Hindawi
Abstract and Applied Analysis
Volume 2017, Article ID 3795120, 11 pages
http://dx.doi.org/10.1155/2017/3795120

Research Article

A Variation on Uncertainty Principle and Logarithmic Uncertainty Principle for Continuous Quaternion Wavelet Transforms

Mawardi Bahri[1] and Ryuichi Ashino[2]

[1]*Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia*
[2]*Division of Mathematical Sciences, Osaka Kyoiku University, Osaka 582-8582, Japan*

Correspondence should be addressed to Ryuichi Ashino; ashino@cc.osaka-kyoiku.ac.jp

Received 23 October 2016; Accepted 5 January 2017; Published 30 January 2017

Academic Editor: Lucas Jodar

The continuous quaternion wavelet transform (CQWT) is a generalization of the classical continuous wavelet transform within the context of quaternion algebra. First of all, we show that the directional quaternion Fourier transform (QFT) uncertainty principle can be obtained using the component-wise QFT uncertainty principle. Based on this method, the directional QFT uncertainty principle using representation of polar coordinate form is easily derived. We derive a variation on uncertainty principle related to the QFT. We state that the CQWT of a quaternion function can be written in terms of the QFT and obtain a variation on uncertainty principle related to the CQWT. Finally, we apply the extended uncertainty principles and properties of the CQWT to establish logarithmic uncertainty principles related to generalized transform.

1. Introduction

As it is known, the classical wavelet transform (WT) is a very useful mathematical tool. It has been discussed extensively in the literature and has been proven to be powerful and useful in the communication theory, quantum mechanics, and many other fields [1–4]. Of great interest is the study of the quaternion wavelet transform, which can be considered as a generalization of the WT using quaternion algebra. Some research papers on the continuous and discrete quaternion wavelet transforms have been published. In [5–7], the authors constructed the continuous quaternion wavelet transform (CQWT) using the quaternionic affine group and similitude group, respectively. Several fundamental properties of this extended wavelet transform, which correspond to classical continuous wavelet transform properties, were also investigated. Further, in regard to a numerical concept of the quaternion wavelet transforms, Bayro-Corrochano [8] developed the discrete quaternion wavelet transform and applied it for optical flow estimation. In [9, 10], the authors studied the discrete reduced biquaternion wavelet transform and applied it to multiscale texture classification. Another approach of the CQWT based on a natural convolution of quaternion-valued functions was recently proposed by Akila and Roopkumar [11, 12]. The essential part in the study of the quaternion wavelet transform, as usual, is to establish its Heisenberg type uncertainty principle. It plays an important role in quaternionic signal processing. Based on the Heisenberg type uncertainty principle for the quaternion Fourier transform (QFT) [13, 14], the authors in [7] proposed a component-wise uncertainty principle associated with the CQWT.

Motivated by the authors in [15–17], in the present paper, we propose the directional uncertainty principle related to the CQWT and then apply this uncertainty to obtain a variation on the Heisenberg type uncertainty principle and the logarithmic uncertainty principle in the context of the CQWT. The uncertainty principle describes the relation between the QFT of a quaternion function and its CQWT. These principles are more general forms of Heisenberg's uncertainty principle related to the CQWT [7]. To achieve the results, our first step is to derive the directional uncertainty principle for the QFT using the component-wise QFT uncertainty principle. Due to this principle, we can easily derive directional QFT uncertainty principle using a representation of polar coordinate from the ones proposed in [18]. We then study an important theorem which describes interactions between the CQWT and QFT in frequency domain. Applying the cyclic multiplication of quaternion, we obtain some useful properties of the CQWT. Based on the relationship between the extended Heisenberg uncertainty principle and properties related to the CQWT, we finally establish the logarithmic uncertainty principles associated with the CQWT.

2. Preliminaries

The concept of the quaternion algebra [19, 20] was introduced by Sir Hamilton in 1842 and is denoted by $\mathbb{H}$ in his honor. It is an extension of the complex numbers to a four-dimensional (4-D) algebra. Every element of $\mathbb{H}$ is a linear combination of a real scalar and three imaginary units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ with real coefficients,

$$\mathbb{H} = \left\{ q = q_0 + \mathbf{i}q_i + \mathbf{j}q_j + \mathbf{k}q_k \mid q_0, q_i, q_j, q_k \in \mathbb{R} \right\}, \quad (1)$$

which obey Hamilton's multiplication rules

$$\begin{aligned} \mathbf{ij} &= -\mathbf{ji} = \mathbf{k}, \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i}, \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j}, \\ \mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \end{aligned} \quad (2)$$

For a quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{H}$, q_0 is called the *scalar* (or *real*) part of q denoted by $\mathrm{Sc}(q)$ and $\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is called the *vector* (or *pure*) part of q. The vector part of q is conventionally denoted by $\mathbf{q}$ or $\mathrm{Vec}(q) = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$.

Let $p, q \in \mathbb{H}$ and $\mathbf{p}$, $\mathbf{q}$ be their vector parts, respectively. Equation (2) yields the quaternionic multiplication qp as

$$qp = q_0 p_0 - \mathbf{q} \cdot \mathbf{p} + q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p}, \quad (3)$$

where

$$\begin{aligned} \mathbf{q} \cdot \mathbf{p} &= q_1 p_1 + q_2 p_2 + q_3 p_3, \\ \mathbf{q} \times \mathbf{p} &= \mathbf{i}(q_2 p_3 - q_3 p_2) + \mathbf{j}(q_3 p_1 - q_1 p_3) \\ &\quad + \mathbf{k}(q_1 p_2 - q_2 p_1). \end{aligned} \quad (4)$$

The conjugate $\overline{q}$ of the quaternion q is the quaternion given by

$$\overline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}. \quad (5)$$

It is an anti-involution; that is,

$$\overline{qp} = \overline{p}\,\overline{q}. \quad (6)$$

From (5), we obtain the norm or modulus of $q \in \mathbb{H}$ defined as

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \quad (7)$$

It is not difficult to see that

$$\begin{aligned} \mathrm{Sc}(q) &\le |q|, \\ |\mathbf{q}| &\le |q|. \end{aligned} \quad (8)$$

Using conjugate (5) and the modulus of q, we can define the inverse of $q \in \mathbb{H} \setminus \{0\}$ as

$$q^{-1} = \frac{\overline{q}}{|q|^2}, \quad (9)$$

which shows that $\mathbb{H}$ is a normed division algebra.

Now we notice that

$$\mathrm{Sc}(p\overline{q}) = \frac{1}{2}(p\overline{q} + q\overline{p}) = q_0 p_0 + \mathbf{q} \cdot \mathbf{p}. \quad (10)$$

These will lead to the cyclic multiplication; that is,

$$\mathrm{Sc}(pqr) = \mathrm{Sc}(rqp) = \mathrm{Sc}(qpr), \quad \forall p, q, r \in \mathbb{H}. \quad (11)$$

Any quaternion q may be split up into

$$q = q_+ + q_-, \quad q_\pm = \frac{1}{2}(q \pm \mathbf{i}q\mathbf{j}). \quad (12)$$

The above gives

$$\begin{aligned} q_\pm &= \{(q_0 \pm q_3) + \mathbf{i}(q_1 \mp q_2)\} \frac{1 \pm \mathbf{k}}{2} \\ &= \frac{1 \pm \mathbf{k}}{2} \{(q_0 \pm q_3) + \mathbf{j}(q_2 \mp q_1)\}. \end{aligned} \quad (13)$$

This leads to the following modulus identity:

$$|q|^2 = |q_-|^2 + |q_+|^2. \quad (14)$$

It is convenient to introduce an inner product for two quaternion functions $f, g : \mathbb{R}^2 \to \mathbb{H}$ as follows:

$$(f, g) = \int_{\mathbb{R}^2} f(\mathbf{x}) \overline{g(\mathbf{x})} \, d\mathbf{x}, \quad (15)$$

where the overline indicates the quaternion conjugation of the function. In particular, for $f = g$, we obtain the $L^p(\mathbb{R}^2; \mathbb{H})$-norm

$$\|f\|_p = \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^p \, d\mathbf{x} \right)^{1/p}, \quad 1 \le p \le 2, \quad (16)$$

where

$$|f| = \sqrt{f_0^2(\mathbf{x}) + f_1^2(\mathbf{x}) + f_2^2(\mathbf{x}) + f_3^2(\mathbf{x})}. \quad (17)$$

