t) - 1}{t^r} = l. \tag{2}$$

Also, Ahmad *et al.* [13] extended the result of Jleli and Samet [12] to metric spaces by applying the following simple condition (θ4) instead of (θ3).

(θ4) θ is continuous on $(0, \infty)$.

Recently, Khojasteh *et al.* [14] introduced the notion of $\mathscr{Z}$-contractions by defining the concept of simulation functions. They unified some existing metric fixed point results. Afterward, many authors ([15–19] and references therein) obtained generalizations of the result of [14].

In the paper, we introduce the concept of a new type of contraction maps, and we establish a new fixed point theorem for such contraction maps in the setting of generalized metric spaces.

Let $\mathscr{L}$ be the family of all mappings $\xi : [1,\infty)\times[1,\infty) \longrightarrow \mathbb{R}$ such that

(ξ1) $\xi(1,1) = 1$;

(ξ2) $\xi(t,s) < s/t \ \forall s,t > 1$;

(ξ3) for any sequence $\{t_n\}, \{s_n\} \subset (1,\infty)$ with $t_n \le s_n \ \forall n = 1,2,3,\cdots$

$$\lim_{n\longrightarrow\infty} t_n = \lim_{n\longrightarrow\infty} s_n > 1 \Longrightarrow \lim_{n\longrightarrow\infty} \sup \xi(t_n, s_n) < 1. \tag{3}$$

We say that $\xi \in \mathscr{L}$ is a *$\mathscr{L}$-simulation function*.

Note that $\xi(t,t) < 1 \ \forall t > 1$.

Example 1. Let $\xi_b, \xi_w, \xi : [1,\infty)\times[1,\infty) \longrightarrow \mathbb{R}$ be functions defined as follows, respectively:

(1) $\xi_b(t,s) = s^k/t \ \forall t,s \ge 1$, where $k \in (0,1)$;

(2) $\xi_w(t,s) = s/t\phi(s)\ \forall t,s \geq 1$, where $\phi : [1,\infty) \longrightarrow [1,\infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$;

$$\xi(t,s) = \begin{cases} 1 & \text{if } (s,t) = (1,1), \\ \dfrac{s}{2t} & \text{if } s < t, \\ \dfrac{s^{\lambda}}{t} & \text{otherwise,} \end{cases} \quad (4)$$

$\forall s,t \geq 1$, where $\lambda \in (0,1)$.

Then $\xi_b, \xi_w, \xi \in \mathscr{L}$.

We recall the following definitions which are in [1].

Let X be a nonempty set, and let $d : X \times X \longrightarrow [0,\infty)$ be a map such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them is different from x and y:

(d1) $d(x,y) = 0$ if and only if $x = y$;

(d2) $d(x,y) = d(y,x)$;

(d3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.

Then d is called a generalized metric on X and (X,d) is called a generalized metric space.

Let (X,d) be a generalized metric space, let $\{x_n\} \subset X$ be a sequence, and $x \in X$.

Then we say that

(1) $\{x_n\}$ is convergent to x (denoted by $\lim_{n\longrightarrow\infty} x_n = x$) if and only if $\lim_{n\longrightarrow\infty} d(x_n,x) = 0$;

(2) $\{x_n\}$ is Cauchy if and only if $\lim_{n,m\longrightarrow\infty} d(x_n,x_m) = 0$;

(3) (X,d) is complete if and only if every Cauchy sequence in X is convergent to some point in X.

Let (X,d) be a generalized metric space.

A map $T : X \longrightarrow X$ is called *continuous* at $x \in X$ if, for any $V \in \tau$ containing Tx, there exists $U \in \tau$ containing x such that $TU \subset V$, where τ is the topology on X induced by the generalized metric d. That is,

$$\begin{aligned} \tau &= \{U \subset X : \ \forall x \in U, \exists B \in \beta, x \in B \subset U\}, \\ \beta &= \{B(x,r) : x \in X, \ \forall r > 0\}, \\ B(x,r) &= \{y \in X : d(x,y) < r\}. \end{aligned} \quad (5)$$

If T is continuous at each point $x \in X$, then it is called *continuous*.

