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Research Article

Approximation Properties of q-Bernoulli Polynomials

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We study the q-analogue of Euler-Maclaurin formula and by introducing a new q-operator we drive to this form. Moreover, approximation properties of q-Bernoulli polynomials are discussed. We estimate the suitable functions as a combination of truncated series of q-Bernoulli polynomials and the error is calculated. This paper can be helpful in two different branches: first we solve the differential equations by estimating functions and second we may apply these techniques for operator theory.

1. Introduction

The present study has sought to investigate the approximation of suitable function $f(x)$ as a linear combination of q-Bernoulli polynomials. This study applies q-operator to expand a function in terms of q-Bernoulli polynomials. In addition, this method leads us to q-analogue of Euler expansion. This expansion offers a proper tool to solve q-difference equations or normal differential equation as well. There are many approaches to approximate the capable functions. According to properties of q-functions many q-functions have been used in order to approximate a suitable function. For example, at [1], some identities and formulae for the q-Bernstein basis function, including the partition of unity property, and formulae for representing the monomials were studied. In addition, a kind of approximation of a function in terms of Bernoulli polynomials is used in several approaches for solving differential equations, such as [2–4]. This paper gives conditions to approximate capable functions as a linear combination of q-Bernoulli polynomials as well as related examples. Also, it introduces the new q-operator to reach the q-analogue of Euler-Maclaurin formula.

This study, first, introduces some q-calculus concepts. There are several types of q-Bernoulli polynomials and numbers that can be generated by different q-exponential functions. Carlitz [5] is pioneer of introducing q-analogue of the Bernoulli numbers; he applied a sequence $\{\beta_m\}_{m\geq 0}$ in the middle of the 20th century;

$$\sum_{k=0}^{m}\binom{m}{k}\beta_k q^{k+1}-\beta_m=\begin{cases}1, & m=1,\\ 0, & m>1.\end{cases} \tag{1}$$

First, this study makes an assumption that $|q| < 1$, and this assumption is going to apply in the rest of the paper as well. If q tends to one from a left side the ordinary form would be reached.

Since Carlitz, there have been many distinct q-analogues of Bernoulli numbers arising from varying motivations. In [6], the authors used improved q-exponential functions to introduce a new class of q-Bernoulli polynomials. In addition, they investigate some properties of these q-Bernoulli polynomials.

Let us introduce a class of q-Bernoulli polynomials in a form of generating function as follows:

$$\frac{ze_q(zt)}{e_q(z)-1}=\sum_{n=0}^{\infty}\beta_{n,q}(t)\frac{z^n}{[n]_q!},\quad |z|<2\pi, \tag{2}$$

where $[n]_q$ is q-number and q-numbers factorial is defined by

$$[a]_q=\frac{1-q^a}{1-q}\quad (q\neq 1);$$

$$[0]!\ =1;$$

$$[n]! = [n][n-1]! \qquad n \in \mathbb{N}, \ a \in \mathbb{C}. \tag{3}$$

The q-shifted factorial for the case $n = 0$, for natural n, and for infinity case and q-polynomial coefficient are defined by the following expressions, respectively:

$$\begin{gathered}
(a;q)_0 = 1, \\
(a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right), \quad n \in \mathbb{N}, \\
(a;q)_\infty = \prod_{j=0}^{\infty}\left(1-q^j a\right), \quad |q| < 1, \ a \in \mathbb{C}; \\
\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k},
\end{gathered} \tag{4}$$

q-standard terminology and notation can be found at [7] and [8]. We call $\beta_{n,q}$ Bernoulli number and $\beta_{n,q} = \beta_{n,q}(0)$. In addition, q-analogue of $(x-a)^n$ is defined as

$$(x-a)_q^n = \begin{cases} 1, & n = 0, \\ (x-a)(x-aq)\cdots\left(x-aq^{n-1}\right), & n \neq 0. \end{cases} \tag{5}$$