As a consequence of the inner product (15), we obtain the *quaternion Cauchy-Schwarz* inequality

$$\begin{aligned} \left| \int_{\mathbb{R}^2} \overline{f} g \, d\mathbf{x} \right|^2 &\le \int_{\mathbb{R}^2} |f|^2 \, d\mathbf{x} \int_{\mathbb{R}^2} |g|^2 \, d\mathbf{x}, \\ &\quad \forall f, g \in L^2(\mathbb{R}^2; \mathbb{H}). \end{aligned} \quad (18)$$

Definition 1. A couple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ of nonnegative integers is called a multi-index. One denotes

$$|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2, \quad \boldsymbol{\alpha}! = \alpha_1!\alpha_2!, \tag{19}$$

and, for $\mathbf{x} \in \mathbb{R}^2$,

$$\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2}. \tag{20}$$

Derivatives are conveniently expressed by multi-indices:

$$\partial^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}. \tag{21}$$

Next, we obtain the Schwartz space as (compared to [21])

$$\mathcal{S}\left(\mathbb{R}^2; \mathbb{H}\right) = \left\{ f \in C^{\infty}\left(\mathbb{R}^2, \mathbb{H}\right) : \sup_{\mathbf{x} \in \mathbb{R}^2} \left(1 + |\mathbf{x}|^k\right) \left|\partial^{\boldsymbol{\alpha}} f(\mathbf{x})\right| < \infty \right\}, \tag{22}$$

where $C^{\infty}(\mathbb{R}^2, \mathbb{H})$ is the set of smooth functions from $\mathbb{R}^2$ to $\mathbb{H}$. Elements in the dual space $\mathcal{S}'(\mathbb{R}^2; \mathbb{H})$ of $\mathcal{S}(\mathbb{R}^2; \mathbb{H})$ are called tempered distribution.

3. Quaternion Fourier Transform and Its Heisenberg Uncertainty Principle

3.1. QFT and Properties. In the following, we introduce the (right-sided) QFT and some of its fundamental properties such as Riemann-Lebesgue lemma and continuity.

Definition 2 (right-sided QFT). The (right-sided) quaternion Fourier transform (QFT) of $f \in L^1(\mathbb{R}^2; \mathbb{H})$ is the transform $\mathcal{F}_Q\{f\}: \mathbb{R}^2 \to \mathbb{H}$ given by

$$\mathcal{F}_Q\{f\}(\boldsymbol{\omega}) = \hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\mathbf{x}) e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x}, \quad d\mathbf{x} = dx_1 dx_2, \tag{23}$$

where $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, $\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2$, and the quaternion exponential product $e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2}$ is the quaternion Fourier kernel.

Theorem 3 (inverse QFT). *Suppose that $f \in L^1(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_Q\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})$. Then, the QFT of f is an invertible transform and its inverse is given by*

$$\mathcal{F}_Q^{-1}\left[\mathcal{F}_Q\{f\}\right](\mathbf{x}) = f(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{F}_Q\{f\}(\boldsymbol{\omega}) e^{\mathbf{j}\omega_2 x_2} e^{\mathbf{i}\omega_1 x_1} d\boldsymbol{\omega}, \tag{24}$$

where the quaternion exponential product $e^{\mathbf{j}\omega_2 x_2} e^{\mathbf{i}\omega_1 x_1}$ is called the inverse (right-sided) quaternion Fourier kernel.

Since $\mathcal{S}(\mathbb{R}^2; \mathbb{H}) \subset L^1(\mathbb{R}^2; \mathbb{H})$, the definition of QFT (23) may be extended to the Schwartz space. It is important to note that $\mathcal{F}_Q\{f\}$ is not necessary in $L^1(\mathbb{R}^2; \mathbb{H})$ even if f is in $L^1(\mathbb{R}^2; \mathbb{H})$, so in general $\mathcal{F}_Q\{f\}$ might not be well defined. However, the QFT of a Schwartz quaternion function is also in the Schwartz space.

Applying (23), we have

$$\begin{aligned}\mathcal{F}_Q\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} \left(f_0(\mathbf{x}) + \mathbf{i} f_1(\mathbf{x}) + \mathbf{j} f_2(\mathbf{x}) + \mathbf{k} f_3(\mathbf{x})\right) \\ &\cdot e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x} = \mathcal{F}_Q\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathcal{F}_Q\{f_1\}(\boldsymbol{\omega}) \\ &+ \mathbf{j}\mathcal{F}_Q\{f_2\}(\boldsymbol{\omega}) + \mathbf{k}\mathcal{F}_Q\{f_3\}(\boldsymbol{\omega}).\end{aligned} \tag{25}$$

Now, we define a module of $\mathcal{F}_Q\{f\}(\boldsymbol{\omega})$ as

$$\left|\mathcal{F}_Q\{f\}(\boldsymbol{\omega})\right|_Q = \left(\left|\mathcal{F}_Q\{f_0\}(\boldsymbol{\omega})\right|^2 + \left|\mathcal{F}_Q\{f_1\}(\boldsymbol{\omega})\right|^2 + \left|\mathcal{F}_Q\{f_2\}(\boldsymbol{\omega})\right|^2 + \left|\mathcal{F}_Q\{f_3\}(\boldsymbol{\omega})\right|^2\right)^{1/p}. \tag{26}$$

Furthermore, we obtain the $L^p(\mathbb{R}^2; \mathbb{H})$-norm

$$\left\|\mathcal{F}_Q\{f\}\right\|_{Q,p} = \left(\int_{\mathbb{R}^2} \left|\mathcal{F}_Q\{f\}(\boldsymbol{\omega})\right|_Q^p d\boldsymbol{\omega}\right)^{1/p}. \tag{27}$$

Remark 4. It is worth noting here that if $\mathcal{F}_Q\{f_i\}$, $i = 0, 2, 3$, is real-valued, (26) can be written in the form

$$\left\|\mathcal{F}_Q\{f\}\right\|_{Q,p} = \left\|\mathcal{F}_Q\{f\}(\boldsymbol{\omega})\right\|_p, \tag{28}$$

where

$$\left|\mathcal{F}_Q\{f\}(\boldsymbol{\omega})\right|_Q = \left(\left(\mathcal{F}_Q\{f_0\}(\boldsymbol{\omega})\right)^2 + \left(\mathcal{F}_Q\{f_1\}(\boldsymbol{\omega})\right)^2 + \left(\mathcal{F}_Q\{f_2\}(\boldsymbol{\omega})\right)^2 + \left(\mathcal{F}_Q\{f_3\}(\boldsymbol{\omega})\right)^2\right)^{1/p}. \tag{29}$$

Some important properties of the QFT are stated in the following lemmas.

Lemma 5 (QFT Plancherel). *If $f \in L^1(\mathbb{R}^2; \mathbb{H}) \cap L^2(\mathbb{R}^2; \mathbb{H})$, then*

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left|\mathcal{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} = \int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^2 d\mathbf{x}. \tag{30}$$

Moreover,

$$\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \left|\mathcal{F}_Q\{f\}(\boldsymbol{\omega})\right|_Q^2 d\boldsymbol{\omega} = \int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^2 d\mathbf{x}. \tag{31}$$

Proof. We prove expression (31) of Lemma 5. Using (26), we immediately get

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|_Q^2 d\boldsymbol{\omega} = \frac{1}{(2\pi)^2} \\
&\cdot\int_{\mathbb{R}^2}\left(\left|\mathscr{F}_Q\{f_0\}(\boldsymbol{\omega})\right|^2 + \left|\mathscr{F}_Q\{f_1\}(\boldsymbol{\omega})\right|^2\right. \\
&\left.+\left|\mathscr{F}_Q\{f_2\}(\boldsymbol{\omega})\right|^2 + \left|\mathscr{F}_Q\{f_3\}(\boldsymbol{\omega})\right|^2\right) d\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2}\left(\int_{\mathbb{R}^2}\left|\mathscr{F}_Q\{f_0\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right. \\
&+\int_{\mathbb{R}^2}\left|\mathscr{F}_Q\{f_1\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} + \int_{\mathbb{R}^2}\left|\mathscr{F}_Q\{f_2\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\
&\left.+\int_{\mathbb{R}^2}\left|\mathscr{F}_Q\{f_3\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right).
\end{aligned} \tag{32}$$

Applying (30) into the right-hand side of the above identity gives

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|_Q^2 d\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^2}\left|f_0(\mathbf{x})\right|^2 d\mathbf{x} + \int_{\mathbb{R}^2}\left|f_1(\mathbf{x})\right|^2 d\mathbf{x} \\
&+ \int_{\mathbb{R}^2}\left|f_2(\mathbf{x})\right|^2 d\mathbf{x} + \int_{\mathbb{R}^2}\left|f_3(\mathbf{x})\right|^2 d\mathbf{x}.
\end{aligned} \tag{33}$$

Since $f_i(\mathbf{x})$, $i = 0, 1, 2, 3$, is real-valued, the above equation can be written in the form

$$\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|_Q^2 d\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^2}\left(f_0^2(\mathbf{x}) + f_1^2(\mathbf{x}) + f_2^2(\mathbf{x}) + f_3^2(\mathbf{x})\right) d\mathbf{x},
\end{aligned} \tag{34}$$

which completes the proof of the theorem. □

Remark 6. Equation (30) shows that the QFT is a bounded linear operator on $L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$. Hence, using standard density arguments, one may extend the QFT in a unique way to the whole of $L^2(\mathbb{R}^2;\mathbb{H})$.