Note that T is continuous if and only if it is sequentially continuous, i.e., $\lim_{n\longrightarrow\infty} d(Tx_n, Tx) = 0$ for any sequence $\{x_n\} \subset X$ with $\lim_{n\longrightarrow\infty} d(x_n,x) = 0$.

Remark 2 (see [6]). If d is a generalized metric on X, then it is not continuous in each coordinate.

Lemma 3 (see [20]). *Let (X,d) be a generalized metric space, let $\{x_n\} \subset X$ be a Cauchy sequence, and $x, y \in X$. If there exists a positive integer N such that*

(1) $x_n \neq x_m\ \forall n,m > N$;

(2) $x_n \neq x\ \forall n > N$;

(3) $x_n \neq y\ \forall n > N$;

(4) $\lim_{n\longrightarrow\infty} d(x_n,x) = \lim_{n\longrightarrow\infty} d(x_n,y)$,

then $x = y$.

2. Fixed Point Theorems

We denote by Θ the class of all functions $\theta : (0,\infty) \longrightarrow (1,\infty)$ such that conditions (θ1) and (θ2) hold.

A mapping $T : X \longrightarrow X$ is called *$\mathscr{L}$-contraction* with respect to ξ if there exist $\theta \in \Theta$ and $\xi \in \mathscr{L}$ such that, for all $x, y \in X$ with $d(Tx,Ty) > 0$,

$$\xi(\theta(d(Tx,Ty)), \theta(d(x,y))) \geq 1. \quad (6)$$

Note that if T is *$\mathscr{L}$-contraction* with respect to ξ, then it is continuous. In fact, let $x \in X$ be a point and let $\{x_n\} \subset X$ be any sequence such that

$$\begin{aligned} &\lim_{n\longrightarrow\infty} d(x_n,x) = 0^+, \\ &d(Tx_n,Tx) > 0 \\ &\qquad \forall n = 1,2,3,\cdots. \end{aligned} \quad (7)$$

Then from (θ2) $\lim_{n\longrightarrow\infty} \theta(d(x_n,x)) = 1$.

It follows from (6) and (ξ2) that

$$1 \leq \xi(\theta(d(Tx_n,Tx)), \theta(d(x_n,x))) < \frac{\theta(d(x_n,x))}{\theta(d(Tx_n,Tx))}, \quad (8)$$

which implies

$$\theta(d(Tx_n,Tx)) < \theta(d(x_n,x)). \quad (9)$$

Since θ is nondecreasing, we have

$$d(Tx_n,Tx) < d(x_n,x), \quad (10)$$

and so

$$\lim_{n\longrightarrow\infty} d(Tx_n,Tx) = 0. \quad (11)$$

Hence T is continuous.

Now, we prove our main result.

Theorem 4. *Let (X,d) be a complete generalized metric space, and let $T : X \longrightarrow X$ be a $\mathscr{L}$-contraction with respect to ξ.*

Then T has a unique fixed point, and for every initial point $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to the fixed point.

Proof. Firstly, we show uniqueness of fixed point whenever it exists.

Assume that w and u are fixed points of T.

If $u \neq z$, then $d(w,u) > 0$, and so it follows from (6) that

$$\begin{aligned} 1 &\leq \xi(\theta(d(Tw,Tu)), \theta(d(w,u))) \\ &= \xi(\theta(d(w,u)), \theta(d(w,u))) < \frac{\theta(d(w,u))}{\theta(d(w,u))}. \end{aligned} \quad (12)$$

Hence

$$\theta\left(d\left(w,u\right)\right)<\theta\left(d\left(w,u\right)\right) \tag{13}$$

which is a contradiction.