In the standard approach to the q-calculus, two exponential functions are used. These q-exponentials are defined by

$$\begin{aligned}
e_q(z) &= \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty}\frac{1}{(1-(1-q)q^k z)}, \\
&\qquad 0 < |q| < 1, \quad |z| < \frac{1}{|1-q|}, \\
E_q(z) &= e_{1/q}(z) = \sum_{n=0}^{\infty}\frac{q^{(1/2)n(n-1)}z^n}{[n]_q!} \\
&= \prod_{k=0}^{\infty}\left(1+(1-q)q^k z\right), \quad 0 < |q| < 1, \ z \in \mathbb{C}.
\end{aligned} \tag{6}$$

q-shifted factorial can be expressed by Heine's binomial formula as follows:

$$(a;q)_n = \sum_{k=0}^{n}\binom{n}{k}_q q^{k(k-1)/2}(-1)^k a^k. \tag{7}$$

For some $0 \le \alpha < 1$, the function $|f(x)x^\alpha|$ is bounded on the interval $(0, A]$, then Jackson integral is defined as [7]

$$\int f(x)\,d_q x = (1-q)\,x\sum_{i=0}^{\infty}q^i f\left(q^i x\right). \tag{8}$$

The above expression converges to a function $F(x)$ on $(0, A]$, which is a q-antiderivative of $f(x)$. Suppose $0 < a < b$; the definite q-integral is defined as

$$\begin{aligned}
\int_0^b f(x)\,d_q x &= (1-q)\,b\sum_{i=0}^{\infty}q^i f\left(q^i b\right), \\
\int_a^b f(x)\,d_q x &= \int_0^b f(x)\,d_q x - \int_0^a f(x)\,d_q x.
\end{aligned} \tag{9}$$

We need to apply some properties of the q-Bernoulli polynomials to prepare the approximation conditions. These properties are listed below as a lemma.

Lemma 1. *The following statements hold true:*

$$\text{(a)} \quad D_q\left(\beta_{n,q}(t)\right) = [n]_q\,\beta_{n-1,q}(t), \tag{10}$$

$$\text{(b)} \quad \beta_{n,q}(t) = \sum_{k=0}^{n}\binom{n}{k}_q \beta_{k,q}t^{n-k}, \tag{11}$$

$$\text{(c)} \quad \sum_{k=0}^{n}\binom{n}{k}_q \frac{\beta_{k,q}}{[n-k+1]_q} = \delta_{0,n} \qquad \textit{where } \delta \textit{ is kronecker delta}, \tag{12}$$

$$\text{(d)} \quad \int_0^1 \beta_{n,q}(t)\,d_q t = 0 \quad n \neq 0. \tag{13}$$

Proof. Taking q-derivative from both sides of (2) to prove part (a). If we apply Cauchy product for generating formula (11), it leads to (b). Put $x = 0$ at part (b) and change boundary of summation to reach (c). If we take Jackson integral directly from $\beta_{n,q}(t)$ with the aid of part (c), we can reach (d). □

A few numbers of these polynomials and related q-Bernoulli numbers can be expressed as follows:

$$\begin{aligned}
\beta_{0,q} &= 1, \\
\beta_{0,q}(t) &= 1, \\
\beta_{1,q} &= -\frac{1}{[2]_q}, \\
\beta_{1,q}(t) &= t - \frac{1}{[2]_q}, \\
\beta_{2,q} &= \frac{q^2}{[3]_q!}, \\
\beta_{2,q}(t) &= t^2 - t + \frac{q^2}{[3]_q!}, \\
\beta_{3,q} &= \frac{q^3(1-q)}{[5]_q + [4]_q - 1}, \\
\beta_{3,q}(t) &= t^3 - \frac{[3]_q}{[2]_q}t^2 + \frac{q^2}{[2]_q}t + \frac{q^3(1-q)}{[5]_q + [4]_q - 1}.
\end{aligned} \tag{14}$$

