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Research Article

Contractibility of Fixed Point Sets of Mean-Type Mappings

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We establish a convergence theorem and explore fixed point sets of certain continuous quasi-nonexpansive mean-type mappings in general normed linear spaces. We not only extend previous works by Matkowski to general normed linear spaces, but also obtain a new result on the structure of fixed point sets of quasi-nonexpansive mappings in a nonstrictly convex setting.

1. Introduction

The theory of mean iteration has been studied long before the 19th century [1]. Johann Carl Friedrich Gauss observed the connection between the arithmetic-geometric mean iteration and an elliptic integral. Indeed, if we recursively define the following sequences of positive real numbers

$$\begin{aligned} a_{n+1} &= \frac{a_n + b_n}{2} \\ b_{n+1} &= \sqrt{a_n b_n}, \end{aligned} \tag{1}$$

we know that both (a_n) and (b_n) converge to the same limit, say, $M(a_0, b_0)$. He found that

$$\frac{1}{M\left(1, \sqrt{2}\right)} = \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^4}}, \tag{2}$$

which is later generalised to

$$\frac{1}{M(a,b)} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}. \tag{3}$$

However, the convergence above is not coincidental as we will see in the next section.

In 1999, Matkowski introduced the notion of mean-type mappings on a real interval and showed the convergence of its Picard iteration if at most one of its coordinate means is not strict. Later in 2009, he showed the same result for mappings with a weaker condition. The fixed point set of such a mapping is exactly the diagonal; however, the fixed point set of a general (continuous) mean-type mapping only covers the diagonal and may not be contractible. On the other hand, in 2012, Chaoha and Chanthorn introduced the concept of virtually stable (fixed point iteration) schemes to connect topological structures of the convergence set of a scheme to those of the fixed point set via a retraction. Many schemes for nonexpansive-type mappings have been proved to be virtually stable.

With those results in mind, in this work, we first extend the concept of mean-type mappings to vector spaces and then explore their fixed point sets using the notion of virtually stable schemes developed in [2]. We are able to establish a convergence theorem for certain continuous quasi-nonexpansive mean-type mappings in general normed linear spaces (which immediately covers the result in [3]) and conclude the contractibility of their fixed point sets. This also gives a new result on the structure of fixed point sets of quasi-nonexpansive mappings outside the strict-convexity setting.

2. Preliminaries

We begin this section by recalling the notion of means and mean-type mappings from [3].

Definition 1. Let I be an interval in $\mathbb{R}$ and $p \geq 2$ an integer. A function $M : I^p \to I$ is said to be a *mean* if

$$\begin{aligned} \min\left(x_1, \ldots, x_p\right) &\leq M\left(x_1, \ldots, x_p\right) \\ &\leq \max\left(x_1, \ldots, x_p\right) \end{aligned} \tag{4}$$

for all $(x_1, \ldots, x_p) \in I^p$.

A mean M is *strict* if, in addition,

$$\min\left(x_1,\ldots,x_p\right) < M\left(x_1,\ldots,x_p\right) < \max\left(x_1,\ldots,x_p\right) \tag{5}$$

for all $(x_1,\ldots,x_p) \in I^p - \Delta$, where Δ denotes the *diagonal*:

$$\Delta = \left\{\left(x_1,\ldots,x_p\right) \in I^p \mid x_1 = \cdots = x_p\right\}. \tag{6}$$

Definition 2. A mapping $\mathbf{M} : I^p \to I^p$ is said to be a *mean-type mapping* if each coordinate function is a mean; i.e., $M_i : I^p \to I$ is a mean for all $i = 1,\ldots,p$, where

$$\mathbf{M}(x) = \left(M_1(x),\ldots,M_p(x)\right). \tag{7}$$

Moreover, $\mathbf{M}$ is *strict* if all coordinate means are strict.

Definition 3. Let $M, N : I^p \to I$ be means. We say that M and N are *comparable* if one of the following holds:

(i) $M(x) \leq N(x)$ for all $x \in I^p$.

(ii) $N(x) \leq M(x)$ for all $x \in I^p$.

We are ready to recall the classical and well-known convergence theorem for a 2-dimensional mean iteration.

Theorem 4 (see [1]). *Let $M, N : I^2 \to I$ be comparable continuous means. Suppose that M or N is strict. Define $\mathbf{M} : I^2 \to I^2$ by*

$$\mathbf{M}(x) = (M(x), N(x)). \tag{8}$$

Then there exists a continuous mean $K : I^2 \to I$ such that

$$\lim_{n\to\infty} \mathbf{M}^n(x) = (K(x), K(x)) \tag{9}$$

for all $x \in I^2$.