Lemma 7 (see [14]). *If $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ and $(\partial/x_k)f$ exists and is also in $L^2(\mathbb{R}^2;\mathbb{H})$, then one has for every $n \in \mathbb{N}$*

$$\mathscr{F}_Q\left\{\frac{\partial^n}{\partial x_1^n} f\right\}(\boldsymbol{\omega}) = \omega_1^n \mathscr{F}_Q\{f\mathbf{i}\}(\boldsymbol{\omega}), \tag{35}$$

$$\mathscr{F}_Q\left\{\frac{\partial^n}{\partial x_2^n} f\right\}(\boldsymbol{\omega}) = \mathscr{F}_Q\{f\}(\boldsymbol{\omega})(\mathbf{j}\omega_2)^n. \tag{36}$$

By Riesz's interpolation theorem, we get that the Hausdorff-Young inequality (see [15])

$$\left\|\mathscr{F}_Q\{f\}\right\|_{Q,p'} \le \|f\|_p \tag{37}$$

holds for $1 \le p \le 2$ with $1/p + 1/p' = 1$. Using inversion formula of the QFT, (37) can be rewritten in the form

$$\|f\|_{p'} \le \left\|\mathscr{F}_Q\{f\}\right\|_{Q,p}. \tag{38}$$

The following theorem is an extension of the Riemann-Lebesgue lemma in the QFT domain.

Theorem 8 (Riemann-Lebesgue lemma of QFT). *For a function in $f \in L^1(\mathbb{R}^2;\mathbb{H})$, one has that*

$$\begin{aligned}
&\lim_{|\omega_1|\to\infty}\left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right| = 0, \\
&\lim_{|\omega_2|\to\infty}\left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right| = 0.
\end{aligned} \tag{39}$$

Proof. Notice first that

$$\begin{aligned}
e^{-\mathbf{i}\omega_1 x_1} &= -e^{-\mathbf{i}\omega_1(x_1+\pi/\omega_1)}, \\
e^{-\mathbf{j}\omega_2 x_2} &= -e^{-\mathbf{j}\omega_2(x_2+\pi/\omega_2)}.
\end{aligned} \tag{40}$$

Applying (40) gives

$$\begin{aligned}
\mathscr{F}_Q\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} f(\mathbf{x})\, e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x} \\
&= -\int_{\mathbb{R}^2} f(\mathbf{x})\, e^{-\mathbf{i}\omega_1(x_1+\pi/\omega_1)} e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x}.
\end{aligned} \tag{41}$$

Representing $\mathscr{F}_Q\{f\} = (1/2)[\mathscr{F}_Q\{f\} + \mathscr{F}_Q\{f\}]$ and changing variable $x_1 + \pi/\omega_1 = t_1$ in the above identity, we immediately obtain

$$\begin{aligned}
&\mathscr{F}_Q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f\left(t_1 - \frac{\pi}{\omega_1}, t_2\right) e^{-\mathbf{i}\omega_1 t_1} e^{-\mathbf{j}\omega_2 t_2} d\mathbf{t} \\
&= \frac{1}{2}\left[\int_{\mathbb{R}^2} f(t_1, t_2)\, e^{-\mathbf{i}\omega_1 t_1} e^{-\mathbf{j}\omega_2 t_2} d\mathbf{t}\right. \\
&\left.- \int_{\mathbb{R}^2} f\left(t_1 - \frac{\pi}{\omega_1}, t_2\right) e^{-\mathbf{i}\omega_1 t_1} e^{-\mathbf{j}\omega_2 t_2} d\mathbf{t}\right] = \frac{1}{2} \\
&\cdot\int_{\mathbb{R}^2}\left[f(t_1, t_2) - f\left(t_1 - \frac{\pi}{\omega_1}, t_2\right)\right] e^{-\mathbf{i}\omega_1 t_1} e^{-\mathbf{j}\omega_2 t_2} d\mathbf{t}.
\end{aligned} \tag{42}$$

This means that

$$\begin{aligned}
&\lim_{|\omega_1|\to\infty}\left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right| \\
&\le \frac{1}{2}\lim_{|\omega_1|\to\infty}\int_{\mathbb{R}^2}\left|f(t_1, t_2) - f\left(t_1 - \frac{\pi}{\omega_1}, t_2\right)\right| d\mathbf{t} \\
&= 0.
\end{aligned} \tag{43}$$

Analogously, it can be shown that

$$\lim_{|\omega_2|\to\infty}\left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right| = 0. \tag{44}$$

The proof is complete. □

Theorem 9 (continuity). *If $f \in L^1(\mathbb{R}^2;\mathbb{H})$, then the quaternion Fourier transform $\mathcal{F}_Q\{f\}(\boldsymbol{\omega})$ is continuous on $\mathbb{R}^2$. Moreover,*

$$\|f\|_{C(\mathbb{R}^2;\mathbb{H})} = \max_{\mathbf{x}\in\mathbb{R}^2} |f(\mathbf{x})| \le \|f\|_1, \tag{45}$$

where $C(\mathbb{R}^2;\mathbb{H})$ is the space of continuous quaternion functions from $\mathbb{R}^2$ to $\mathbb{H}$.

Proof. From the QFT definition (23), we readily see that

$$\begin{aligned}
&|\mathcal{F}_Q\{f\}(\boldsymbol{\omega}+\mathbf{h}) - \mathcal{F}_Q\{f\}(\boldsymbol{\omega})| = \Big|\int_{\mathbb{R}^2} f(\mathbf{x}) \\
&\cdot e^{-\mathbf{i}(\omega_1+h_1)x_1} e^{-\mathbf{j}(\omega_2+h_2)x_2} d\mathbf{x} - \int_{\mathbb{R}^2} f(\mathbf{x}) \\
&\cdot e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x}\Big| = \Big|\int_{\mathbb{R}^2} f(\mathbf{x}) \\
&\cdot e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{i}h_1 x_1} e^{-\mathbf{j}\omega_2 x_2} e^{-\mathbf{j}h_2 x_2} d\mathbf{x} - \int_{\mathbb{R}^2} f(\mathbf{x}) \\
&\cdot e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x}\Big| = \Big|\int_{\mathbb{R}^2} f(\mathbf{x}) \\
&\cdot \left(e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{i}h_1 x_1} e^{-\mathbf{j}\omega_2 x_2} e^{-\mathbf{j}h_2 x_2}\right. \\
&\left. - e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2}\right) d\mathbf{x}\Big| \le \int_{\mathbb{R}^2} \Big|f(\mathbf{x}) \\
&\cdot \left(e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{i}h_1 x_1} e^{-\mathbf{j}h_2 x_2} e^{-\mathbf{j}\omega_2 x_2}\right. \\
&\left. - e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2}\right)\Big| d\mathbf{x} = \int_{\mathbb{R}^2} \Big|f(\mathbf{x}) \\
&\cdot e^{-\mathbf{i}\omega_1 x_1} \left(e^{-\mathbf{i}h_1 x_1} e^{-\mathbf{j}h_2 x_2} - 1\right) e^{-\mathbf{j}\omega_2 x_2}\Big| d\mathbf{x} \\
&= \int_{\mathbb{R}^2} |f(\mathbf{x})| \left|e^{-\mathbf{i}h_1 x_1} e^{-\mathbf{j}h_2 x_2} - 1\right| d\mathbf{x}.
\end{aligned} \tag{46}$$