Definition 2. q-Bernoulli polynomials of two variables are defined as a generating function as follows:

$$\frac{te_q(xt)\,e_q(yt)}{e_q(t)-1}=\sum_{n=0}^{\infty}\beta_{n,q}(x,y)\frac{t^n}{[n]_q!},\quad |t|<2\pi. \tag{15}$$

A simple calculation of generating function leads us to

$$\begin{aligned}te_q(xt)+\frac{t}{e_q(t)-1}e_q(tx)&=\frac{te_q(xt)}{e_q(t)-1}e_q(t)\\&=\sum_{n=0}^{\infty}\beta_{n,q}(x,1)\frac{t^n}{[n]_q!},\quad |t|<2\pi.\end{aligned} \tag{16}$$

The LHS can be written as

$$\sum_{n=0}^{\infty}\frac{t^{n+1}x^n}{[n]_q!}+\sum_{n=0}^{\infty}\beta_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{17}$$

By comparing the nth coefficients, we drive to difference equations as follows:

$$\beta_{n,q}(x,1)-\beta_{n,q}(x)=[n]_q!x^{n-1}\quad n\geq 1. \tag{18}$$

If we put $x=0$, then $\beta_{n,q}(0,1)-\beta_{n,q}(0)=\delta_{n,1}$.

2. q-Analogue of Euler-Maclaurin Formula

This section introduces q-operator to find q-analogue of Euler-Maclaurin formula. The q-analogue of Euler-Maclaurin formula has been studied in [9]. The authors of [9] applied q-integral by parts to reach q-analogue of Euler-Maclaurin formula. We can not apply that approach to approximation, because it was written in terms of $p(x)=\beta_{n,q}(x-[x])$. Moreover, the errors have not been studied and our q-operator, which is totally new, leads us to the more applicable function. That is why, this study situates q-Taylor theorem first [8].

Theorem 3. *If the function $f(x)$ is capable of expansion as a convergent power series and if $q\neq$ root of unity, then*

$$f(x)=\sum_{n=0}^{\infty}\frac{(x-a)_q^n}{[n]_q!}\left(D_q^n f\right)(a), \tag{19}$$

where $D_q(f(x))=(f(xq)-f(x))/x(q-1)$ is q-derivative of $f(x)$, and for $x\neq 0$, we can define it at $x=0$ as a normal derivative.

Definition 4. One defines H_q and D_h operators as follows:

$$\begin{aligned}H_q^n(x)&=\frac{h}{1-q}\left(\frac{h}{1-q}+x\right)\\&\quad\cdot\left(\frac{h}{1-q}+[2]_q x\right)\cdots\left(\frac{h}{1-q}+[n-1]_q x\right)(1-q)^n\\&=(x+h)^n\left(\frac{x}{x+h};n\right)_q\\&=\sum_{k=0}^{n}\binom{n}{k}_q q^{k(k-1)/2}(x+h)^{n-k}(-x)^k\\D_h(F(x))&=\frac{F(x+h)-F(x)}{h}=f(x).\end{aligned} \tag{20}$$

In this definition, we assume that $n\in\mathbb{N}$ and the functions and values are well defined ($h\neq 0, q\neq 1$). The first equality holds because of Heine's binomial formula.

For a long time, mathematicians have worked on the area of operators and they solved several types of differential equations by using shifted-operators. The H-operator rules like a bridge between the ordinary expansions and q-analogue of these expansions. We may rewrite several formulae of these areas to the form of q-calculus such as [10–14]. Actually we may write q-expansion of these functions. In a letter, Bernoulli concerned the importance of this expansion for Leibnitz by these words "Nothing is more elegant than the agreement, which you have observed between the numerical power of the binomial and differential expansions, there is no doubt that something is hidden there" [13].