Remark 5. The convergence of the arithmetic-geometric mean iteration follows directly from Theorem 4 by letting

$$\mathbf{M}(a,b) = \left(\frac{a+b}{2}, \sqrt{ab}\right) \tag{10}$$

for all $a, b \in (0,\infty)$. In this case,

$$\frac{1}{K(a,b)} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}. \tag{11}$$

In 1999, Matkowski showed that the comparability between means is not necessary. He also extended the convergence theorem to a p-dimensional mean iteration as follows.

Theorem 6 (see [4]). *Let $\mathbf{M} = (M_1,\ldots,M_p) : I^p \to I^p$ be a continuous mean-type mapping such that at most one of the coordinate means M_i is not strict. Then there exists a continuous mean $K : I^p \to I$ such that*

$$\lim_{n\to\infty} \mathbf{M}^n(x) = (K(x),\ldots,K(x)) \tag{12}$$

for all $x \in I^p$.

Again, in 2009, he improved his earlier result to include the larger class of nonstrict mean-type mappings.

Theorem 7 (see [3]). *Let $\mathbf{M} = (M_1,\ldots,M_p) : I^p \to I^p$ be a continuous mean-type mapping such that*

$$\max\left(M_1(x),\ldots,M_p(x)\right) - \min\left(M_1(x),\ldots,M_p(x)\right) < \max(x) - \min(x) \tag{13}$$

for all $x \in I^p - \Delta$. Then there exists a continuous mean $K : I^p \to I$ such that

$$\lim_{n\to\infty} \mathbf{M}^n(x) = (K(x),\ldots,K(x)) \tag{14}$$

for all $x \in I^p$.

Remark 8. If a mean-type mapping $\mathbf{M}$ satisfies the condition in Theorem 7, then

$$\operatorname{Fix}(\mathbf{M}) = \Delta. \tag{15}$$

Next, we recall the concept of virtual stability of fixed point iteration schemes [2]. We use this to conclude the contractibility of the fixed point sets of nonexpansive-type mappings.

Let $\mathcal{S} = (s_n)$ be a sequence of self-maps on a Hausdorff space X. Define the *fixed point set* of $\mathcal{S}$ and the *convergence set* of $\mathcal{S}$ as follows:

$$F(\mathcal{S}) = \bigcap_{n\in\mathbb{N}} \operatorname{Fix}(s_n)$$
$$C(\mathcal{S}) = \left\{x \in X \mid \lim s_n(x) \text{ exists}\right\}. \tag{16}$$

Define a function $r : C(\mathcal{S}) \to X$ by $r(x) = \lim_{n\to\infty} s_n(x)$.

Definition 9 (see [2]). Let (f_n) be a sequence of self-maps on X and

$$s_n = \prod_{i=1}^{n} f_i = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1. \tag{17}$$

The sequence $\mathcal{S} = (s_n)$ is called a *(fixed point iteration) scheme* if $\emptyset \neq F(\mathcal{S}) = r(C(\mathcal{S}))$.

Definition 10 (see [2]). Let $\mathcal{S} = (\prod_{i=1}^{n} f_i)$ be a scheme. A fixed point $p \in F(\mathcal{S})$ is called *virtually stable* if, for each neighbourhood U of p, there exist a neighbourhood V of p and a strictly increasing sequence $(k_n) \subseteq \mathbb{N}$ such that $\prod_{i=j}^{k_n} f_i(V) \subseteq U$ for all $n \in \mathbb{N}$ and $j \leq k_n$.

The scheme $\mathcal{S}$ is called *virtually stable* if all its common fixed points are virtually stable.

Theorem 11 (see [2]). *If X is a regular space and $\mathcal{S}$ is a virtually stable scheme having a subsequence consisting of continuous mappings, then the function r defined above is continuous and hence $F(\mathcal{S})$ is a retract of $C(\mathcal{S})$.*

Theorem 12 (see [5]). *A retract subspace of a contractible space is contractible.*

Lastly, we recall a very well-known fact about the structure of the fixed point sets of quasi-nonexpansive mappings.

Definition 13. A normed linear space X is called *strictly convex* if $\|(x+y)/2\| < 1$ for all $x, y \in X$ such that $x \neq y$ and $\|x\| = \|y\| = 1$; equivalently the boundary of the unit ball does not contain any line segment.

Definition 14. Let X be a subset of a normed linear space. A mapping $T : X \to X$ is said to be

(i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in X$;

(ii) *quasi-nonexpansive* if $\|Tx - q\| \leq \|x - q\|$ for all $x \in X$ and $q \in \mathrm{Fix}(T)$.

It is easy to see that nonexpansive mappings are continuous and quasi-nonexpansive while continuous quasi-nonexpansive mappings may not be nonexpansive.