Using the triangle inequality for quaternions, we easily get

$$\left|e^{-\mathbf{i}h_1 x_1} e^{-\mathbf{j}h_2 x_2} - 1\right| \le \left|e^{-\mathbf{i}h_1 x_1} e^{-\mathbf{j}h_2 x_2}\right| + 1 = 2. \tag{47}$$

This means that we have

$$|\mathcal{F}_Q\{f\}(\boldsymbol{\omega}+\mathbf{h}) - \mathcal{F}_Q\{f\}(\boldsymbol{\omega})| \le 2\int_{\mathbb{R}^2} |f(\mathbf{x})| d\mathbf{x}. \tag{48}$$

The quaternion function $f(\mathbf{x})$ is integrable and the Lebesgue dominated convergence theorem with (46) then gives

$$\lim_{\mathbf{h}\to 0} |\mathcal{F}_Q\{f\}(\boldsymbol{\omega}+\mathbf{h}) - \mathcal{F}_Q\{f\}(\boldsymbol{\omega})| = 0. \tag{49}$$

This proves that $\mathcal{F}_Q\{f\}(\boldsymbol{\omega})$ is continuous on $\mathbb{R}^2$. Again, since (48) is independent of $\boldsymbol{\omega}, \mathcal{F}_Q\{f\}(\boldsymbol{\omega})$ is, in fact, uniformly continuous on $\mathbb{R}^2$. □

3.2. Uncertainty Principle for QFT. In what follows, we investigate the uncertainty principles associated with the QFT. These results will be needed in the next section.

Theorem 10 (QFT component-wise uncertainty principle [14]). *If $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ and $(\partial/x_k)f$ exists and is also in $L^2(\mathbb{R}^2;\mathbb{H})$, then*

$$\begin{aligned}
&\int_{\mathbb{R}^2} x_k^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_k^2 |\mathcal{F}_Q\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} \\
&\quad \ge \frac{(2\pi)^2}{4} \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\right)^2, \quad k = 1, 2.
\end{aligned} \tag{50}$$

Remark 11. An alternative form of Theorem 10 is

$$\begin{aligned}
&\int_{\mathbb{R}^2} x_k^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_l^2 |\mathcal{F}_Q\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} \\
&\quad \ge \frac{(2\pi)^2}{4} \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\right)^2.
\end{aligned} \tag{51}$$

Notice that for $1 \le p \le 2$ we can replace the L^2 norms to L^p norms on the left-hand side of (50) and obtain the following theorem.

Theorem 12. *Under the assumptions of Theorem 10, one has*

$$\begin{aligned}
&\left(\int_{\mathbb{R}^2} x_1^p |f(\mathbf{x})|^p d\mathbf{x}\right)^{1/p} \\
&\quad \cdot \left(\int_{\mathbb{R}^2} \omega_1^p |\mathcal{F}_Q\{\mathbf{i}f\}(\boldsymbol{\omega})|_Q^p d\boldsymbol{\omega}\right)^{1/p} \ge \frac{1}{2} \\
&\quad \cdot \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}, \\
&\left(\int_{\mathbb{R}^2} x_2^p |f(\mathbf{x})|^p d\mathbf{x}\right)^{1/p} \\
&\quad \cdot \left(\int_{\mathbb{R}^2} \omega_2^p |\mathcal{F}_Q\{f\}(\boldsymbol{\omega})|_Q^p d\boldsymbol{\omega}\right)^{1/p} \ge \frac{1}{2} \\
&\quad \cdot \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}.
\end{aligned} \tag{52}$$

Proof. It is not difficult to check that

$$\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} = -2\mathrm{Sc}\left(\int_{\mathbb{R}} x_k f(\mathbf{x}) \overline{\frac{\partial}{\partial x_k} f(\mathbf{x})} d\mathbf{x}\right). \tag{53}$$

Using Holder's inequality, we further get

$$\begin{aligned}
&\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} \le 2\left|\int_{\mathbb{R}} x_k f(\mathbf{x}) \overline{\frac{\partial}{\partial x_k} f(\mathbf{x})} d\mathbf{x}\right| \\
&\le 2\left(\int_{\mathbb{R}} |x_k f(\mathbf{x})|^p d\mathbf{x}\right)^{1/p} \left(\int_{\mathbb{R}} \left|\frac{\partial}{\partial x_k} f(\mathbf{x})\right|^{p'} d\mathbf{x}\right)^{1/p'} \\
&= 2\|x_k f\|_p \left\|\frac{\partial}{\partial x_k} f\right\|_{p'}.
\end{aligned} \tag{54}$$

By application of the Hausdorff-Young inequality (38) and then integration by parts, we obtain

$$\begin{aligned} \left\|\frac{\partial}{\partial x_k} f\right\|_{p'} &\leq \left\|\mathscr{F}_Q\left\{\frac{\partial}{\partial x_k}\right\}\right\|_{Q,p} \\ &= \left(\int_{\mathbb{R}^2}\left(\left|\mathscr{F}_Q\left\{\frac{\partial}{\partial x_k} f_0\right\}\right|^p + \left|\mathscr{F}_Q\left\{\frac{\partial}{\partial x_k} f_1\right\}\right|^p\right.\right. \\ &\quad \left.\left. + \left|\mathscr{F}_Q\left\{\frac{\partial}{\partial x_k} f_2\right\}\right|^p \left|\mathscr{F}_Q\left\{\frac{\partial}{\partial x_k} f_3\right\}\right|^p\right) d\boldsymbol{\omega}\right)^{1/p}. \end{aligned} \tag{55}$$

We set $k = 1$. Using (35) gives

$$\begin{aligned} \left\|\mathscr{F}_Q\left\{\frac{\partial}{\partial x_1}\right\}\right\|_{Q,p} &= \left(\int_{\mathbb{R}^2}\left(\left|\omega_1 \mathscr{F}_Q\{\mathbf{i} f_0\}\right|^p\right.\right. \\ &\quad + \left|\omega_1 \mathscr{F}_Q\{\mathbf{i} f_1\}\right|^p + \left|\omega_1 \mathscr{F}_Q\{\mathbf{i} f_2\}\right|^p \\ &\quad \left.\left. + \left|\omega_1 \mathscr{F}_Q\{\mathbf{i} f_3\}\right|^p\right) d\boldsymbol{\omega}\right)^{1/p} \\ &= \left(\int_{\mathbb{R}^2}\left(\omega_1^p \left|\mathscr{F}_Q\{\mathbf{i} f_0\}\right|^p + \omega_1^p \left|\mathscr{F}_Q\{\mathbf{i} f_1\}\right|^p\right.\right. \\ &\quad \left.\left. + \omega_1^p \left|\mathscr{F}_Q\{\mathbf{i} f_2\}\right|^p + \omega_1^p \left|\mathscr{F}_Q\{\mathbf{i} f_3\}\right|^p\right) d\boldsymbol{\omega}\right)^{1/p} \\ &= \left(\int_{\mathbb{R}^2} \omega_1^p \left|\mathscr{F}_Q\{\mathbf{i} f\}\right|_Q^p d\boldsymbol{\omega}\right)^{1/p}. \end{aligned} \tag{56}$$

For $k = 2$, we can take similar steps as above using (36) and get

$$\left\|\mathscr{F}_Q\left\{\frac{\partial}{\partial x_2}\right\}\right\|_{Q,p} = \left(\int_{\mathbb{R}^2} \omega_2^p \left|\mathscr{F}_Q\{f\}\right|_Q^p d\boldsymbol{\omega}\right)^{1/p}. \tag{57}$$

This concludes the proof of the theorem. □

A generalized version of Theorem 10 is directional uncertainty principle for the QFT given by the following.