Theorem 5 (fundamental theorem of h-calculus). *If $F(x)$ is an h-derivative of $f(x)$ and $b-a\in h\mathbb{Z}$, one has [7]*

$$\int f(x)\,d_hx=F(b)-F(a), \tag{21}$$

where we define h-integral as follows:

$$\int f(x)\,d_hx=\begin{cases}h\left(f(a)+f(a+h)+\cdots+f(b-h)\right) & \text{if } a<b\\ 0 & \text{if } a=b\\ -h\left(f(b)+f(b+h)+\cdots+f(a-h)\right) & \text{if } a>b.\end{cases} \tag{22}$$

Now in the aid of q-Taylor expansion, we may write $F(x+h)$ as follows:

$$\begin{aligned}F(x+h)&=\sum_{j=0}^{\infty}\frac{((x+h)-x)_q^j}{[j]_q!}\left(D_q^jF\right)(x)\\&=\sum_{j=0}^{\infty}\frac{D_q^jH_q^j}{[j]_q!}(F(x))=e_q\left(D_qH_q\right)(F(x)).\end{aligned} \tag{23}$$

Here, e_q is q-shifted operator and can be expressed as expansion of D_q and H_q. We may assume that $D_h(F(x))=f(x)$ and then

$$f(x)=\frac{F(x+h)-F(x)}{h}=\frac{e_q\left(D_qH_q\right)-1}{H_q}F(x). \tag{24}$$

Since the Jackson integral is q-antiderivative, we have

$$\frac{H_qD_q}{e_q\left(D_qH_q\right)-1}\int f(x)\,d_qx=F(x). \tag{25}$$

And in the aid of (2), the left hand side of the equation can be expressed as a q-Bernoulli numbers; therefore

$$F(x) = \sum_{n=0}^{\infty} \beta_{n,q} \frac{(H_q D_q)^n}{[n]_q!} \int f(x)\, d_q x \tag{26}$$

$$\begin{aligned} &= \int f(x)\, d_q x - \frac{h}{[2]_q} f(x) \\ &+ \sum_{n=1}^{\infty} \frac{\beta_{n,q}}{[n]_q!} \left(D_q^{n-1} f(x)\right) H_q^n(x) \end{aligned} \tag{27}$$

$$\begin{aligned} &= \int f(x)\, d_q x - \frac{h}{[2]_q} f(x) \\ &+ \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-1)/2} (h+x)^{n-k} (-x)^k \right) \\ &\cdot \frac{\left(D_q^{n-1} f(x)\right) \beta_{n,q}}{[n]_q!}. \end{aligned} \tag{28}$$

Thus we can state the following q-analogue of Euler-Maclaurin formula.

Theorem 6. *If the function $f(x)$ is capable of expansion as a convergent power series and $f(x)$ decreases so rapidly with x such that all normal derivatives approach zero as $x \to \infty$, then one can express the series of function $f(x)$ as follows:*

$$\begin{aligned} \sum_{n=a}^{\infty} f(n) &= \int_a^{\infty} f(x)\, d_q x + \frac{1}{[2]_q} (f(a)) \\ &- \lim_{b\to\infty} \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}_q \right. \\ &\left. \cdot q^{k(k-1)/2} \left((1+b)^{n-k} (-b)^k - (1+a)^{n-k} (-a)^k\right) \right) \\ &\cdot \frac{h^n \left(D_q^{n-1} f(a)\right) \beta_{n,q}}{[n]_q!}. \end{aligned} \tag{29}$$

Proof. In the aid of Theorem (12) and (24), if we suppose that $h = 1$ and $b - a \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{n=a}^{b-1} f(n) &= \int_a^b f(x)\, d_1 x = F(b) - F(a) \\ &= \int_a^b f(x)\, d_q x - \frac{1}{[2]_q} (f(b) - f(a)) \\ &+ \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}_q \right. \\ &\left. \cdot q^{k(k-1)/2} \left((1+b)^{n-k} (-b)^k - (1+a)^{n-k} (-a)^k\right) \right) \\ &\cdot \frac{\left(D_q^{n-1} f(b) - D_q^{n-1} f(a)\right) \beta_{n,q}}{[n]_q!}. \end{aligned} \tag{30}$$

