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Research Article

Axioms for Consensus Functions on the n-Cube

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A p value of a sequence $\pi = (x_1, x_2, \ldots, x_k)$ of elements of a finite metric space (X, d) is an element x for which $\sum_{i=1}^{k} d^p(x, x_i)$ is minimum. The ℓ_p–function with domain the set of all finite sequences on X and defined by $\ell_p(\pi) = \{x\colon x \text{ is a } p \text{ value of } \pi\}$ is called the ℓ_p–function on (X, d). The ℓ_1 and ℓ_2 functions are the well-studied median and mean functions, respectively. In this note, simple characterizations of the ℓ_p–functions on the n-cube are given. In addition, the center function (using the minimax criterion) is characterized as well as new results proved for the median and antimedian functions.

1. Introduction

A *consensus function* (aka *location function*) on a finite connected graph $G = (X, E)$ is a mapping $L : X^* \to 2^X \setminus \{\emptyset\}$, where 2^X denotes the set of all subsets of X, and $X^* = \bigcup_{k\geq 1} X^k$ with $X^k = \overbrace{X \times \cdots \times X}^{k \text{ times}}$. The elements of X^* are called *profiles* and a generic one of *length* k is denoted by $\pi = (x_1, x_2, \ldots, x_k)$. Let d denote the usual geodesic distance, where $d(x, y)$ is the length of a minimum length path joining vertices x and y. Suppose the graph $G = (X, E)$ represents the totality of possible locations. Then a profile $\pi = (x_1, \ldots, x_k)$ is formed where x_i represents the best location from the point-of-view of client (voter, customer, and user) i. A typical approach in location theory is to find those vertices (locations) in X that are "closest" to the profile π. There has been much work in this area of research, ranging from practical computational methods to more theoretical aspects. Since Holzman's paper in 1990 [1], there have been many axiomatic studies of the procedures themselves which resulted in a much better understanding of the process of location (for a small sample, see [2–4] and references within). Now suppose the vertex set X is the set of all linear orders (preference ranking) on a given set of alternatives. In this consensus situation, a profile $\pi = (x_1, \ldots, x_k)$ could represent the collection of ballots of the voters labeled by the set $\{1, \ldots, k\}$; that is, x_i is the preferred ranking of alternatives by voter i. Here a closest vertex to π would represent the entire group's preferred consensus ranking. Many references for this classical situation can be found in [5] and other books on voting theory. Another classic situation, and one pertinent to our study, is the process of selecting a committee from a slate of n candidates. Here each of k voters is to nominate a subset of candidates, so a ballot is simply a profile $\pi = (x_1, \ldots, x_k)$ where each x_i is a subset of the candidates [6, 7]. The vertices of the graph G are the subsets of candidates and the committee consensus function will return one or more subsets closest to the profile.

Four popular measures of the closeness, or remoteness, of a vertex x to a profile $\pi = (x_1, \ldots, x_k)$ are as follows:

(1) The *eccentricity* of x, $e(x, \pi) = \max\{d(x, x_1)d(x, x_2), \ldots, d(x, x_k)\}$

(2) The *status* of x, $S_\pi(x) = \sum_{i=1}^{k} d(x, x_i)$

(3) The *square status* of x, $SS_\pi(x) = \sum_{i=1}^{k} d^2(x, x_i)$

(4) The ℓ_p *status* of x, $\ell_p S_\pi(x) = \sum_{i=1}^{k} d^p(x, x_i)$

The consensus functions based on the these measures of remoteness have been defined as follows:

(a) The *center function*, denoted by Cen, is defined by

$$\mathrm{Cen}(\pi) = \{x \in X : e(x, \pi) \text{ is minimum}\}. \tag{1}$$

(b) The *median function*, denoted by Med, is defined by

$$\mathrm{Med}(\pi) = \{x \in X : S_\pi(x) \text{ is minimum}\}. \tag{2}$$

(c) The *mean function*, denoted by Mean, is defined by

$$\mathrm{Mean}(\pi) = \{x \in X : SS_\pi(x) \text{ is minimum}\}. \tag{3}$$

(d) The ℓ_p-*function*, denoted by ℓ_p, is defined by

$$\ell_p(\pi) = \{x \in X : \ell_p S_\pi(x) \text{ is minimum}\}. \tag{4}$$

The median and mean functions are special cases of the ℓ_p–function, but earlier work [8–10] shows a striking difference between the case of $p = 1$ and $p > 1$.