Hence $w=u$, and fixed point of T is unique.

Secondly, we prove existence of fixed point.

Let $x_0\in X$ be a point. Define a sequence $\{x_n\}\subset X$ by $x_n=Tx_{n-1}=T^n x_0$ $\forall n=1,2,3\cdots$.

If $x_{n_0}=x_{n_0+1}$ for some $n_0\in\mathbb{N}$, then x_{n_0} is a fixed point of T, and the proof is finished.

Assume that

$$x_{n-1}\neq x_n \quad \forall n=1,2,3\cdots. \tag{14}$$

It follows from (6) and (14) that $\forall n=1,2,3,\cdots$

$$\begin{aligned}1&\leq\xi\left(\theta\left(d\left(Tx_{n-1},Tx_n\right)\right),\theta\left(d\left(x_{n-1},x_n\right)\right)\right)\\&=\xi\left(\theta\left(d\left(x_n,x_{n+1}\right)\right),\theta\left(d\left(x_{n-1},x_n\right)\right)\right)\\&<\frac{\theta\left(d\left(x_{n-1},x_n\right)\right)}{\theta\left(d\left(x_n,x_{n+1}\right)\right)}.\end{aligned} \tag{15}$$

Consequently, we obtain that

$$\theta\left(d\left(x_n,x_{n+1}\right)\right)<\theta\left(d\left(x_{n-1},x_n\right)\right) \quad \forall n=1,2,3,\cdots \tag{16}$$

which implies

$$d\left(x_n,x_{n+1}\right)<d\left(x_{n-1},x_n\right) \quad \forall n=1,2,3,\cdots. \tag{17}$$

Hence $\{d(x_{n-1},x_n)\}$ is a decreasing sequence, and so there exists $r\geq 0$ such that

$$\lim_{n\longrightarrow\infty}d\left(x_{n-1},x_n\right)=r. \tag{18}$$

We now show that $r=0$.

Assume that $r\neq 0$.

Then it follows from (θ2) that

$$\lim_{n\longrightarrow\infty}\theta\left(d\left(x_{n-1},x_n\right)\right)\neq 1, \tag{19}$$

and so

$$\lim_{n\longrightarrow\infty}\theta\left(d\left(x_{n-1},x_n\right)\right)>1. \tag{20}$$

Let $s_n=\theta(d(x_{n-1},x_n))$ and $t_n=\theta(d(x_n,x_{n+1}))$ $\forall n=1,2,3,\cdots$.

From (ξ3) we obtain

$$1\leq\lim_{n\longrightarrow\infty}\sup\xi\left(t_n,s_n\right)<1 \tag{21}$$

which is a contradiction.

Thus we have

$$\lim_{n\longrightarrow\infty}d\left(x_{n-1},x_n\right)=0 \tag{22}$$

and so

$$\lim_{n\longrightarrow\infty}\theta\left(d\left(x_{n-1},x_n\right)\right)=1. \tag{23}$$

We show that

$$\lim_{n\longrightarrow\infty}d\left(x_{n-1},x_{n+1}\right)=0. \tag{24}$$

We consider three cases.

Case 1. $x_n\neq x_{n+2}$ $\forall n=1,2,3,\cdots$.

From (6) and (14) we obtain that $\forall n=1,2,3,\cdots$

$$\begin{aligned}1&\leq\xi\left(\theta\left(d\left(Tx_{n-1},Tx_{n+1}\right)\right),\theta\left(d\left(x_{n-1},x_{n+1}\right)\right)\right)\\&=\xi\left(\theta\left(d\left(x_n,x_{n+2}\right)\right),\theta\left(d\left(x_{n-1},x_{n+1}\right)\right)\right)\\&<\frac{\theta\left(d\left(x_{n-1},x_{n+1}\right)\right)}{\theta\left(d\left(x_n,x_{n+2}\right)\right)},\end{aligned} \tag{25}$$

and so

$$\theta\left(d\left(x_n,x_{n+2}\right)\right)<\theta\left(d\left(x_{n-1},x_{n+1}\right)\right) \quad \forall n=1,2,3,\cdots \tag{26}$$

which implies

$$d\left(x_n,x_{n+2}\right)<d\left(x_{n-1},x_{n+1}\right) \quad \forall n=1,2,3,\cdots. \tag{27}$$

Hence $\{d(x_{n-1},x_{n+1})\}$ is decreasing.