Let $f(x)$ decrease so rapidly with x, since $D_q f(x) = (f(xq) - f(x))/x(q-1))$, then by applying the limits when x tends to infinity, $D_q f(x)$ reach zero. Now if normal derivatives of $f(x)$ tend to zero, then q-derivatives are also tending to zero. The same discussion makes $D_q^{n-1} f(b) = 0$ for big enough b. Therefore

$$\begin{aligned} \sum_{n=a}^{\infty} f(n) &= \int_a^{\infty} f(x)\, d_q x + \frac{1}{[2]_q} (f(a)) \\ &- \lim_{b\to\infty} \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}_q \right. \\ &\left. \cdot q^{k(k-1)/2} \left((1+b)^{n-k} (-b)^k - (1+a)^{n-k} (-a)^k\right) \right) \\ &\cdot \frac{h^n \left(D_q^{n-1} f(a)\right) \beta_{n,q}}{[n]_q!}. \end{aligned} \tag{31}$$

□

Example 7. Let $f(x) = x^s$, where s is a positive integer, then we can apply this function at (28), where $a = 0$ and $b - 1 \in \mathbb{N}$

$$\begin{aligned} \sum_{n=0}^{b-1} n^s &= \frac{b^{s+1}}{[s+1]_q} - \left(\frac{b^s}{[2]_q}\right) \\ &+ \sum_{n=2}^{s+1} \left(\sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-1)/2} \left((1+b)^{n-k} (-b)^k\right) \right) \\ &\cdot \binom{s}{n-1}_q \frac{b^{s-n+1} \beta_{n,q}}{[n]_q}. \end{aligned} \tag{32}$$

This relation shows the sum of power in the combination of q-Bernoulli numbers. When $q \to 1$ from the right side, we have an ordinary form of this relation. For the another forms of sum of power, see [15].

Example 8. Let $f(x) = e_q^{-x} = 1/E_q^x$, then this function decreases so rapidly with x such that all normal derivatives approach zero as $x \to \infty$. For confirming this, let us mention that E_q^x, for some fixed $0 < |q| < 1$ and $|-x| < 1/|1-q|$, is increasing rapidly, since $(d/dx)(E_q^x) = \sum_{j=1}^{\infty}(1-q)q^j\prod_{k=0,\,k\neq j}^{\infty}(1+(1-q)q^k x) > 0$.

$$\sum_{n=0}^{\infty} e_q^{-n} = 1 + \frac{1}{[2]_q} - \lim_{b\to\infty}\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q q^{k(k-1)/2}\left((1+b)^{n-k}(-b)^k\right)\right) \cdot \frac{\beta_{n,q}}{[n]_q!}. \tag{33}$$

Moreover, this is q-analogue of $\sum_{n=0}^{\infty} e^{-n} = -1/(e^{-1}-1) = 3/2 + \sum_{n=1}^{\infty}(\beta_{2n}/(2n)!)$, where β_{2n} is a normal Bernoulli number that is generated by $-1/(e^{-1}-1)$.

3. Approximation by q-Bernoulli Polynomial

We know that the class of q-Bernoulli polynomials are not in a form of orthogonal polynomials. In addition, we may apply Gram-Schmidt algorithm to make these polynomials orthonormal, then, according to Theorem 8.11 [16], we have the best approximation in the form of Fourier series. Now, instead of using that algorithm, we apply properties of Lemma 1 to achieve an approximation. Let $H = L_q^2[0,1] = \{f_q : [0,1] \to [0,1] \mid \int_0^1 |f_q^2(t)|d_q t < \infty\}$ be the Hilbert space [17], then $\beta_{n,q}(x) \in H$ for $n = 0,\ldots,N$ and $Y = \operatorname{Span}\{\beta_{0,q}(x), \beta_{1,q}(x), \ldots, \beta_{N,q}(x)\}$ is finite dimensional vector subspace of H. Unique best approximation for any arbitrary elements of H like h is $\widehat{h} \in Y$ such that for any $y \in Y$ the inequality $\|h-\widehat{h}\|_{2,q} \le \|h-y\|_{2,q}$, where the norm is defined by $\|f\|_{2,q} := (\int_0^1 |f_q^2(t)|d_q t)^{1/2}$. Following proposition determine the coefficient of q-Bernoulli polynomials, when we estimate any $f \in L_q^2[0,1]$ by truncated q-Bernoulli series. In fact, in order to see how well a certain partial sum approximates the actual value, we would like to drive a formula similar to (25), but with the infinite sum on the right hand side repla by the Nth partial sum S_N, plus an additional term R_N.