Theorem 15 (see [6]). *The fixed point set of a quasi-nonexpansive mapping defined on a convex subset of a strictly convex space is convex.*

3. Main Results

In this section, we extend the notions of means and mean-type mappings to general vector spaces. Then we prove a convergence theorem as well as the contractibility of fixed point sets for certain continuous quasi-nonexpansive mean-type mappings.

Let X be a convex subset of a vector space (over $\mathbb{R}$) and p an integer with $p \geq 2$. As usual, the diagonal in X^p is simply $\Delta = \{(x_1, \ldots, x_p) \in X^p \mid x_1 = \cdots = x_p\}$.

Definition 16. A function $M : X^p \to X$ is said to be a *mean* if, for each $i = 1, \ldots, p$, there is a function $\alpha_i : X \to [0,1]$ such that

$$M(x) = M(x_1, \ldots, x_p) = \alpha_1(x)x_1 + \cdots + \alpha_p(x)x_p \quad (18)$$

with $\alpha_1(x) + \cdots + \alpha_p(x) = 1$ for each $x = (x_1, \ldots, x_p) \in X^p$. For simplicity, we usually write $M(x) = \sum_n \alpha_n x_n$. We also call M *strict* if, in addition, $\alpha_i(x) \in (0,1)$ for all $x \in X^p$ and $i = 1, \ldots, p$.

We note that Definition 16 is equivalent to Definition 1 when X is an interval in $\mathbb{R}$.

Definition 17. A mapping $\mathbf{M} : X^p \to X^p$ is said to be a *mean-type mapping* if each coordinate function is a mean; that is

$$\mathbf{M} = (M_1, \ldots, M_p), \quad (19)$$

where $M_i : X^p \to X$ is a mean for all $i = 1, \ldots, p$. Moreover, $\mathbf{M}$ is *strict* if each M_i is strict.

Remark 18. For any mean $M : X^p \to X$, we clearly have $M(x, \ldots, x) = x$, for each $x \in X$. Consequently, for any mean-type mapping $\mathbf{M} : X^p \to X^p$, we have $\Delta \subseteq \mathrm{Fix}(\mathbf{M})$.

Example 19. Define $\mathbf{M} : (0,\infty)^2 \to (0,\infty)^2$ by

$$\begin{aligned}\mathbf{M}(x,y) &= \left(\frac{x+y}{2}, \sqrt{xy}\right) \\ &= \left(\frac{x+y}{2}, \frac{x\sqrt{y}}{\sqrt{x}+\sqrt{y}} + \frac{y\sqrt{x}}{\sqrt{x}+\sqrt{y}}\right).\end{aligned} \quad (20)$$

Clearly, $\mathbf{M}$ is a nonlinear mean-type mapping and it is straightforward to verify that $\mathbf{M}$ is strict.

Theorem 20. *If* $\mathbf{M} : X^p \to X^p$ *is a strict mean-type mapping, then* $\mathrm{Fix}(\mathbf{M}) = \Delta$.

Proof. Let $x \in \mathrm{Fix}(\mathbf{M})$. We can form the following system of linear equations:

$$\begin{aligned}
x_1 &= M_1(x) = \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1(p-1)}x_{p-1} + \left(1 - \sum_{j=1}^{p-1}\alpha_{1j}\right)x_p \\
x_2 &= M_2(x) = \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2(p-1)}x_{p-1} + \left(1 - \sum_{j=1}^{p-1}\alpha_{2j}\right)x_p \\
&\vdots \\
x_p &= M_p(x) = \alpha_{p1}x_1 + \alpha_{p2}x_2 + \cdots + \alpha_{p(p-1)}x_{p-1} + \left(1 - \sum_{j=1}^{p-1}\alpha_{pj}\right)x_p,
\end{aligned} \quad (21)$$

which is equivalent to

$$\begin{pmatrix}
1-\alpha_{11} & -\alpha_{12} & -\alpha_{13} & \cdots & -\alpha_{1(p-1)} \\
-\alpha_{21} & 1-\alpha_{22} & -\alpha_{23} & \cdots & -\alpha_{2(p-1)} \\
-\alpha_{31} & -\alpha_{32} & 1-\alpha_{33} & \cdots & -\alpha_{3(p-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\alpha_{(p-1)1} & -\alpha_{(p-1)2} & -\alpha_{(p-1)3} & \cdots & 1-\alpha_{(p-1)(p-1)} \\
-\alpha_{p1} & -\alpha_{p2} & -\alpha_{p3} & \cdots & -\alpha_{p(p-1)}
\end{pmatrix}
\begin{bmatrix} x_1 - x_p \\ x_2 - x_p \\ x_3 - x_p \\ \vdots \\ x_{p-1} - x_p \end{bmatrix} = \bar{0}. \quad (22)$$

Since $\mathbf{M}$ is strict, we can apply Gauss-Jordan elimination to the coefficient matrix to obtain

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{bmatrix} x_1 - x_p \\ x_2 - x_p \\ x_3 - x_p \\ \vdots \\ x_{p-1} - x_p \end{bmatrix} = \overline{0}, \tag{23}$$

which implies $x_1 = \cdots = x_p$. □

When $\mathbf{M}$ is nonstrict, $\mathrm{Fix}(\mathbf{M})$ may still be the diagonal Δ, or even the whole space X^p.