Theorem 13 (QFT directional uncertainty principle). *If* $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ *and* $(\partial/x_k) f$ *exists and is also in* $L^2(\mathbb{R}^2;\mathbb{H})$*, then*

$$\begin{aligned} &\int_{\mathbb{R}^2} |\mathbf{x}|^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} |\boldsymbol{\omega}|^2 \left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\ &\quad \geq (2\pi)^2 \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\right)^2. \end{aligned} \tag{58}$$

Proof. We have

$$\begin{aligned} &\int_{\mathbb{R}^2} |\mathbf{x}|^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} |\boldsymbol{\omega}|^2 \left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\ &= \int_{\mathbb{R}^2} \left(x_1^2 + x_2^2\right) |f(\mathbf{x})|^2 d\mathbf{x} \\ &\quad \cdot \int_{\mathbb{R}^2} \left(\omega_1^2 + \omega_2^2\right) \left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\ &= \int_{\mathbb{R}^2} x_1^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_1^2 \left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\ &\quad + \int_{\mathbb{R}^2} x_1^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_2^2 \left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\ &\quad + \int_{\mathbb{R}^2} x_2^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_1^2 \left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\ &\quad + \int_{\mathbb{R}^2} x_2^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_2^2 \left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}. \end{aligned} \tag{59}$$

Using (51) gives

$$\begin{aligned} &\int_{\mathbb{R}^2} |\mathbf{x}|^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} |\boldsymbol{\omega}|^2 \left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\ &\geq \frac{(2\pi)^2}{4}\left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\right)^2 \\ &\quad + \frac{(2\pi)^2}{4}\left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\right)^2 \\ &\quad + \frac{(2\pi)^2}{4}\left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\right)^2 \\ &\quad + \frac{(2\pi)^2}{4}\left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\right)^2 \\ &= (2\pi)^2 \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\right)^2. \end{aligned} \tag{60}$$

The result follows. □

Remark 14. A different proof of Theorem 13 using the logarithmic uncertainty principle for the QFT can be found in [15].

Using the polar coordinate form of quaternion function f, Yang et al. [18] obtained an alternative form of the directional uncertainty principle for the QFT as follows.

Theorem 15. *If* $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ *and* $(\partial/x_k) f$ *exists and is also in* $L^2(\mathbb{R}^2;\mathbb{H})$*, then*

$$\begin{aligned} &\int_{\mathbb{R}^2} x_k^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_k^2 \left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\ &\quad \geq (2\pi)^2 \left(\frac{1}{4} + \mathrm{COV}_{x_k\omega_k}^2\right), \\ &\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} = 1, \end{aligned} \tag{61}$$

where $\mathrm{COV}_{x_k\omega_k}$ *is the absolute covariance.*

Applying (51), we can easily generalize the uncertainty principle (61) to the directional QFT uncertainty principle; that is,

$$\begin{aligned}\int_{\mathbb{R}^2} |\mathbf{x}|^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} |\boldsymbol{\omega}|^2 |\mathscr{F}_Q\{f\}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} \\ \geq (2\pi)^2 \left(4\left(\frac{1}{4} + \mathrm{COV}^2_{x_k\omega_k}\right)\right) \\ \geq (2\pi)^2 \left(1 + \mathrm{COV}^2_{x_k\omega_k}\right).\end{aligned} \tag{62}$$

It is obvious that the result is the same as Theorem 4.3 in [18].

4. Continuous Quaternion Wavelet Transform

This section briefly introduces the continuous quaternion wavelet transform (CQWT). We shall derive two theorems of the CQWT which will be used in proving the main theorem. A more complete and detailed discussion of the properties of the CQWT can be found in [5, 7–9].

Definition 16. A quaternion-valued function is admissible if and only if it satisfies the following admissibility condition:

$$C_\psi = \int_{\mathbb{R}^+} |\mathscr{F}_Q\{\psi\}(a\boldsymbol{\omega})|^2 \frac{da}{a} < \infty. \tag{63}$$

Here, C_ψ is a real positive constant independent of $\boldsymbol{\omega}$ satisfying $|\boldsymbol{\omega}| = 1$.

Let $\psi \in L^1(\mathbb{R}^2; \mathbb{H})$ be a quaternion mother wavelet. We consider the family of the wavelets $\psi_{a,\mathbf{b}}$ defined by

$$\psi_{a,\mathbf{b}}(\mathbf{x}) = T_{\mathbf{b}}\psi_a(\mathbf{x}) = \frac{1}{a}\psi\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right), \quad a \in \mathbb{R}^+, \tag{64}$$

where $T_{\mathbf{b}} f(\mathbf{x}) = f(\mathbf{x}-\mathbf{b})$ and $\psi_a(\mathbf{x}) = (1/a)\psi(\mathbf{x}/a)$. Here, a is a dilation parameter and $\mathbf{b}$ is a translation vector parameter.

The relationship between (64) and its QFT is given in the following lemma.

Lemma 17. *Let ψ be an admissible quaternion function. The family of the wavelets (64) can be written in terms of the QFT as*

$$\begin{aligned}\mathscr{F}_Q\{\psi_{a,\mathbf{b}}\}(\boldsymbol{\omega}) \\ = a e^{-\mathbf{i}\omega_1 b_1} \{\mathscr{F}_Q\{\psi_0\}(a\boldsymbol{\omega}) + \mathbf{i}\mathscr{F}_Q\{\psi_1\}(a\boldsymbol{\omega})\} e^{-\mathbf{j}\omega_2 b_2} \\ + a e^{\mathbf{i}\omega_1 b_1} \{\mathbf{j}\mathscr{F}_Q\{\psi_2\}(a\boldsymbol{\omega}) + \mathbf{k}\mathscr{F}_Q\{\psi_3\}(a\boldsymbol{\omega})\} e^{-\mathbf{j}\omega_2 b_2}.\end{aligned} \tag{65}$$

If we assume that $\mathscr{F}_Q\{\psi_i\}(\boldsymbol{\omega})$, $i = 0, 1, 2, 3$, is real-valued (in the next section, we will always assume that the QFT of quaternion mother wavelet, i.e., $\mathscr{F}_Q\{\psi\}(\boldsymbol{\omega}) = \widehat{\psi}(\boldsymbol{\omega})$, is real-valued), (65) can be rewritten in the form

$$\begin{aligned}\mathscr{F}_Q\{\psi_{a,\mathbf{b}}\}(\boldsymbol{\omega}) \\ = a\left[\widehat{\psi_0}(a\boldsymbol{\omega}) + \mathbf{i}\widehat{\psi_1}(a\boldsymbol{\omega}) + \mathbf{j}\widehat{\psi_2}(a\boldsymbol{\omega}) + \mathbf{k}\widehat{\psi_3}(a\boldsymbol{\omega})\right] \\ \cdot e^{-\mathbf{i}\omega_1 b_1} e^{-\mathbf{j}\omega_2 b_2} \overset{(25)}{=} a\mathscr{F}_Q\{\psi\}(a\boldsymbol{\omega}) e^{-\mathbf{i}\omega_1 b_1} e^{-\mathbf{j}\omega_2 b_2}.\end{aligned} \tag{66}$$

Definition 18 (CQWT). The CQWT of a quaternion function $f \in L^2(\mathbb{R}^2; \mathbb{H})$ with respect to the quaternion mother wavelet ψ is defined by

$$T_\psi f(a, \mathbf{b}) = \int_{\mathbb{R}^2} f(\mathbf{x}) \frac{1}{a}\overline{\psi\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right)} d\mathbf{x}. \tag{67}$$

As an easy consequence of the above definition, we further obtain the following useful theorem.

Theorem 19. *Let $\psi \in L^2(\mathbb{R}^2; \mathbb{H})$ be a quaternion admissible wavelet; then, CQWT (67) can be expressed as*

$$T_\psi f(a, \mathbf{b}) = \frac{a}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{f}(\boldsymbol{\omega}) e^{\mathbf{j}b_2\omega_2} e^{\mathbf{i}b_1\omega_1} \overline{\widehat{\psi}(a\boldsymbol{\omega})} d\boldsymbol{\omega}. \tag{68}$$

We need the following two important results, which will be useful in proving the logarithmic uncertainty principle for the CQWT.