In a manner similar to that which proved (22), we have

$$\lim_{n\longrightarrow\infty}d\left(x_{n-1},x_{n+1}\right)=0. \tag{28}$$

Case 2. There exists $n_0\geq 1$ such that $x_{n_0}=x_{n_0+2}$.

From the first term to the n_0 th term shall be removed, and let $x_n=x_{n_0+n}$ $\forall n=1,2,3,\cdots$.

Then $x_n\neq x_{n+2}$ $\forall n=1,2,3,\cdots$. By Case 1, we have

$$\lim_{n\longrightarrow\infty}d\left(x_{n-1},x_{n+1}\right)=0. \tag{29}$$

Case 3. $x_n=x_{n+2}$ $\forall n=0,1,2,\cdots$.

We have

$$d\left(x_{n-1},x_{n+1}\right)=0 \quad \forall n=1,2,3,\cdots. \tag{30}$$

Hence

$$\lim_{n\longrightarrow\infty}d\left(x_{n-1},x_{n+1}\right)=0. \tag{31}$$

In all cases, (24) is satisfied.

Now, we show that $\{x_n\}$ is bounded.

If $\{x_n\}$ is not bounded, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n_1=1$ and $\forall k=1,2,3,\ldots,n(k+1)$ is the minimum integer greater than $n(k)$ with

$$\begin{gathered}d\left(x_{n(k+1)},x_{n(k)}\right)>1,\\d\left(x_l,x_{n(k)}\right)\leq 1\end{gathered} \tag{32}$$

for $n(k)\leq l\leq n(k+1)-1$.

Then we have

$$\begin{aligned}1&<d\left(x_{n(k+1)},x_{n(k)}\right)\\&\leq d\left(x_{n(k+1)},x_{n(k+1)-2}\right)+d\left(x_{n(k+1)-2},x_{n(k+1)-1}\right)\\&\quad+d\left(x_{n(k+1)-1},x_{n(k)}\right)\\&\leq d\left(x_{n(k+1)},x_{n(k+1)-2}\right)+d\left(x_{n(k+1)-2},x_{n(k+1)-1}\right)+1.\end{aligned} \tag{33}$$

By letting $k \longrightarrow \infty$ in the above, we obtain

$$\lim_{k\longrightarrow\infty} d\left(x_{n(k+1)}, x_{n(k)}\right) = 1. \tag{34}$$

By using (22), (34), and condition (d3), we deduce that

$$\lim_{k\longrightarrow\infty} d\left(x_{n(k+1)-1}, x_{n(k)-1}\right) = 1. \tag{35}$$

It follows from (θ2), (34), and (35) that

$$\lim_{k\longrightarrow\infty} \theta\left(d\left(x_{n(k+1)}, x_{n(k)}\right)\right) > 1, \tag{36}$$

$$\lim_{n\longrightarrow\infty} \theta\left(d\left(x_{n(k+1)-1}, x_{n(k)-1}\right)\right) > 1. \tag{37}$$