Example 21. Consider $\mathrm{Id}, \mathbf{M} : X^p \to X^p$ defined by

$$\begin{gathered} \mathrm{Id}\left(x_1, \ldots, x_p\right) = \left(x_1, x_2, \ldots, x_p\right), \\ \mathbf{M}\left(x_1, \ldots, x_p\right) = \left(x_p, x_1, \ldots, x_{p-1}\right). \end{gathered} \tag{24}$$

Clearly, we have $\mathrm{Fix(Id)} = X^p$ and $\mathrm{Fix}(\mathbf{M}) = \Delta$.

When X is also a metric space, the next theorem surprisingly gives an explicit construction of a continuous mean-type mapping, whose fixed point set is any closed subset of X^p containing the diagonal.

Recall that the distance between a point x in the metric space (X, d) and $\emptyset \neq A \subseteq X$ is defined to be

$$d(x, A) = \inf_{a \in A} d(x, a). \tag{25}$$

When A is closed, we also have $d(x, A) = 0$ iff $x \in A$.

Theorem 22. *Suppose further that (X, d) is a metric space. For any closed subset F of X^p such that $\Delta \subseteq F$, there exists a continuous mean-type mapping $\mathbf{M} : X^p \to X^p$ such that* $\mathrm{Fix}(\mathbf{M}) = F$.

Proof. Define $t : X^p \to [0, 1]$ and $\mathbf{M} : X^p \to X^p$ by

$$\begin{aligned} t(x) &= \frac{d(x, F)}{1 + d(x, F)}, \\ \mathbf{M}(x) &= \left(x_1, t(x)\, x_1 + [1 - t(x)]\, x_2, \ldots, t(x)\, x_1 \right. \\ &\quad \left. + [1 - t(x)]\, x_p\right). \end{aligned} \tag{26}$$

It is easy to verify that $t(x) = 0$ iff $x \in F$, and $\mathbf{M}$ is continuous with $\mathrm{Fix}(\mathbf{M}) = F$. □

From now on, let X be a convex subset of a normed linear space $(E, \|\ \|)$, and we will always use the maximum norm

$$\left\|\left(x_1, \ldots, x_p\right)\right\| = \max_n \left\|x_n\right\| \tag{27}$$

on X^p. Notice that X^p together with maximum norm may not be strictly convex. This prevents us from using Theorem 15 to conclude the convexity (and hence contractibility) of fixed point sets of quasi-nonexpansive mean-type mappings.

The following theorem shows that, under a simple condition, a mean-type mapping is always quasi-nonexpansive.

Theorem 23. *Let $\mathbf{M} : X^p \to X^p$ be a mean-type mapping. If* $\mathrm{Fix}(\mathbf{M}) = \Delta$, *then $\mathbf{M}$ is quasi-nonexpansive. In particular, strict mean-type mappings are always quasi-nonexpansive.*

Proof. Let $x = (x_1, \ldots, x_p) \in X^p$, $\widehat{q} = (q, \ldots, q) \in \Delta = \mathrm{Fix}(\mathbf{M})$, and write $\mathbf{M} = (M_1, \ldots, M_p)$. Consider, for each $i = 1, \ldots, p$,

$$\begin{aligned} \left\|M_i(x) - q\right\| &= \left\|\sum_n \alpha_{in} x_n - \sum_n \alpha_{in} q\right\| \leq \sum_n \alpha_{in} \left\|x_n - q\right\| \\ &\leq \max_n \left\|x_n - q\right\| \sum_n \alpha_{in} = \max_n \left\|x_n - q\right\|, \end{aligned} \tag{28}$$

and hence,

$$\begin{aligned} \left\|\mathbf{M}(x) - \widehat{q}\right\| &= \max_i \left\|M_i(x) - q\right\| \leq \max_n \left\|x_n - q\right\| \\ &= \left\|x - \widehat{q}\right\|. \end{aligned} \tag{29}$$

□

Unfortunately, the converse of the previous theorem is not true (see Example 36 below). Moreover, the last line in the proof of the previous theorem shows that any mean-type mapping is continuous on the diagonal. However, the continuity may not hold at other points as in the next example.