In this paper we focus on consensus functions on the *n-dimensional hypercube* $Q_n = (X, E)$ whose vertex set is $X = \{(w_1, \dots, w_n) : w_i \in \{0, 1\}\}$. Of course the natural realization of Q_n is the set of all subsets of an n-element set. Recall that, for $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ vertices in Q_n, uv is an edge of Q_n if and only if $\sum_{i=1}^{n} |u_i - v_i| = 1$. We set $u \le v$ if and only if $u_i \le v_i$ for all i. Let d be the usual Hamming distance, where $d(u, v) = \sum_{i=1}^{n} |u_i - v_i|$, so that uv is an edge if and only if $d(u, v) = 1$. Let $\oplus$ denote the addition *modulo* 2, and define $u \oplus v = (u_1 \oplus v_1, \dots, u_n \oplus v_n)$. For a profile $\pi = (x_1, \dots, x_k)$ and $u \in Q_n$ let $\pi \oplus u = (x_1 \oplus u, \dots, x_k \oplus u)$. Let $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$. Note that $x \oplus x = \mathbf{0}$ for all $x \in Q_n$. Also it is easy to see that, for x, y and z vertices in Q_n, $d(x, y) = d(x \oplus z, y \oplus z)$. We set $e_j \in Q_n$ to be the vertex with 0's everywhere except 1 in the jth coordinate. So, for example, in Q_5

$$\begin{aligned} &(0,0,1,1,0) = e_3 \oplus e_4, \\ &(0,0,1,0,1) \oplus e_3 = (0,0,0,0,1) \\ &(0,1,0,1,1) \oplus (0,0,1,1,0) = (0,1,1,0,1) \\ &\quad = e_2 \oplus e_3 \oplus e_5. \end{aligned} \tag{5}$$

Let $\langle \pi \rangle$ denote the subgraph induced by the vertices comprising π. Note that $\langle \pi \oplus v \rangle$ is isomorphic to $\langle \pi \rangle$ for all $v \in Q_n$, and so intuitively $\langle \pi \oplus v \rangle$ is simply a "translation" of $\langle \pi \rangle$ to another position within Q_n. Our goal is to use the particular structure of Q_n to present a very simple unifying approach to give axiomatic characterizations of the consensus functions Cen, Med, and ℓ_p on these graphs. Mulder and Novick [10, 11] have given an elegant set of axioms characterizing the function Med on all median graphs (of which Q_n is a special case) whereas our axioms are essentially straightforward properties that follow from the definitions. At present the most general graph for which characterizations exist for Cen, Mean, and ℓ_p is a tree [9, 12–14]. An interesting weighted version of Cen on Q_n is studied in [6].

We mention that the following results can be framed in the more abstract context of finite Boolean algebras, as it is done in [15–17]. We prefer to work in the more specific situation of the n-cube where properties become quite easy to visualize, and yet we are working without loss of generality because every finite Boolean algebra is isomorphic to an n-cube.

2. The Axioms and Characterizations of Cen, Med, and ℓ_p-Function

In this section we give two very simple properties that will allow us to establish a general result that can be used to give a new way to view Cen, Med, and ℓ_p defined on Q_n. Let $f : X^* \to 2^X \setminus \{\emptyset\}$ be a consensus function on $Q_n = (X, E)$. Our key axiom for a consensus function f is the following.

Translation (T). For any profile π and vertices u and v of Q_n,

$$\begin{gathered} u \in f(\pi) \\ \text{implies that } u \oplus v \in f(\pi \oplus v). \end{gathered} \tag{6}$$

Note that this is equivalent to $u \in f(\pi)$ if and only if $u \oplus v \in f(\pi \oplus v)$.