From (6) and (32) we infer that

$$\begin{aligned}1 \leq \xi\left(\theta\left(d\left(Tx_{n(k+1)-1}, Tx_{n(k)-1}\right)\right),\right.\\ \left.\theta\left(d\left(x_{n(k+1)-1}, x_{n(k)-1}\right)\right)\right) = \xi\left(\theta\left(d\left(x_{n(k+1)}, x_{n(k)}\right)\right),\right.\\ \left.\theta\left(d\left(x_{n(k+1)-1}, x_{n(k)-1}\right)\right)\right) < \frac{\theta\left(d\left(x_{n(k+1)-1}, x_{n(k)-1}\right)\right)}{\theta\left(d\left(x_{n(k+1)}, x_{n(k)}\right)\right)}\end{aligned} \tag{38}$$

which implies

$$\theta\left(d\left(x_{n(k+1)}, x_{n(k)}\right)\right) < \theta\left(d\left(x_{n(k+1)-1}, x_{n(k)-1}\right)\right). \tag{39}$$

Let

$$\begin{aligned}s_n &= \theta(d\left(x_{n(k+1)-1}, x_{n(k)-1}\right),\\ t_n &= \theta\left(d\left(x_{n(k+1)}, x_{n(k)}\right)\right) \quad \forall n = 1, 2, 3, \cdots.\end{aligned} \tag{40}$$

Then $t_k < s_k$ $\forall n = 1, 2, 3, \cdots$ and $\lim_{k\longrightarrow\infty} t_k = \lim_{k\longrightarrow\infty} s_k > 1$.

It follows from (ξ3) that

$$1 \leq \lim_{k\longrightarrow\infty} \sup \xi\left(t_k, s_k\right) < 1 \tag{41}$$

which is a contradiction.

Thus $\{x_n\}$ is bounded.

Now, we show that $\{x_n\}$ is a Cauchy sequence.

Let

$$M_n = \sup\left\{d\left(x_i, x_j\right) : \ i, j \geq n\right\}. \tag{42}$$

Clearly,

$$0 \leq M_{n+1} \leq M_n < \infty \quad \forall n = 1, 2, 3, \cdots \tag{43}$$

and so there exists $M \geq 0$ such that

$$\lim_{n\longrightarrow\infty} M_n = M. \tag{44}$$

Assume that $M > 0$.

It follows from (42) that $\forall k = 1, 2, 3, \cdots$ there exist $n(k), m(k) \geq k$ with

$$M_k - \frac{1}{k} < d\left(x_{m(k)}, x_{n(k)}\right) \leq M_k. \tag{45}$$

So

$$\lim_{k\longrightarrow\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \lim_{k\longrightarrow\infty} M_k = M. \tag{46}$$

It follows from (6) and (14) that

$$\begin{aligned}&\xi\left(\theta\left(d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)\right), \theta\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right)\right)\\ &= \xi\left(\theta\left(d\left(x_{m(k)}, x_{n(k)}\right)\right), \theta\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right)\right)\\ &< \frac{\theta\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right)}{\theta\left(d\left(x_{m(k)}, x_{n(k)}\right)\right)}\end{aligned} \tag{47}$$

which implies

$$\theta\left(d\left(x_{m(k)}, x_{n(k)}\right)\right) < \theta\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right). \tag{48}$$

Hence we have

$$\begin{aligned}d\left(x_{m(k)}, x_{n(k)}\right) &< d\left(x_{m(k)-1}, x_{n(k)-1}\right)\\ &\leq d\left(x_{m(k)-1}, x_{m(k)}\right) + d\left(x_{m(k)}, x_{n(k)}\right)\\ &\quad + d\left(x_{n(k)}, x_{n(k)-1}\right).\end{aligned} \tag{49}$$

Letting $k \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n\longrightarrow\infty} d\left(x_{m(k)-1} x_{n(k)-1}\right) = M. \tag{50}$$

Let

$$\begin{aligned}s_k &= \theta\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right),\\ t_k &= \theta\left(d\left(x_{m(k)}, x_{n(k)}\right)\right) \quad \forall k = 1, 2, 3, \cdots.\end{aligned} \tag{51}$$

Then $t_k < s_k$ $\forall k = 1, 2, 3, \cdots$. Since $M > 0$,

$$\lim_{k\longrightarrow\infty} t_k = \lim_{k\longrightarrow\infty} s_k > 1. \tag{52}$$

Thus we have

$$1 \leq \lim_{k\longrightarrow\infty} \sup \xi\left(t_k, s_k\right) < 1 \tag{53}$$

which is a contradiction.