Suppose $a \in \mathbb{Z}$ and $b = a+1$. Consider the Nth partial sum

$$S_N = \sum_{k=0}^{N}\frac{\beta_{k,q}}{[k]_q!}\left(H_q^k D_q^k(f)(a+1) - H_q^k D_q^k(f)(a)\right). \tag{34}$$

Let $g(x,y) = \beta_{N,q}(x,y)/[N]_q!$, then in the aid of Lemma 1 we have $D_q^{n-k} g(x,y) = \beta_{k,q}(x,y)/[k]_q!$ and we have

$$\begin{aligned}
S_N &= \sum_{k=0}^{N}\left(\frac{\beta_{k,q}(0,0)}{[k]_q!}H_q^k D_q^k(f)(a+1) - \frac{\beta_{k,q}(0,1)}{[k]_q!}H_q^k D_q^k(f)(a)\right) - D_q(f)(a) \\
&= \sum_{k=0}^{N}\left(D_q^{N-k} g(0,0) H_q^k D_q^k(f)(a+1) - D_q^{N-k} g(0,1) H_q^k D_q^k(f)(a)\right) - D_q(f)(a) \\
&= D_q(f)(a) - \sum_{k=0}^{N} D_q^{N-k} g(0,x) H_q^k D_q^k(f)(a+1-x)\Big|_{x=0}^{x=1}.
\end{aligned} \tag{35}$$

By taking q-derivative of the summation, we may rewrite this partial sum by integral presentation. Moreover this relation gives a boundary for $R_N(x)$.

Proposition 9. *Let $f \in L_q^2[0,1]$ and be estimated by truncated q-Bernoulli series $\sum_{n=0}^{N} C_n\beta_{n,q}(x)$. Then C_n coefficients for $n = 0,1,\ldots,N$ can be calculated as an integral form $C_n = (1/[n]_q!)\int_0^1 D_q^{(n)} f(x)d_q x$. In addition one can write $f(x)$ in the following form:*

$$f(x) = \int_0^1 f(x)\,d_q x - \frac{h}{[2]_q}f(x) + \sum_{n=1}^{N}\left(\frac{1}{[n]_q!}\int_0^1 D_q^{(n)} f(x)\,d_q x\right)\beta_{n,q}(x) + R_{q,n}(x). \tag{36}$$

Moreover $R_{q,n}(x)$ as a reminder part is bounded by $(2^n/[N]_q!)\sup_{x\in[0,1]}|\beta_{n,q}(x)|\sup_{x\in[0,1]}|D_q^{(n)} f(x)|$.

Proof. In fact we approximate $f(x)$ as a linear combination of q-Bernoulli polynomials and we assume that $f(x) \simeq \sum_{n=0}^{N} C_n\beta_{n,q}(x)$, so taking the Jackson integral from both sides leads us to

$$\begin{aligned}
\int_0^1 f(x)\,d_q x &\simeq \sum_{n=0}^{N} C_n\int_0^1 \beta_{n,q}(x)\,d_q x \\
&= C_0\int_0^1 \beta_{0,q}(x)\,d_q x + C_1\int_0^1 \beta_{1,q}(x)\,d_q x + \cdots + C_N\int_0^1 \beta_{N,q}(x)\,d_q x.
\end{aligned} \tag{37}$$