Theorem 20. *Let $\psi \in L^2(\mathbb{R}^2; \mathbb{H})$ be a quaternion admissible wavelet; then, CQWT (67) has a quaternion Fourier representation of the form*

$$\mathscr{F}_Q\left\{T_\psi f(a, \mathbf{b})\right\}(\boldsymbol{\omega}) = a\widehat{f}(\boldsymbol{\omega})\overline{\widehat{\psi}(a\boldsymbol{\omega})}. \tag{69}$$

Proof. From the definition of QFT (23), it follows that

$$\begin{aligned}\mathscr{F}_Q\left\{T_\psi f(a, \mathbf{b})\right\}(\boldsymbol{\omega}') = \frac{a}{(2\pi)^2} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \widehat{f}(\boldsymbol{\omega}) \\ \cdot e^{\mathbf{j}b_2\omega_2} e^{\mathbf{i}b_1\omega_1} \overline{\widehat{\psi}(a\boldsymbol{\omega})} e^{-\mathbf{i}b_1\omega_1'} e^{-\mathbf{j}b_2\omega_2'} d\boldsymbol{\omega} d\mathbf{b}.\end{aligned} \tag{70}$$

Using the assumption that the QFT of quaternion mother wavelet is real-valued and then applying Fubini's theorem, we obtain

$$\begin{aligned}\mathscr{F}_Q\left\{T_\psi f(a, \mathbf{b})\right\}(\boldsymbol{\omega}') = \frac{a}{(2\pi)^2} \\ \cdot \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \widehat{f}(\boldsymbol{\omega}) \overline{\widehat{\psi}(a\boldsymbol{\omega})} e^{\mathbf{j}b_2\omega_2} e^{\mathbf{i}b_1\omega_1} e^{-\mathbf{i}b_1\omega_1'} e^{-\mathbf{j}b_2\omega_2'} d\boldsymbol{\omega} d\mathbf{b} \\ = \frac{a}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{f}(\boldsymbol{\omega}) \overline{\widehat{\psi}(a\boldsymbol{\omega})} \\ \cdot \int_{\mathbb{R}^2} e^{\mathbf{j}b_2\omega_2} e^{\mathbf{i}b_1(\omega_1-\omega_1')} e^{-\mathbf{j}b_2\omega_2'} d\mathbf{b} d\boldsymbol{\omega} \\ = a\int_{\mathbb{R}^2} \widehat{f}(\boldsymbol{\omega}) \overline{\widehat{\psi}(a\boldsymbol{\omega})} \delta(\boldsymbol{\omega}-\boldsymbol{\omega}') d\boldsymbol{\omega} = a\widehat{f}(\boldsymbol{\omega}') \\ \cdot \overline{\widehat{\psi}(a\boldsymbol{\omega}')}, \quad \forall \omega' \in \mathbb{R}^2,\end{aligned} \tag{71}$$

where $\delta(\boldsymbol{\omega}-\boldsymbol{\omega}') = \delta(\omega_1-\omega_1')\delta(\omega_2-\omega_2')$. The proof is complete. □

Theorem 21. *Let $\psi \in L^2(\mathbb{R}^2; \mathbb{H})$ be a quaternion admissible wavelet which satisfies the admissibility condition defined by (63). Then, for every $f \in L^2(\mathbb{R}^2; \mathbb{H})$, one has*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^2} \left|T_\psi f(a, \mathbf{b})\right|^2 d\mathbf{b}\frac{da}{a^3} = C_\psi \int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}. \tag{72}$$

Proof. Applying Plancherel's theorem for QFT (30) to the $\mathbf{b}$-integral into the left-hand side of (72) yields

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\left|T_\psi f(a,\mathbf{b})\right|^2 d\mathbf{b}\frac{da}{a^3} &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\left|\mathscr{F}_Q\{T_\psi f(a,\mathbf{b})\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\frac{da}{a^3} \\ &\overset{(69)}{=} \frac{1}{(2\pi)^2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\left|a\widehat{f}(\boldsymbol{\omega})\overline{\widehat{\psi}(a\boldsymbol{\omega})}\right|^2 d\boldsymbol{\omega}\frac{da}{a^3}.\end{aligned} \tag{73}$$

Taking into consideration Fubini's theorem about the inversion of order of integration and applying Plancherel's theorem (30), we get

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\left|T_\psi f(a,\mathbf{b})\right|^2 d\mathbf{b}\frac{da}{a^3} &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\left|\widehat{f}(\boldsymbol{\omega})\right|^2\left|\overline{\widehat{\psi}(a\boldsymbol{\omega})}\right|^2 d\boldsymbol{\omega}\frac{da}{a} \\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^+}\left|\widehat{f}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\int_{\mathbb{R}^2}\left|\overline{\widehat{\psi}(a\boldsymbol{\omega})}\right|^2\frac{da}{a} \\ &= \frac{C_\psi}{(2\pi)^2}\int_{\mathbb{R}^2}\left|\widehat{f}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} = C_\psi\int_{\mathbb{R}^2}\left|f(\mathbf{x})\right|^2 d\mathbf{x},\end{aligned} \tag{74}$$

which gives the desired result. □

5. Logarithmic Uncertainty Principle for CQWT

The simplest formulation of the uncertainty principle in harmonic analysis is Heisenberg-Weyl inequality, which gives us the information that a nontrivial function and its Fourier transform cannot both be simultaneously sharply localized [1, 22]. In this section, we first derive a variation on uncertainty principle associated with the CQWT. From this, we establish the logarithmic uncertainty principle which is valid for the QFT [14] to the setting of the CQWT.

Due to the uncertainty principle for QFT (58), we have the logarithmic uncertainty principle for the QFT [15] as follows.

Theorem 22 (QFT logarithmic uncertainty principle). *For* $f \in \mathcal{S}(\mathbb{R}^2;\mathbb{H})$,

$$\begin{aligned}&\int_{\mathbb{R}^2}\ln|\mathbf{x}|\left|f(\mathbf{x})\right|^2 d\mathbf{x} + \int_{\mathbb{R}^2}\ln|\boldsymbol{\omega}|\left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\ &\quad\geq \left(\frac{\Gamma'(t)}{\Gamma(t)} - \ln\pi\right)\int_{\mathbb{R}^2}\left|f(\mathbf{x})\right|^2 d\mathbf{x},\end{aligned} \tag{75}$$

where $\Gamma'(t) = (d/dt)[\Gamma(t)]$ *and* $\Gamma(t)$ *is Gamma function. Here,* $\mathcal{S}(\mathbb{R}^2;\mathbb{H})$ *denotes the Schwartz class on quaternion function.*

Applying Plancherel's theorem for QFT (30) to the right-hand side of (75), we easily obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}\ln|\mathbf{x}|\left|f(\mathbf{x})\right|^2 d\mathbf{x} + \int_{\mathbb{R}^2}\ln|\boldsymbol{\omega}|\left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\ &\quad\geq \frac{1}{(2\pi)^2}\left(\frac{\Gamma'(t)}{\Gamma(t)} - \ln\pi\right)\int_{\mathbb{R}^2}\left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}.\end{aligned} \tag{76}$$

It is proved that, for every $f,\psi \in \mathcal{S}(\mathbb{R}^2;\mathbb{H})$, the Heisenberg type uncertainty principle for the CQWT is given [7]:

$$\begin{aligned}&\left[\int_{\mathbb{R}^+}\int_{\mathbb{R}^2} b_k^2\left|T_\psi f(a,\mathbf{b})\right|^2 d\mathbf{b}\frac{da}{a^3}\right]^{1/2} \\ &\quad\cdot\left[\int_{\mathbb{R}^2}\omega_k^2\left|\widehat{f}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\right]^{1/2} \geq \frac{\sqrt{C_\psi}}{2}\int_{\mathbb{R}^2}\left|f(\mathbf{x})\right|^2 d\mathbf{x}.\end{aligned} \tag{77}$$

A generalization of the above uncertainty principle is given in the following theorem.

Theorem 23. *Let* $\psi, f \in \mathcal{S}(\mathbb{R}^2;\mathbb{H})$ *be a quaternion admissible wavelet. Let* $T_\psi f(a,\mathbf{b})$ *be the CQWT of* f. *If* $1 \leq p \leq 2$, *then*

$$\begin{aligned}&\left[\int_{\mathbb{R}^+}\int_{\mathbb{R}^2} b_k^p\left|T_\psi f(a,\mathbf{b})\right|^p d\mathbf{b}\frac{da}{a^3}\right]^{1/p} \\ &\quad\cdot\left[\int_{\mathbb{R}^2}\omega_k^p\left|\widehat{f}(\boldsymbol{\omega})\right|^p d\boldsymbol{\omega}\right]^{1/p} \geq \frac{a^{(6-3p)/2p}\sqrt{C_\psi}}{2} \\ &\quad\cdot\int_{\mathbb{R}^2}\left|f(\mathbf{x})\right|^2 d\mathbf{x}, \quad k = 1,2.\end{aligned} \tag{78}$$

For the proof of Theorem 23, we use the following lemma.