Example 24. Define $\mathbf{M} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{M}(x, y) = \begin{cases} \left(\dfrac{2x + y}{3}, \dfrac{x + 2y}{3}\right); & (x, y) \in \mathbb{Q}^2, \\ \left(\dfrac{x + 2y}{3}, \dfrac{2x + y}{3}\right); & (x, y) \notin \mathbb{Q}^2. \end{cases} \tag{30}$$

It is easy to see that $\mathbf{M}$ is a strict mean-type mapping. However, $\mathbf{M}$ is not continuous at each $(x, y) \notin \Delta$.

Definition 25. For each $k \in \mathbb{N}$, $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k) \in [0, 1]^{k+1}$ with $\sum_{i=0}^{k} \lambda_i = 1$, and $T : X \to X$, we define the *λ-combination* of T as follows:

$$T_\lambda = \lambda_0 \mathrm{Id} + \lambda_1 T + \cdots + \lambda_k T^k = \sum_{i=0}^{k} \lambda_i T^i, \tag{31}$$

where T^0 denotes the identity mapping (Id). The *λ-combination* of a mean-type mapping $\mathbf{M} : X^p \to X^p$ is defined similarly.

From the above definition, it is easy to verify that the λ-combination of a mean-type mapping $\mathbf{M} : X^p \to X^p$ is also a mean-type mapping, and it is continuous if $\mathbf{M}$ is continuous. Before we establish a convergence theorem, let us recall some basic facts about quasi-nonexpansive mappings and the distance between points in a convex hull.

Lemma 26. *Let X be a convex subset of a normed linear space, $T : X \to X$ a quasi-nonexpansive mapping, $p \in \mathrm{Fix}(T)$, and $x_0, y, z \in X$. For a given $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k) \in [0,1]^{k+1}$ with $\sum_{i=0}^{k} \lambda_i = 1$, define a sequence $(x_n) \subseteq X$ by*

$$x_n = T_\lambda x_{n-1}. \quad (32)$$

(1) *If y and z are limit points of (x_n), then $\|y-p\| = \|z-p\|$.*

(2) *If p is a limit point of (x_n), then $\lim_{n\to\infty} x_n = p$.*

Proof. Since T is quasi-nonexpansive, we have for each $n \in \mathbb{N}$,

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| \sum_{i=0}^{k} \lambda_i T^i x_n - p \right\| = \left\| \sum_{i=0}^{k} \lambda_i \left(T^i x_n - p\right) \right\| \\ &\le \sum_{i=0}^{k} \lambda_i \left\| T^i x_n - p \right\| = \|x_n - p\|. \end{aligned} \quad (33)$$

(1) If y and z are limit points of (x_n), then there exist subsequences (x_{n_k}), (x_{n_l}) of (x_n) such that $\lim_{k\to\infty} x_{n_k} = y$ and $\lim_{l\to\infty} x_{n_l} = z$. From (33), for each $l \in \mathbb{N}$, there exist $k_0, l_0 \in \mathbb{N}$ such that $n_l \le n_{k_0} \le n_{l_0}$, and hence

$$\left\| x_{n_{l_0}} - p \right\| \le \left\| x_{n_{k_0}} - p \right\| \le \left\| x_{n_l} - p \right\|. \quad (34)$$

This implies, as $l \to \infty$, that $\|z-p\| \le \|y-p\| \le \|z-p\|$.

(2) If p is a limit point of (x_n), then there exists a subsequence (x_{n_k}) of (x_n) such that $\lim_{k\to\infty} x_{n_k} = p$. Again, by (33), we must have $\lim_{n\to\infty} x_n = p$.

□

For each $x = (x_1, \ldots, x_p) \in X^p$ and $A \subseteq X$, recall that the *convex hull* of x, the *algebraic interior* of the convex hull of x, and the *diameter* of A are, respectively, defined as follows (see [7] for details):

$$\begin{aligned} \mathrm{co}(x) &= \left\{ \alpha_1 x_1 + \cdots + \alpha_p x_p : \alpha_i \in [0,1],\ \alpha_1 + \cdots + \alpha_p = 1 \right\}, \\ \mathrm{algint}(\mathrm{co}(x)) &= \left\{ \alpha_1 x_1 + \cdots + \alpha_p x_p : \alpha_i \in (0,1),\ \alpha_1 + \cdots + \alpha_p = 1 \right\}, \\ \mathrm{diam}(A) &= \sup\left\{ \|x - y\| : x, y \in A \right\}. \end{aligned} \quad (35)$$

Note that $\mathrm{diam}(A)$ always exists when A is nonempty and bounded. Otherwise, let $\mathrm{diam}(A) = \infty$.