Now let f and g be consensus functions on Q_n and let x_0 be a vertex. We say f and g *agree* at x_0 if for any profile π

$$x_0 \in f(\pi) \quad \text{iff } x_0 \in g(\pi). \tag{7}$$

Theorem 1. *If the consensus functions f and g on Q_n both satisfy (T) and agree at a vertex x_0, then $f = g$.*

Proof. Let π be a profile and $v \in X$. Then there exists $v' \in X$ such that $v \oplus v' = x_0$. Since f satisfies (T), we have

$$v \in f(\pi) \quad \text{iff } v \oplus v' = x_0 \in f(\pi \oplus v'). \tag{8}$$

Because f and g agree at x_0,

$$x_0 \in f(\pi \oplus v') \quad \text{iff } x_0 \in g(\pi \oplus v'). \tag{9}$$

Since g satisfies (T),

$$v \oplus v' = x_0 \in g(\pi \oplus v') \quad \text{iff } v \in g(\pi). \tag{10}$$

Hence $v \in f(\pi)$ if and only if $v \in g(\pi)$. □

Theorem 1 implies that if f and g are consensus functions on Q_n and both satisfy (T); then $f = g$ if the conditions placing $\mathbf{0}$ in $f(\pi)$ are the same as the conditions placing $\mathbf{0}$ in $g(\pi)$.

As observed before, $d(x, y) = d(x \oplus z, y \oplus z)$ for x, y, and z vertices in Q_n. Using this and the definitions it is easy to see that Cen, Med, and ℓ_p all satisfy (T). Therefore, characterizations will follow once the conditions are obtained for when $\mathbf{0} \in \mathrm{Cen}(\pi)$, $\mathbf{0} \in \mathrm{Med}(\pi)$, and $\mathbf{0} \in \ell_p(\pi)$. We present these results in a series of lemmas and corollaries.

Let $u \in Q_n$ and set $\|u\| = d(\mathbf{0}, u)$, that is, the number of ones that appear in the representation u as a vertex of Q_n. Let $\pi = (x_1, x_2, \ldots, x_k)$ be a profile on Q_n. Then $\|\pi\|$ is defined to be

$$\|\pi\| = \max\{\|x_1\|, \|x_2\|, \ldots, \|x_k\|\}. \tag{11}$$

Lemma 2. *Let Cen be the center function on Q_n and π a profile. Then*

$$\mathbf{0} \in Cen(\pi) \quad \text{iff } \|\pi\| \le \|\pi \oplus u\|, \ \forall u \in Q_n. \tag{12}$$

Proof. The result is clear because $d(x, y) = d(x \oplus z, y \oplus z)$ in Q_n, and $e(\mathbf{0}, \pi) = \|\pi\|$ for any profile π. □

Corollary 3. *Let f be a consensus function on Q_n. Then $f = Cen$ if and only f satisfies (T) and for every profile π and $u \in Q_n$*

$$\mathbf{0} \in f(\pi) \quad \text{iff } \|\pi\| \le \|\pi \oplus u\|. \tag{13}$$

Mulder and Novick [10] give an elegant characterization of Med on Q_n, which was extended to all median graphs in [11]. We will give another characterization using the approach given by Theorem 1. For a profile $\pi = (x_1, \ldots, x_k)$ let $x_i = (x_1^i, \ldots, x_n^i)$. The next result has been noted in [10].

Lemma 4. *Let Med be the median function on Q_n and $\pi = (x_1, \ldots, x_k)$ a profile. Then*

$$\mathbf{0} \in Med(\pi) \quad \text{iff } \sum_{j=1}^{k} x_i^j \le \frac{k}{2} \ \forall i. \tag{14}$$

Corollary 5. *Let f be a consensus function on Q_n. Then $f = Med$ if and only f satisfies (T) and for any profile $\pi = (x_1, \ldots, x_k)$,*

$$\mathbf{0} \in f(\pi) \quad \text{iff } \sum_{j=1}^{k} x_i^j \le \frac{k}{2} \ \forall i. \tag{15}$$

For the function ℓ_p it is easy to see from the definitions that, for any profile π and a in Q_n,

$$\mathbf{0} \in \ell_p(\pi) \quad \text{iff } a = \mathbf{0} \oplus a \in \ell_p(\pi \oplus a). \tag{16}$$

As in [17] we consider the *p-characteristic* of a profile $\pi = (x_1, x_2, \ldots, x_k)$ to be the number

$$\mathrm{Char}_p(\pi) = \sum_{i=1}^{n} \|x_i\|^p. \tag{17}$$

Lemma 3.12 in [17] gives the following result.