In the aid of Lemma 1 part (d), all the terms at the right side except the first one have to be zero. We calculate $\beta_{0,q}(x) = 1$, so $C_0 = (1/[0]_q!)\int_0^1 f(x)d_q x$. Using part (a) of that lemma for q-derivative of q-Bernoulli polynomial gathered by taking q-derivative of $f(x)$ leads to

$$\begin{aligned}
\int_0^1 D_q(f(x))\,d_q x &\simeq \sum_{n=0}^{N} C_n [n]_q! \int_0^1 \beta_{n-1,q}(x)\,d_q x \\
&= C_1\int_0^1 \beta_{0,q}(x)\,d_q x = C_1.
\end{aligned} \tag{38}$$

Repeating this procedure n-times yields the form of C_n as we mention it at theorem. We mention that if $x \in [0,1]$ and $0 < |q| < 1$, then $H_q^n(x)$ for $h = 1$ is bounded by 2^n. In addition, the reminder part can be presented by integral forms and this boundary can be found easily. □

3.1. Further Works. In this paper, we introduced a proper tool to approximate a given capable function by combinations of q-Bernoulli polynomials. In spite of a lot of investigations on q-Bernoulli polynomials, approximation properties of the q-Bernoulli polynomials are not studied. We may apply these results to solve q-difference equation, q-analogue of the things that is done in [3] or [4]. The techniques of H-operator can be applied in operator theory.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R. Goldman, P. Simeonov, and Y. Simsek, "Generating functions for the q-Bernstein bases," *SIAM Journal on Discrete Mathematics*, vol. 28, no. 3, pp. 1009–1025, 2014.

[2] E. Tohidi and A. Kiliçman, "A collocation method based on the bernoulli operational matrix for solving nonlinear BVPs which arise from the problems in calculus of variation," *Mathematical Problems in Engineering*, vol. 2013, Article ID 757206, 2013.

[3] E. Tohidi and M. Khorsand Zak, "A new matrix approach for solving second-order linear matrix partial differential equations," *Mediterranean Journal of Mathematics*, vol. 13, no. 3, pp. 1353–1376, 2016.

[4] B. Zogheib and E. Tohidi, "A new matrix method for solving two-dimensional time-dependent diffusion equations with Dirichlet boundary conditions," *Applied Mathematics and Computation*, vol. 291, pp. 1–13, 2016.

[5] L. Carlitz, "q-Bernoulli numbers and polynomials," *Duke Mathematical Journal*, vol. 15, pp. 987–1000, 1948.

[6] N. I. Mahmudov and M. Momenzadeh, "On a class of q -Bernoulli, q -Euler, and q -genocchi polynomials," *Abstract and Applied Analysis*, vol. 2014, Article ID 696454, 2014.

[7] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, NY, USA, 2002.

[8] T. Ernst, "The history of q-calculus and a new method," USA:Uppsala.

[9] A. S. Hegazi and M. Mansour, "A note on q-Bernoulli numbers and polynomials," *Journal of Nonlinear Mathematical Physics*, vol. 13, no. 1, pp. 9–18, 2006.

[10] R. Carmichael, "A treatise on the calculus of operations," *Longman, brown, green, and longmans*, pp. 92–103, 1855.

[11] P. E. Ricci and I. Tavkhelidze, "An introduction to operational techniques and special polynomials," *Journal of Mathematical Sciences*, vol. 157, no. 1, pp. 161–189, 2009.

[12] C. Cassisa, P. E. Ricci, and I. Tavkhelidze, "Operational methods and solutions of boundary-value problems with periodic data," *Journal of Mathematical Sciences*, vol. 157, no. 1, pp. 85–97, 2009.

[13] H. Davis, *The Thory of Linear Operators*, Principia, Elsah, Ill, USA, 1936.

[14] G. Boole, "A treatise on the calculus of finite differences," England, London, 1880.

[15] O. Chan and D. Manna, "A new q-analogue for Bernoulli numbers," *Ams clas*, vol. 11, no. 68, pp. 1–6, 2010.

[16] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill Book Co., New York, NY, USA, 1964.

[17] M. H. Annaby and Z. S. Mansour, *q-Fractional Calculus and Equation*, Springer, NY, USA, 2012.