Hence $M = 0$, and hence $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $z \in X$ such that

$$\lim_{n\longrightarrow\infty} d\left(x_n, z\right) = 0. \tag{54}$$

Because T is continuous,

$$\lim_{n\longrightarrow\infty} d\left(x_n, Tz\right) = \lim_{n\longrightarrow\infty} d\left(Tx_{n-1}, Tz\right) = 0. \tag{55}$$

By Lemma 3, $z = Tz$. □

We give an example to illustrate Theorem 4.

Example 5. Let $X = \{1, 2, 3, 4\}$ and define $d : X \times X \longrightarrow [0, \infty)$ as follows:

$$\begin{aligned}&d(1,2) = d(2,1) = 3,\\ &d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1,\\ &d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4)\\ &\quad = d(4,3) = 4,\\ &d(x,x) = 0 \quad \forall x \in X.\end{aligned} \tag{56}$$

Then (X, d) is a complete generalized metric space, but not a metric space (see [21]).

Define a map $T : X \longrightarrow X$ by

$$Tx = \begin{cases} 3 & (x \neq 4), \\ 1 & (x = 4). \end{cases} \quad (57)$$

And define a function $\theta : (0, \infty) \longrightarrow (1, \infty)$ by

$$\theta(t) = e^t. \quad (58)$$

We now show that T is a $\mathscr{L}$-contraction with respect to ξ_b, where $\xi_b(t, s) = s^k/t\ \forall t, s \geq 1, k = 1/2$.

We have

$$d(Tx, Ty) = \begin{cases} d(1,3) = 1 & (x = 4, y \neq 4), \\ d(1,1) = 0 & (x = 4, y = 4), \\ d(3,3) = 0 & (x \neq 4, y \neq 4) \end{cases} \quad (59)$$

so

$$d(Tx, Ty) > 0 \Longleftrightarrow x = 4,\ \ y \neq 4. \quad (60)$$

We have, for $x = 4$ and $y \neq 4$,

$$d(x, y) = 4, \qquad d(Tx, Ty) = 1. \quad (61)$$

We deduce that, for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\xi_b(\theta(d(Tx, Ty)), \theta(d(x, y))) = \frac{[\theta(d(x, y))]^k}{\theta(d(Tx, Ty))} = \frac{[e^4]^{1/2}}{e^1} = e > 1. \quad (62)$$

Thus all hypotheses of Theorem 4 are satisfied, and T has a fixed point $x_* = 3$.

Note that Banach's contraction principle is not satisfied with the usual metric $\rho(x, y) = |x - y|\ \forall x, y \in X$. In fact, if $x = 2$, $y = 4$, then

$$\rho(T2, T4) \leq k\rho(2, 4), \quad k \in (0, 1) \quad (63)$$

which implies

$$k \geq 1. \quad (64)$$

Also, note that the θ-contraction condition [13] does not hold.

Let $\theta(t) = e^t,\ \forall t > 0$.

Then $\theta(t)$ satisfies conditions $(\theta 1)$, $(\theta 2)$, and $(\theta 4)$.

If

$$\theta(\rho(T2, T4)) \leq [\theta(\rho(2, 4))]^k, \quad \text{where } k \in (0, 1) \quad (65)$$

then

$$e^2 \leq [e^2]^k \quad (66)$$

and so $k \geq 1$. Hence T is not θ-contraction map.

By taking $\xi = \xi_b$ in Theorem 4, we obtain Corollary 6.

Corollary 6. *Let (X, d) be a complete generalized metric space, and let $T : X \longrightarrow X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$*

$$\theta(d(Tx, Ty)) \leq \theta(d(x, y))^k \quad (67)$$

where $\theta \in \Theta$ and $k \in (0, 1)$.