Lemma 6. *Consider the function ℓ_p on Q_n, and let $\pi = (x_1, \ldots, x_k)$ be a profile. Then*

$$\begin{gathered}\mathbf{0} \in \ell_p(\pi) \\ \text{iff } Char_p(\pi) \le Char_p(\pi \oplus a) \ \text{ for every } a \text{ in } Q_n.\end{gathered} \tag{18}$$

Corollary 7. *Let f be a consensus function on Q_n. Then $f = \ell_p$ if and only f satisfies (T) and for any profile $\pi = (x_1, \ldots, x_k)$,*

$$\begin{gathered}\mathbf{0} \in f(\pi) \\ \text{iff } Char_p(\pi) \le Char_p(\pi \oplus a) \ \text{ for every vertex } a \text{ in } Q_n.\end{gathered} \tag{19}$$

Here are three other examples of consensus functions that satisfy the Translation property. However it is clear that these functions would not be useful in committee elections or as location functions, for instance.

Example 1. Let f_1 be the consensus function on Q_n defined by $f(\pi) = \{x_1\}$ for any profile $\pi = (x_1, \ldots, x_k)$. That is, f_1 is a standard projection function. Then clearly f_1 satisfies (T).

Example 2. Let f_2 be the consensus function on Q_n defined by $f_2(\pi) = X$ for all profiles π. That is, f_2 is the constant function with ouput being the entire vertex set X. Then f_2 satisfies (T), and moreover it can be easily shown that it is the only constant function that satisfies (T).

Example 3. Let f_3 be the consensus function on Q_n defined by $f_3(\pi) = \{\pi\}$ for all π where $\{\pi\}$ is the set of vertices appearing in the profile π. Then clearly f_3 satisfies (T).

The function f_2 allows us to see some of the implications of imposing (T). First we need to recall one of the crucial axioms for the characterization of the consensus function Med [10, 11, 18].

Consistency (C). The consensus function f satisfies (C) if, for profiles π_1 and π_2,

$$\begin{gathered} f(\pi_1) \cap f(\pi_2) \neq \emptyset \\ \text{implies } f(\pi_1 \pi_2) = f(\pi_1) \cap f(\pi_2). \end{gathered} \tag{20}$$

Proposition 8. *A consensus function f on Q_n satisfies (T), (C), and*

$$\bigcap_{x \in X} f(x) \neq \emptyset \tag{21}$$

if and only if $f = f_2$.

Proof. Clearly f_2 satisfies the conditions, so now let f be a consensus function that satisfies (T), (C), and the intersection condition. Let $v \in f(x)$ for all $x \in X$. Then since f satisfies (T) we have $v \oplus x \in f(x \oplus x) = f(\mathbf{0})$ for all $x \in$ X. Now let w be an arbitrary vertex. Then $w = v \oplus (v \oplus w) \in f(\mathbf{0})$ and thus $f(\mathbf{0}) = X$. So if z is any vertex in X, $z \oplus x \in f(\mathbf{0})$ and since f satisfies (T) we have

$$z = (z \oplus x) \oplus x \in f(\mathbf{0} \oplus x) = f(x). \tag{22}$$

Therefore $f(x) = X$ for all $x \in X$, which means that $f(\pi) = X$ for all profiles π of length 1. Using (C) and induction we conclude that $f(\pi) = X$ for all profiles π, that is, $f = f_2$. □