Lemma 24. *Suppose that* $f,\psi \in \mathcal{S}(\mathbb{R}^2;\mathbb{H})$. *Then,*

$$\begin{aligned}&\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\omega_k^p\left|\mathscr{F}_Q\{T_\psi f(a,\mathbf{b})\}\right|^p d\boldsymbol{\omega}\frac{da}{a^3} \\ &\quad\leq \int_{\mathbb{R}^2}\omega_k^p\left|\widehat{f}(\boldsymbol{\omega})\right|^p a^{(3p-6)/2} C_\psi^{p/2} d\boldsymbol{\omega}.\end{aligned} \tag{79}$$

Proof. A straightforward computation yields

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\omega_k^p\left|\mathcal{F}_Q\left\{T_\psi f(a,\mathbf{b})\right\}\right|^p d\boldsymbol{\omega}\frac{da}{a^3} \\
&\overset{(69)}{=}\int_{\mathbb{R}^2}\int_{\mathbb{R}^+}\omega_k^p\left|a\widehat{f}(\boldsymbol{\omega})\overline{\widehat{\psi}(a\boldsymbol{\omega})}\right|^p\frac{da}{a^3}d\boldsymbol{\omega} \\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^+}\omega_k^p\left|\widehat{f}(\boldsymbol{\omega})\right|^p\left|\overline{\widehat{\psi}(a\boldsymbol{\omega})}\right|^p\frac{da}{a^{3-p}}d\boldsymbol{\omega} \\
&\le\int_{\mathbb{R}^2}\omega_k^p\left|\widehat{f}(\boldsymbol{\omega})\right|^p \\
&\cdot\left(\int_{\mathbb{R}^+}\left|\overline{\widehat{\psi}(a\boldsymbol{\omega})}\right|^2\frac{da}{a^{6/p-2}}\right)^{p/2}d\boldsymbol{\omega} \\
&=\int_{\mathbb{R}^2}\omega_k^p\left|\widehat{f}(\boldsymbol{\omega})\right|^p \\
&\cdot\left(a^{(3p-6)/p}\int_{\mathbb{R}^+}\left|\overline{\widehat{\psi}(a\boldsymbol{\omega})}\right|^2\frac{da}{a}\right)^{p/2}d\boldsymbol{\omega} \\
&=\int_{\mathbb{R}^2}\omega_k^p\left|\widehat{f}(\boldsymbol{\omega})\right|^p a^{(3p-6)/2}C_\psi^{p/2}d\boldsymbol{\omega}.
\end{aligned} \tag{80}$$

The proof is complete. □

Proof. Using the uncertainty principle in Theorem 12, we immediately obtain

$$\begin{aligned}
&\left[\int_{\mathbb{R}^2}b_k^p\left|T_\psi f(a,\mathbf{b})\right|^p d\mathbf{b}\right]^{1/p} \\
&\cdot\left[\int_{\mathbb{R}^2}\omega_k\left|\mathcal{F}_Q\left\{T_\psi f(a,\mathbf{b})\right\}\right|^p d\boldsymbol{\omega}\right]^{1/p}\ge\frac{1}{2} \\
&\cdot\int_{\mathbb{R}^2}\left|T_\psi f(a,\mathbf{b})\right|^2 d\mathbf{b}.
\end{aligned} \tag{81}$$

Now, integrating both sides of (81) with respect to the Haar measure da/a^3, we obtain

$$\begin{aligned}
&\left[\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}b_k^p\left|T_\psi f(a,\mathbf{b})\right|^p d\mathbf{b}\frac{da}{a^3}\right]^{1/p} \\
&\cdot\left[\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\omega_k^p\left|\mathcal{F}_Q\left\{T_\psi f(a,\mathbf{b})\right\}\right|^p d\boldsymbol{\omega}\frac{da}{a^3}\right]^{1/p}\ge\frac{1}{2} \\
&\cdot\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\left|T_\psi f(a,\mathbf{b})\right|^2 d\mathbf{b}\frac{da}{a^3}.
\end{aligned} \tag{82}$$

Then, inserting Lemma 24 into the second term of (82), we easily obtain

$$\begin{aligned}
&\left[\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}b_k^p\left|T_\psi f(a,\mathbf{b})\right|^p d\mathbf{b}\frac{da}{a^3}\right]^{1/p} \\
&\cdot a^{(3p-6)/2p}\sqrt{C_\psi}\left[\int_{\mathbb{R}^2}\omega_k^p\left|\widehat{f}(\boldsymbol{\omega})\right|^p d\boldsymbol{\omega}\right]^{1/p}\ge\frac{1}{2} \\
&\cdot\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\left|T_\psi f(a,\mathbf{b})\right|^2 d\mathbf{b}\frac{da}{a^3}.
\end{aligned} \tag{83}$$

Substituting (72) into the right-hand side of (83) and simplifying it, we finally get

$$\begin{aligned}
&\left[\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}b_k^p\left|T_\psi f(a,\mathbf{b})\right|^p d\mathbf{b}\frac{da}{a^3}\right]^{1/p} \\
&\cdot\left[\int_{\mathbb{R}^2}\omega_k^p\left|\widehat{f}(\boldsymbol{\omega})\right|^p d\boldsymbol{\omega}\right]^{1/p}\ge\frac{a^{(6-3p)/2p}\sqrt{C_\psi}}{2} \\
&\cdot\int_{\mathbb{R}^2}\left|f(\mathbf{x})\right|^2 d\mathbf{x},
\end{aligned} \tag{84}$$

which concludes the proof of Theorem 23. □

Let us derive a logarithmic uncertainty principle associated with the continuous quaternion wavelet transform (CQWT).

Theorem 25 (CQWT logarithmic uncertainty principle). *Let $\psi\in\mathcal{S}(\mathbb{R}^2;\mathbb{H})$ be a quaternion admissible wavelet. Let $T_\psi f(a,\mathbf{b})$ be the CQWT of $f\in\mathcal{S}(\mathbb{R}^2;\mathbb{H})$. Then, the following inequality is satisfied:*

$$\begin{aligned}
&C_\psi\int_{\mathbb{R}^2}\ln|\boldsymbol{\omega}|\left|\widehat{f}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \\
&\quad+\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\ln|\mathbf{b}|\left|T_\psi f(a,\mathbf{b})\right|^2 d\mathbf{b}\frac{da}{a^3} \\
&\quad\ge\frac{C_\psi}{2}\left(\frac{\Gamma'(t)}{\Gamma(t)}-\ln\pi\right)\int_{\mathbb{R}^2}\left|f(\mathbf{x})\right|^2 d\mathbf{x}.
\end{aligned} \tag{85}$$

Next, we need the following lemma to assist the proof of the above theorem.

Lemma 26. *Under the same conditions as in Theorem 25, one has*

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\ln|\boldsymbol{\omega}|\left|\mathcal{F}_Q\left\{T_\psi f(a,\mathbf{b})\right\}\right|^2 d\boldsymbol{\omega}\frac{da}{a^3} \\
&\quad=C_\psi\int_{\mathbb{R}^2}\ln|\boldsymbol{\omega}|\left|\widehat{f}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}.
\end{aligned} \tag{86}$$

Proof. A simple calculation reveals

$$\begin{aligned}
&\int_{\mathbb{R}^+}\int_{\mathbb{R}^2}\ln|\boldsymbol{\omega}|\left|\mathcal{F}_Q\left\{T_\psi f(a,\mathbf{b})\right\}\right|^2 d\boldsymbol{\omega}\frac{da}{a^3} \\
&\overset{(69)}{=}\int_{\mathbb{R}^2}\int_{\mathbb{R}^+}\ln|\boldsymbol{\omega}|\left|a\widehat{f}(\boldsymbol{\omega})\overline{\widehat{\psi}(a\boldsymbol{\omega})}\right|^2\frac{da}{a^3}d\boldsymbol{\omega} \\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^+}a^2\ln|\boldsymbol{\omega}|\,\widehat{f}(\boldsymbol{\omega})\,\widehat{\psi}(a\boldsymbol{\omega})\,\overline{\widehat{\psi}(a\boldsymbol{\omega})}\,\overline{\widehat{f}(\boldsymbol{\omega})}\frac{da}{a^3}d\boldsymbol{\omega} \\
&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^+}\ln|\boldsymbol{\omega}|\left|\widehat{f}(\boldsymbol{\omega})\right|^2\left|\widehat{\psi}(a\boldsymbol{\omega})\right|^2\frac{da}{a}d\boldsymbol{\omega} \\
&\overset{(63)}{=}C_\psi\int_{\mathbb{R}^2}\ln|\boldsymbol{\omega}|\left|\widehat{f}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}.
\end{aligned} \tag{87}$$