Then T has a unique fixed point.

Remark 7. Corollary 6 is a generalization of Theorem 2.1 of [12] without condition $(\theta 3)$ and Theorem 2.2 of [13] without condition $(\theta 4)$.

By taking $\xi = \xi_w$ in Theorem 4, we obtain Corollary 8.

Corollary 8. *Let (X, d) be a complete generalized metric space, and let $T : X \longrightarrow X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$*

$$\theta(d(Tx, Ty)) \leq \frac{\theta(d(x, y))}{\phi(\theta(d(x, y)))} \quad (68)$$

where $\theta \in \Theta$ and $\phi : [1, \infty) \longrightarrow [1, \infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$.

Then T has a unique fixed point.

Corollary 9. *Let (X, d) be a complete generalized metric space, and let $T : X \longrightarrow X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \quad (69)$$

where $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is nondecreasing and lower semicontinuous such that $\varphi^{-1}(\{0\}) = 0$.

Then T has a unique fixed point.

Proof. Condition (69) implies T is continuous.

Let $\theta(t) = e^t,\ \forall t > 0$.

From (69) we have that, for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\theta(d(Tx, Ty)) = e^{d(Tx,Ty)} \leq e^{d(x,y) - \varphi(d(x,y))} = \frac{e^{d(x,y)}}{e^{\varphi(d(x,y))}}. \quad (70)$$

Let $\varphi(t) = \ln(\phi(\theta(t))),\ \forall t \geq 0$, where $\phi : [1, \infty) \longrightarrow [1, \infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$.

Then φ is nondecreasing and lower semicontinuous, and

$$\varphi(t) = 0 \Longleftrightarrow \phi(\theta(t)) = 1 \Longleftrightarrow \theta(t) = e^t = 1 \Longleftrightarrow t = 0. \quad (71)$$

It follows from (70) that, for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\theta(d(Tx, Ty)) \leq \frac{\theta(d(x, y))}{e^{\ln(\phi(\theta(d(x,y))))}} = \frac{\theta(d(x, y))}{\phi(\theta(d(x, y)))}. \quad (72)$$

By Corollary 8, T has a unique fixed point. □

By taking $\theta(t) = 2 - (2/\pi)\arctan(1/t^\alpha)$, where $\alpha \in (0, 1), t > 0$ in Corollary 8, we obtain the following result.

Corollary 10. *Let (X, d) be a complete generalized metric space, and let $T : X \longrightarrow X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$*

$$2-\frac{2}{\pi}\arctan\left(\frac{1}{\left[d\left(Tx,Ty\right)\right]^{\alpha}}\right)\leq\frac{2-(2/\pi)\arctan\left(1/\left[d\left(x,y\right)\right]^{\alpha}\right)}{\phi\left(2-(2/\pi)\arctan\left(1/\left[d\left(x,y\right)\right]^{\alpha}\right)\right)} \tag{73}$$

where $\alpha\in(0,1)$ *and* $\phi:[1,\infty)\longrightarrow[1,\infty)$ *is nondecreasing and lower semicontinuous such that* $\phi^{-1}(\{1\})=1$.

Then T has a unique fixed point.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The author express his gratitude to the referees for careful reading and giving variable comments. This research was supported by Hanseo University.

References

[1] A. Branciari, "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces," *Publicationes Mathematicae*, vol. 57, no. 1-2, pp. 31–37, 2000.

[2] H. Aydi, E. Karapınar, and B. Samet, "Fixed points for generalized (α,ψ) contractions on generalized metric spaces," *Journal of Inequalities and Applications*, vol. 2014, article 229, 2014.

[3] C. Di Bari and P. Vetro, "Common fixed points in generalized metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 13, pp. 7322–7325, 2012.