3. Alternative Characterizations of the Median and Antimedian Functions on Q_n

For any profile $\pi = (v_1, \ldots, v_k)$ such that

$$v_i = \left(x_1^i, \ldots, x_n^i\right) \in \{0,1\}^n \quad (23)$$

for $i = 1, \ldots k$, let $\mathrm{Maj}(\pi) = (w_1, \ldots, w_n)$ be the vertex in X such that

$$w_i = 1 \quad \text{iff} \sum_{j=1}^{k} x_i^j > \frac{k}{2} \quad (24)$$

for $i = 1, \ldots, n$. We will say that a location function f satisfies the condition (Maj) if

$$\mathrm{Maj}(\pi) \in f(\pi) \quad (25)$$

for any profile π. We have previously noted that the median function satisfies (T) and we will show below that, as expected, Med satisfies (Maj). However, there are other location functions that satisfy these two conditions, such as f_2, for example. But, arguably f_2 is not a very reasonable method of consensus or location. So our next step is to invoke a condition that restricts the range of a location function.

For any profile $\pi = (v_1, \ldots, v_k)$ such that

$$v_i = \left(x_1^i, \ldots, x_n^i\right) \in \{0,1\}^n \quad (26)$$

for $i = 1, \ldots, k$ define the *Condorcet score* of π to be

$$\mathrm{Cs}(\pi) = \left|\left\{ i : \sum_{j=1}^{k} x_i^j = \frac{k}{2} \right\}\right|. \quad (27)$$

Observe that if the profile length k is odd, then $\mathrm{Cs}(\pi) = 0$. A location function f satisfies *Restricted Range* (RR) if

$$|f(\pi)| \le 2^{\mathrm{Cs}(\pi)} \quad (28)$$

for any profile π.

We can now give a completely different characterization of Med from that found in [10].

Theorem 9. *Let f be a location function on Q_n. Then f = Med if and only if f satisfies (T), (Maj), and (RR).*

Proof. Assume f = Med. We already know that f satisfies (T), so we only need to show that Med satisfies (Maj) and (RR).

We will follow the notation given above. Let $\pi = (v_1, \ldots, v_k)$ be a profile such that

$$v_i = \left(x_1^i, \ldots, x_n^i\right) \in \{0,1\}^n \quad (29)$$

for $i = 1, \ldots, k$ and let $\mathrm{Maj}(\pi) = (w_1, \ldots, w_n) = w$. Now let $a = (y_1, \ldots, y_n) \neq w$ be such that $y_m \neq w_m$ for some m. First note that, for every j, because w_j and x_j^i are equal for at least $k/2$ of the i's,

$$\sum_{i=1}^{k} \left|y_j - x_j^i\right| \ge \sum_{i=1}^{k} \left|w_j - x_j^i\right|. \quad (30)$$

Since

$$S_\pi(a) = \sum_{i=1}^{k} d(a, v_i) \quad \text{where } d(a, v_i) = \sum_{j=1}^{n} \left|y_j - x_j^i\right| \quad (31)$$

we have

$$\begin{aligned} S_\pi(a) &= \sum_{i=1}^{k}\sum_{j=1}^{n} \left|y_j - x_j^i\right| = \sum_{j=1}^{n}\sum_{i=1}^{k} \left|y_j - x_j^i\right| \\ &\ge \sum_{j=1}^{n}\sum_{i=1}^{k} \left|w_j - x_j^i\right| = S_\pi(w). \end{aligned} \quad (32)$$

Therefore $w \in \mathrm{Med}(\pi)$ and f satisfies (Maj).

Let $u = (u_1, \ldots, u_n)$ be the vertex in X such that

$$u_i = 1 \quad \text{iff} \sum_{j=1}^{k} x_i^j \ge \frac{k}{2} \quad (33)$$

for $i = 1, \ldots, n$. For any vertex $a = (y_1, \ldots, y_n)$ such that $w \le a \le u$ and for any $i \in \{1, \ldots, n\}$ such that $\sum_{j=1}^{k} x_i^j = k/2$ we get that $w_i = 0$, $u_i = 1$, and of course $y_i \in \{0,1\}$. Observe that

$$\sum_{i=1}^{k} \left|y_j - x_j^i\right| = \frac{k}{2} = \sum_{i=1}^{k} \left|w_j - x_j^i\right|. \quad (34)$$