Lemma 27. *Let X be a subset of a normed linear space.* $\mathrm{co}(x)$ *is compact for each $x \in X^p$.*

Proof. Let $H = \{(\alpha_1, \ldots, \alpha_p) \in [0,1]^p : \alpha_1 + \cdots + \alpha_p = 1\}$. Define $T : H \to \mathrm{co}(x)$ by

$$T\left(\alpha_1, \ldots, \alpha_p\right) = \alpha_1 x_1 + \cdots + \alpha_p x_p. \quad (36)$$

It is easy to verify that $\mathrm{co}(x)$ is the image of the compact set H under the continuous function T. □

Lemma 28. *Let X be a subset of a normed linear space and $x \in X^p$.*

(1) $\mathrm{diam}(\mathrm{co}(x)) = \max_{i,j} \|x_i - x_j\|$.

(2) *If $x \notin \Delta$, $y \in \mathrm{algint}(\mathrm{co}(x))$ and $z \in \mathrm{co}(x)$, then* $\|y - z\| < \mathrm{diam}(\mathrm{co}(x))$.

Proof. (1) It is clear that $\max_{i,j}\|x_i - x_j\| \le \mathrm{diam}(\mathrm{co}(x))$. Conversely, let $y, z \in \mathrm{co}(x)$; say $y = \sum_{n=1}^{p} \alpha_n x_n$ and $z = \sum_{n=1}^{p} \beta_n x_n$. Then

$$\begin{aligned} \|y - z\| &= \left\| \sum_n \alpha_n x_n - \sum_n \beta_n x_n \right\| \\ &= \left\| \sum_n \beta_n \sum_m \alpha_m x_m - \sum_n \beta_n x_n \right\| \\ &\le \sum_n \beta_n \left\| \sum_m \alpha_m x_m - x_n \right\| \\ &= \sum_n \beta_n \left\| \sum_m \alpha_m x_m - \sum_m \alpha_m x_n \right\| \\ &\le \sum_n \beta_n \sum_m \alpha_m \|x_m - x_n\| \\ &\le \sum_n \beta_n \sum_m \alpha_m \max_{i,j} \left\| x_i - x_j \right\| = \max_{i,j} \left\| x_i - x_j \right\|. \end{aligned} \quad (37)$$

(2) From the proof of (1), also notice that

$$\begin{aligned} \|y - z\| &\le \sum_n \beta_n \sum_m \alpha_m \|x_m - x_n\| = \sum_n \beta_n \\ &\cdot \sum_{m\ne n} \alpha_m \|x_m - x_n\| \le \sum_n \beta_n \sum_{m\ne n} \alpha_m \max_{i,j} \left\| x_i - x_j \right\| \end{aligned} \quad (38)$$

If $y \in \mathrm{algint}(\mathrm{co}(x))$, we have $\alpha_i \in (0,1)$ for all $i = 1, \ldots, p$, which implies $\sum_{m\ne n} \alpha_m < 1$, and hence

$$\|y - z\| < \max_{i,j} \left\| x_i - x_j \right\| = \mathrm{diam}(\mathrm{co}(x)). \quad (39)$$

□

Lemma 28(1) naturally leads us to the notion of *diameter* of $x = (x_1, \ldots, x_p) \in X^p$:

$$\mathrm{diam}(x) = \mathrm{diam}\left\{x_1, \ldots, x_p\right\} = \max_{i,j} \left\| x_i - x_j \right\|, \quad (40)$$

which will replace $\mathrm{diam}(\mathrm{co}(x))$ until the end of this paper.

Remark 29. From the above notation, we clearly have

$$\mathrm{diam}(x) = \max_i \left\| x_i - x_{j_0} \right\| \quad (41)$$

for some $j_0 \in \{1, \ldots, p\}$, and $\mathrm{diam}(x) = 0$ iff $x \in \Delta$.

Definition 30. A mean-type mapping $\mathbf{M} : X^p \to X^p$ is called *diametrically contractive* if

$$\mathrm{diam}(\mathbf{M}x) < \mathrm{diam}(x) \quad (42)$$

for all $x \in X^p - \mathrm{Fix}(\mathbf{M})$.

Example 31. It is easy to see that the mean-type mapping $\mathbf{M}: X^p \to X^p$ defined by

$$\mathbf{M}(x_1, \ldots, x_p) = (x_p, x_1, \ldots, x_{p-1}) \quad (43)$$

is not diametrically contractive.

Theorem 32. *Suppose X is a convex subset of a normed linear spaces, and $\mathbf{M}: X^p \to X^p$ is a continuous quasi-nonexpansive mean-type mapping such that $\mathbf{M}_\lambda$ is diametrically contractive for some $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_k) \in [0,1]^{k+1}$ with $\sum_{i=0}^{k} \lambda_i = 1$ and $\mathrm{Fix}(\mathbf{M}_\lambda) = \mathrm{Fix}(\mathbf{M})$.*

Let $x_0 \in X^p$ and define a sequence $(x_n) \subseteq X^p$ by

$$x_{n+1} = \mathbf{M}_\lambda x_n. \quad (44)$$

Then (x_n) converges to a fixed point of $\mathbf{M}$.