[4] P. Das, "A fixed point theorem on a class of generalized metric spaces," *The Korean Journal of Mathematics*, vol. 9, pp. 29–33, 2002.

[5] W. A. Kirk and N. Shahzad, "Generalized metrics and Caristi's theorem," *Fixed Point Theory and Applications*, vol. 2013, article 129, 2013.

[6] B. Samet, "Discussion on "A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces" by A. Branciari," *Publicationes Mathematicae*, vol. 76, no. 3-4, pp. 493-494, 2010.

[7] I. R. Sarma, J. M. Rao, and S. S. Rao, "Contractions over generalized metric spaces," *The Journal of Nonlinear Science and its Applications*, vol. 2, no. 3, pp. 180–182, 2009.

[8] W. Shatanawi, A. Al-Rawashdeh, Hassen Aydi, and H. K. Nashine, "On a Fixed Point for Generalized Contractions in Generalized Metric Spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 246085, 13 pages, 2012.

[9] T. Suzuki, B. Alamri, and L. A. Khan, "Some notes on fixed point theoems in ν-generalized metric spaces," *Bulletin of the Kyushu Institute of Technology, Mathematics, Natural Science*, no. 62, pp. 15–23, 2015.

[10] M. Abtahi, Z. Kadelburg, and S. Radenovic, "Fixed points of Ciric-Matakowski-type contractions in ν-generalized metric spaces," *RACSAM*.

[11] Z. D. Mitrović and S. Radenović, "The Banach and Reich contractiond in bν(s)-metric spaces," *Journal of Fixed Point Theory and Applications*, vol. 19, no. 4, pp. 3087–3095, 2017.

[12] M. Jleli and B. Samet, "A new generalization of the Banach contraction principle," *Journal of Inequalities and Applications*, vol. 2014, no. 1, article no. 38, 8 pages, 2014.

[13] J. Ahmad, A. E. Al-Mazrooei, Y. J. Cho, and Y.-O. Yang, "Fixed point results for generalized θ-contractions," *Journal of Nonlinear Sciences and Applications. JNSA*, vol. 10, no. 5, pp. 2350–2358, 2017.

[14] F. Khojasteh, S. Shukla, and S. Radenovic, "A new approach to the study of fixed point theorems via simulation functions," *Filomat*, vol. 29, no. 6, pp. 1189–1194, 2015.

[15] H. H. Alsulami, E. Karapinar, F. Khojasteh, and A.-F. Roldán-López-de-Hierro, "A proposal to the study of contractions in quasi-metric spaces," *Discrete Dynamics in Nature and Society*, vol. 2014, Article ID 269286, pp. 1–10, 2014.

[16] H. Isik, N. Gungor, C. Park, and S. Jang, "Fixed Point Theorems for Almost Z-Contractions with an Application," *Mathematics*, vol. 6, no. 3, p. 37, 2018.

[17] E. Karapınar, "Fixed points results via simulation functions," *Filomat*, vol. 30, no. 8, pp. 2343–2350, 2016.

[18] A.-F. Roldan-Lopez-de-Hierro, E. Karapinar, C. Roldan-Lopez-de-Hierro, and J. Martinez-Moreno, "Coincidence point theorems on metric spaces via simulation functions," *Journal of Computational and Applied Mathematics*, vol. 275, pp. 345–355, 2015.

[19] S. Radenović, F. Vetro, and J. Vujaković, "An alternative and easy approach to fixed point results via simulation functions," *Demonstratio Mathematica*, vol. 50, no. 1, pp. 223–230, 2017.

[20] M. Jleli and B. Samet, "The Kannan's fixed point theorem in a cone rectangular metric space," *The Journal of Nonlinear Science and its Applications*, vol. 2, no. 3, pp. 161–167, 2009.

[21] A. Azam and M. Arshad, "Kannan fixed point theorem on generalized metric spaces," *The Journal of Nonlinear Science and its Applications*, vol. 1, no. 1, pp. 45–48, 2008.