Since $y_i = w_i$ whenever $\sum_{j=1}^{k} x_i^j \neq k/2$ it follows that $S_\pi(a) = S_\pi(w)$ and so $a \in \mathrm{Med}(\pi)$. Moreover, if $b = (z_1, \ldots, z_n)$ is vertex in X such that $z_m \neq w_m$ for some $m \in \{1, \ldots, n\}$ where $\sum_{j=1}^{k} x_m^j \neq k/2$, then

$$\sum_{i=1}^{k} \left|z_m - x_m^i\right| > \frac{k}{2} > \sum_{i=1}^{k} \left|w_m - x_m^i\right|. \quad (35)$$

In this case, $S_\pi(b) > S_\pi(w)$ and so $b \notin \mathrm{Med}(\pi)$. It now follows that

$$\mathrm{Med}(\pi) = \left\{ \mathrm{Maj}(\pi) \oplus \sum_{\alpha \in A} i_\alpha : A \subseteq S \right\}, \quad (36)$$

where

$$S = \left\{ \alpha \in \{1, \ldots, n\} : \sum_{j=1}^{k} x_\alpha^j = \frac{k}{2} \right\}. \quad (37)$$

Therefore, $|\mathrm{Med}(\pi)| = 2^{|S|} = 2^{\mathrm{Cs}(\pi)}$ and hence Med satisfies (RR).

For the converse, assume that f satisfies (T), (Maj), and (RR). We will show that f = Med. Let $\pi = (v_1, \ldots, v_k)$ be a profile. Then, using Theorem 9,

$$v \in \mathrm{Med}(\pi) \text{ iff } \mathbf{0} \in \mathrm{Med}(\pi \oplus v) \text{ iff } \sum_{j=1}^{k} y_i^j \le \frac{k}{2} \quad \forall i, \quad (38)$$

where $v_j \oplus v = (y_1^j, \dots, y_n^j)$ for $j = 1, \dots, k$. Observe that $\mathrm{Maj}(\pi \oplus v) = \mathbf{0}$, and since f satisfies (Maj) it follows that $\mathbf{0} \in f(\pi \oplus v)$. Since f satisfies (T) we get

$$v = \mathbf{0} \oplus v \in f(\pi). \quad (39)$$

It now follows that $\mathrm{Med}(\pi) \subseteq f(\pi)$ for any profile π. Therefore,

$$|\mathrm{Med}(\pi)| \le |f(\pi)| \quad (40)$$

for any profile π. We know that $\mathrm{Med}(\pi) = 2^{\mathrm{Cs}(\pi)}$ and, by (RR), that $|f(\pi)| \le 2^{\mathrm{Cs}(\pi)}$ for any profile π. Hence $f(\pi) = \mathrm{Med}(\pi)$ for any profile π and we are done. □

The three consensus functions we have considered all minimize a criterion in order to produce vertices that are close to a given profile of vertices, and as such are useful in location theory. When finding locations to place noxious entities, it is more appropriate to maximize rather than minimize these objective functions, and the resulting "anti"-functions have also been well-studied. Because we have proved Theorem 9 about the median function, we mention the *antimedian function*, denoted by AM, defined by

$$\mathrm{AM}(\pi) = \{x \in X : S_\pi(x) \text{ is maximum}\}. \quad (41)$$

AM has been characterized on Q_n in [19], but we will give an alternate characterization as a corollary to Theorem 9. As before $\pi = (v_1, \dots, v_k)$ is a profile such that

$$v_i = (x_1^i, \dots, x_n^i) \in \{0,1\}^n \quad (42)$$

for $i = 1, \dots, k$. Let $\mathrm{Min}(\pi) = (m_1, \dots, m_n)$ be the vertex in X such that

$$m_i = 1 \quad \text{iff} \sum_{j=1}^{k} x_i^j < \frac{k}{2} \quad (43)$$

for $i = 1, \dots, n$. We will say that a location function f satisfies condition (Min) if

$$\mathrm{Min}(\pi) \in f(\pi) \quad (44)$$

for any profile π. Corollary 3 now follows from the proof of Theorem 9 in the obvious way by reversing the inequalities.