Proof. Let $K = \mathrm{co}(x_0) \subseteq X$. Then $x_0 \in K^p$ and $x_n \in K^p$ for all $n \in \mathbb{N}$ because $\mathbf{M}_\lambda$ is a mean-type mapping. Since K^p is compact by Lemma 27, (x_n) has a subsequence (x_{n_k}) converging to some $z \in K^p$. It follows that $x_{n_k+1} = \mathbf{M}_\lambda x_{n_k} \to \mathbf{M}_\lambda z$, and hence both z and $\mathbf{M}_\lambda z$ are limit points of (x_n). By Lemma 26(1), we have

$$\|\mathbf{M}_\lambda z - \hat{q}\| = \|z - \hat{q}\| \quad (45)$$

for all $\hat{q} = (q, \ldots, q) \in \mathrm{Fix}(\mathbf{M}_\lambda)$. By writing $z = (z_1, \ldots, z_p)$ and $\mathbf{M}_\lambda = (M_{\lambda 1}, \ldots, M_{\lambda p})$, the above equation becomes

$$\max_i \|M_{\lambda i} z - q\| = \max_i \|z_i - q\|, \quad (46)$$

for all $q \in K$.

By Remark 29, there exists $j_0 \in \{1, \ldots, p\}$ such that $\max_i \|z_i - z_{j_0}\| = \mathrm{diam}(z)$.

By letting $q = z_{j_0}$ in (46), we have

$$\max_i \|M_{\lambda i} z - z_{j_0}\| = \mathrm{diam}(z), \quad (47)$$

and hence

$$\|M_{\lambda i_0} z - z_{j_0}\| = \mathrm{diam}(z), \quad (48)$$

for some $i_0 \in \{1, \ldots, p\}$.

Again, by letting $q = M_{\lambda i_0} z$ in (46), we have

$$\begin{aligned} \mathrm{diam}(M_\lambda z) &\geq \max_i \|M_{\lambda i} z - M_{\lambda i_0} z\| \\ &= \max_i \|z_i - M_{\lambda i_0} z\| \geq \|z_{j_0} - M_{\lambda i_0} z\| \\ &= \mathrm{diam}(z), \end{aligned} \quad (49)$$

which contradicts the diametrical contractivity unless $z \in \mathrm{Fix}(\mathbf{M}_\lambda) = \mathrm{Fix}(\mathbf{M})$.

Therefore, by Lemma 26(2), we have $\lim_{n\to\infty} x_n = z$. □

We now combine the above convergence theorem with virtual stability to obtain the contractibility of fixed point sets of certain continuous quasi-nonexpansive mean-type mappings.

Corollary 33. *If $\mathbf{M}: X^p \to X^p$ is a continuous quasi-nonexpansive mean-type mapping such that $(1-\alpha)\mathrm{Id} + \alpha\mathbf{M}$ is diametrically contractive for some $\alpha \in (0,1]$, then there is a continuous mean-type mapping $r: X^p \to \mathrm{Fix}(\mathbf{M}) \subseteq X^p$ such that*

$$r(x) = \lim_{n\to\infty} [(1-\alpha)\mathrm{Id} + \alpha\mathbf{M}]^n x, \quad (50)$$

and hence $\mathrm{Fix}(\mathbf{M})$ is contractible.

Proof. For each $x_0 \in X^p$ and $n \in \mathbb{N}$, let $x_{n+1} = \mathbf{M}_\lambda x_n$, where $\lambda = (1-\alpha, \alpha)$. Notice that $\mathrm{Fix}(\mathbf{M}) = \mathrm{Fix}(\mathbf{M}_\lambda)$ and $x_n = s_n(x_0)$, where $s_n = [\mathbf{M}_\lambda]^n = [(1-\alpha)\mathrm{Id} + \alpha\mathbf{M}]^n$. Then, by Theorem 32, $r(x_0) = \lim_{n\to\infty} s_n(x_0) = \lim_{n\to\infty} x_n \in \mathrm{Fix}(\mathbf{M})$, and hence $\mathcal{S} = (s_n)$ is a scheme consisting of continuous mappings, where $F(\mathcal{S}) = \mathrm{Fix}(\mathbf{M})$ and $C(\mathcal{S}) = X^p$, which is contractible. It is easy to see that $(1-\alpha)\mathrm{Id} + \alpha\mathbf{M}$ is a mean-type mapping. So is r. Moreover, for each $x \in X^p$ and $q \in \mathrm{Fix}(\mathbf{M})$, we have

$$\begin{aligned} \|s_1 x - q\| &= \|(1-\alpha)x + \alpha\mathbf{M}x - (1-\alpha)q - \alpha q\| \\ &\leq (1-\alpha)\|x - q\| + \alpha\|\mathbf{M}x - q\| \leq \|x - q\| \end{aligned} \quad (51)$$

and hence

$$\|s_n x - q\| = \|s_1^n x - q\| \leq \|s_1^{n-1} x - q\| \leq \cdots \leq \|x - q\|. \quad (52)$$