The proof is complete. □

Proof. It is known that

$$\int_{\mathbb{R}^2} \ln|\mathbf{x}| \left|f(\mathbf{x})\right|^2 d\mathbf{x} + \int_{\mathbb{R}^2} \ln|\boldsymbol{\omega}| \left|\mathscr{F}_Q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \geq \left(\frac{\Gamma'(t)}{\Gamma(t)} - \ln\pi\right) \int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^2 d\mathbf{x}. \tag{88}$$

Notice that $f, \psi \in \mathcal{S}(\mathbb{R}^2; \mathbb{H})$. This implies that $T_\psi f(a, \mathbf{b}) \in \mathcal{S}(\mathbb{R}^2; \mathbb{H})$. Therefore, we may replace f by $T_\psi f(a, \mathbf{b})$ on both sides of (88) and get

$$\int_{\mathbb{R}^2} \ln|\boldsymbol{\omega}| \left|\mathscr{F}_Q\left\{T_\psi f(a, \mathbf{b})\right\}\right|^2 d\boldsymbol{\omega} + \int_{\mathbb{R}^2} \ln|\mathbf{b}| \left|T_\psi f(a, \mathbf{b})\right|^2 d\mathbf{b} \geq \left(\frac{\Gamma'(t)}{\Gamma(t)} - \ln\pi\right) \int_{\mathbb{R}^2} \left|T_\psi f(a, \mathbf{b})\right|^2 d\mathbf{b}. \tag{89}$$

Integrating both sides of this equation with respect to da/a^3 yields

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \ln|\boldsymbol{\omega}| \left|\mathscr{F}_Q\left\{T_\psi f(a, \mathbf{b})\right\}\right|^2 d\boldsymbol{\omega} \frac{da}{a^3} + \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \ln|\mathbf{b}| \left|T_\psi f(a, \mathbf{b})\right|^2 d\mathbf{b} \frac{da}{a^3} \geq \left(\frac{\Gamma'(t)}{\Gamma(t)} - \ln\pi\right) \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \left|T_\psi f(a, \mathbf{b})\right|^2 d\mathbf{b} \frac{da}{a^3}. \tag{90}$$

Inserting Lemma 26 into the first term on the left-hand side of (90), we obtain

$$C_\psi \int_{\mathbb{R}^2} \ln|\boldsymbol{\omega}| \left|\widehat{f}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} + \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \ln|\mathbf{b}| \left|T_\psi f(a, \mathbf{b})\right|^2 d\mathbf{b} \frac{da}{a^3} \geq \left(\frac{\Gamma'(t)}{\Gamma(t)} - \ln\pi\right) \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \left|T_\psi f(a, \mathbf{b})\right|^2 d\mathbf{b} \frac{da}{a^3}. \tag{91}$$

Finally, substituting (72) into the right-hand side of (91), we have

$$C_\psi \int_{\mathbb{R}^2} \ln|\boldsymbol{\omega}| \left|\widehat{f}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} + \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \ln|\mathbf{b}| \left|T_\psi f(a, \mathbf{b})\right|^2 d\mathbf{b} \frac{da}{a^3} \geq \frac{C_\psi}{2} \left(\frac{\Gamma'(t)}{\Gamma(t)} - \ln\pi\right) \int_{\mathbb{R}^2} \left|f(\mathbf{x})\right|^2 d\mathbf{x}, \tag{92}$$

which was to be proved. □

Remark 27. It is worth nothing that, following the steps of the proof of the above theorem, we can also obtain Theorem 25 using (76).

6. Conclusion

Based on the logarithmic uncertainty principle in the quaternion Fourier domain, we have established a logarithmic uncertainty principle related to the CQWT. It is a more general form of component-wise uncertainty principle associated with the CQWT, which describes the relationship between the QFT of a quaternion function and its CQWT. We also presented a variation on uncertainty principle related to the QFT and then found the variation on uncertainty principle related to the CQWT.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

Acknowledgments

This work is partially supported by WCU Percepatan Publikasi Internasional Tahun 2016 (no. 3834/UN4.21/PL.09/2016) from the Hasanuddin University, Indonesia. The second author is partially supported by JSPS KAKENHI (C)25400202 of Japan.

References

[1] C. K. Chui, *An Introduction to Wavelets*, Academic Press, New York, NY, USA, 1992.

[2] K. Gröchenig, *Foundation of Time-Frequency Analysis*, Birkhäuser, Boston, Mass, USA, 2001.

[3] J. X. He and B. Yu, "Continuous wavelet transforms on the space $L^2(\mathrm{R}, \mathbb{H}; dx)$," *Applied Mathematics Letters*, vol. 17, no. 1, pp. 111–121, 2001.

[4] R. S. Pathak, *The Wavelet Transform*, Atlantis Press, Amsterdam, The Netherlands, 2009.

[5] S. T. Ali and K. Thirulogasanthar, "The quaternionic affine group and related continuous wavelet transforms on complex and quaternionic Hilbert spaces," *Journal of Mathematical Physics*, vol. 55, no. 6, 16 pages, 2014.

[6] M. Bahri, R. Ashino, and R. Vaillancourt, "Continuous quaternion Fourier and wavelet transforms," *International Journal of Wavelets, Multiresolution and Information Processing*, vol. 12, no. 4, Article ID 1460003, 21 pages, 2014.

[7] M. Bahri, R. Ashino, and R. Vaillancourt, "Two-dimensional quaternion wavelet transform," *Applied Mathematics and Computation*, vol. 218, no. 1, pp. 10–21, 2011.

[8] E. Bayro-Corrochano, "The theory and use of the quaternion wavelet transform," *Journal of Mathematical Imaging and Vision*, vol. 24, no. 1, pp. 19–35, 2006.

[9] Y. Xi, X. Yang, L. Song, L. Traversoni, and W. Lu, "QWT: retrospective and new application," in *Applied Geometric Algebras in Computer Science and Engineering*, E. Bayro-Corrochano and G. Scheuermann, Eds., pp. 249–273, Springer, London, UK, 2010.

[10] S. Gai, G. Yang, and S. Zhang, "Multiscale texture classification using reduced quaternion wavelet transform," *International Journal of Electronics and Communications*, vol. 67, no. 3, pp. 233–241, 2013.

[11] L. Akila and R. Roopkumar, "Ridgelet transform for quaternion-valued functions," *International Journal of Wavelets,*

Multiresolution and Information Processing, vol. 14, no. 1, 18 pages, 2016.

[12] L. Akila and R. Roopkumar, "Multidimensional quaternionic Gabor transforms," *Advances in Applied Clifford Algebras*, vol. 26, no. 3, pp. 985–1011, 2016.

[13] E. M. Hitzer, "Quaternion Fourier transform on quaternion fields and generalizations," *Advances in Applied Clifford Algebras*, vol. 17, no. 3, pp. 497–517, 2007.

[14] M. Bahri, E. S. Hitzer, A. Hayashi, and R. Ashino, "An uncertainty principle for quaternion Fourier transform," *Computers and Mathematics with Applications*, vol. 56, no. 9, pp. 2398–2410, 2008.

[15] L.-P. Chen, K. I. Kou, and M.-S. Liu, "Pitt's inequality and the uncertainty principle associated with the quaternion Fourier transform," *Journal of Mathematical Analysis and Applications*, vol. 423, no. 1, pp. 681–700, 2015.

[16] E. M. Hitzer, "Directional uncertainty principle for quaternion Fourier transform," *Advances in Applied Clifford Algebras*, vol. 20, no. 2, pp. 271–284, 2010.

[17] M. Bahri, "A modified uncertainty principle for two-sided quaternion Fourier transform," *Advances in Applied Clifford Algebras*, vol. 26, no. 2, pp. 513–527, 2016.

[18] Y. Yang, P. Dang, and T. Qian, "Tighter uncertainty principles based on quaternion Fourier transform," *Advances in Applied Clifford Algebras*, vol. 26, no. 1, pp. 479–497, 2016.

[19] T. Bülow, *Hypercomplex spectral signal representations for the processing and analysis of images [Ph.D. thesis]*, University of Kiel, Kiel, Germany, 1999.

[20] J. P. Morais, S. Georgiev, and W. Sprößig, *Real Quaternionic Calculus Handbook*, Birkhäuser, New York, NY, USA, 2014.

[21] K. I. Kou and J. Morais, "Asymptotic behaviour of the quaternion linear canonical transform and the Bochner-Minlos theorem," *Applied Mathematics and Computation*, vol. 247, no. 15, pp. 675–688, 2014.

[22] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover, New York, NY, USA, 2nd edition, 1950.