Corollary 10. *Let f be a location function on Q_n. Then f = AM if and only if f satisfies (T), (Min), and (RR).*

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] R. Holzman, "An axiomatic approach to location on networks," *Mathematics of Operations Research*, vol. 15, no. 3, pp. 553–563, 1990.

[2] P. Hansen and F. S. Roberts, "An impossibility result in axiomatic location theory," *Mathematics of Operations Research*, vol. 21, no. 1, pp. 195–208, 1996.

[3] F. R. McMorris, H. M. Mulder, and R. V. Vohra, "Axiomatic characterization of location functions," in *Advances in Interdisciplinary Applied Discrete Mathematics*, H. Kaul and H. M. Mulder, Eds., pp. 71–91, World Scientific, Singapore, 2011.

[4] P. B. Mirchandani and R. L. Francis, Eds., *Discrete Location Theory*, John Wiley & Sons, New York, NY, USA, 1990.

[5] W. H. E. Day and F. R. McMorris, *Axiomatic consensus theory in group choice and biomathematics*, vol. 29 of *Frontiers in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, USA, 2003.

[6] S. J. Brams, D. M. Kilgour, and M. R. Sanver, "A minimax procedure for electing committees," *Public Choice*, vol. 132, no. 3, pp. 401–420, 2007.

[7] D. M. Kilgour, S. J. Brams, and M. R. Sanver, "How to elect a representative committee using approval balloting," in *Mathematics and Democracy*, B. Simeone and F. Pukelsheim, Eds., pp. 83–95, Springer, Berlin, Germany, 2006.

[8] O. Ortega and G. Kriston, "The median function on trees," *Discrete Mathematics, Algorithms and Applications*, vol. 5, no. 4, Article ID 1350033, 14 pages, 2013.

[9] F. R. McMorris, H. M. Mulder, and O. Ortega, "The ℓ_p-function on trees," *Networks*, vol. 60, no. 2, pp. 94–102, 2012.

[10] H. M. Mulder and B. Novick, "An axiomatization of the median procedure on the n-cube," *Discrete Applied Mathematics*, vol. 159, no. 9, pp. 939–944, 2011.

[11] H. M. Mulder and B. Novick, "A tight axiomatization of the median procedure on median graphs," *Discrete Applied Mathematics*, vol. 161, no. 6, pp. 838–846, 2013.

[12] F. R. McMorris, F. S. Roberts, and C. Wang, "The center function on trees," *Networks*, vol. 38, no. 2, pp. 84–87, 2001.

[13] F. R. McMorris, H. M. Mulder, and O. Ortega, "Axiomatic characterization of the mean function on trees," *Discrete Mathematics, Algorithms and Applications*, vol. 2, no. 3, pp. 313–329, 2010.

[14] H. M. Mulder, M. J. Pelsmajer, and K. B. Reid, "Axiomization of the center function on trees," *The Australasian Journal of Combinatorics*, vol. 41, pp. 223–226, 2008.

[15] A. Kezdy and R. C. Powers, *The Center Function on Boolean Algebras*, Department of Mathematics, University of Louisville, 2001.

[16] O. Ortega and C. Garcia-Martinez, "The median function on Boolean lattices," *Discrete Mathematics, Algorithms and Applications*, vol. 6, no. 4, Article ID 1450056, 21 pages, 2014.

[17] O. Ortega, C. Garcia-Martinez, and K. Adamski, "The ℓ_p function on finite Boolean lattices," *Discrete Mathematics, Algorithms and Applications*, vol. 8, Article ID 1650044, 15 pages, 2016.

[18] F. R. McMorris, H. M. Mulder, and F. S. Roberts, "The median procedure on median graphs," *Discrete Applied Mathematics*, vol. 84, no. 1–3, pp. 165–181, 1998.

[19] K. Balakrishnan, M. Changat, H. M. Mulder, and A. R. Subhamathi, "Axiomatic characterization of the antimedian function on paths and hypercubes," *Discrete Mathematics, Algorithms and Applications*, vol. 4, no. 4, 1250054, 20 pages, 2012.