Therefore, $\mathcal{S}$ is virtually stable. By Theorem 11, r is continuous and $\mathrm{Fix}(\mathbf{M})$ is a retract of X^p. Hence, by Theorem 12, $\mathrm{Fix}(\mathbf{M})$ is contractible. □

Remark 34. The previous corollary immediately extends Theorem 7 to general normed linear spaces because when X is an interval $I \subseteq \mathbb{R}$, the condition

$$\begin{aligned} &\max(M_1(x), \ldots, M_p(x)) \\ &\quad - \min(M_1(x), \ldots, M_p(x)) \\ &\quad < \max(x) - \min(x), \end{aligned} \quad (53)$$

for all $x \in I^p - \Delta$, implies $\mathrm{Fix}(\mathbf{M}) = \Delta$. Hence, $\mathbf{M}$ is quasi-nonexpansive and diametrically contractive, and Theorem 7 follows (with $\alpha = 1$). Notice that, in this case, the existence of K in Theorem 7 is the consequence of the fact that $r(x) \in \Delta$ for each $x \in I^p$.

Moreover, it should be pointed out here that, for any mean-type mapping $\mathbf{M}: X^2 \to X^2$ and $\alpha \in (0,1)$, the combination $(1-\alpha)\mathrm{Id} + \alpha\mathbf{M}$ is always diametrically contractive. If $x = (x_1, x_2) \in X^2$ such that $x \notin \mathrm{Fix}(\mathbf{M})$, we have

$$\begin{aligned} &(1-\alpha)x_1 + \alpha M_1 x \neq x_1 \\ &\text{or } (1-\alpha)x_2 + \alpha M_2 x \neq x_2. \end{aligned} \quad (54)$$

Since $\alpha \neq 1$, we must have

$$\begin{aligned} &(1-\alpha)x_1 + \alpha M_1 x \in \mathrm{algint}(\mathrm{co}(x)) \\ &\text{or } (1-\alpha)x_2 + \alpha M_2 x \in \mathrm{algint}(\mathrm{co}(x)). \end{aligned} \quad (55)$$

Then, by Lemma 28(2), $\mathrm{diam}((1-\alpha)x + \alpha\mathbf{M}x) < \mathrm{diam}(x)$.

The next example shows that, without quasi-nonexpansiveness, the diametrical contractivity alone may not be sufficient to obtain the contractibility of fixed point sets.

Example 35. Define $\mathbf{M} : [0,1]^2 \to [0,1]^2$ by

$$\begin{aligned} \mathbf{M}(x,y) &= (x, [x(1-y)]x + [1 - x(1-y)]y) \\ &= \left(x, y + x^2 - xy + xy^2 - x^2 y\right). \end{aligned} \quad (56)$$

Clearly, $\mathbf{M}$ is a continuous mean-type mapping and it is diametrically contractive by the previous remark. However, $\mathbf{M}$ is not quasi-nonexpansive because

$$\begin{aligned} \left\|\mathbf{M}\left(\frac{1}{4},\frac{1}{2}\right) - \left(\frac{1}{4},1\right)\right\| &= \left\|\left(\frac{1}{4},\frac{15}{32}\right) - \left(\frac{1}{4},1\right)\right\| = \frac{17}{32} \\ &> \frac{1}{2} = \left\|\left(\frac{1}{4},\frac{1}{2}\right) - \left(\frac{1}{4},1\right)\right\|, \end{aligned} \quad (57)$$

and $\mathrm{Fix}(\mathbf{M}) = \{(t,1),(0,t),(t,t) \mid t \in [0,1]\}$ is not contractible.

We now end this paper by giving an example showing that, under the condition of Theorem 32, we only have the contractibility of fixed point sets but not the convexity.

Example 36. Let $\mathbb{R}^2$ be equipped with the maximum norm. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x,y) = \begin{cases} (x, |x|); & y \ge |x|, \\ (|y|, y); & x \ge |y|, \\ (x, -|x|); & y \le -|x|, \\ (-|y|, y); & x \le -|y|. \end{cases} \quad (58)$$

It is easy to see that f is a nonexpansive, hence quasi-nonexpansive, mean-type mapping. The previous remark assures that $(1-\alpha)\mathrm{Id} + \alpha f$ is diametrically contractive for any $\alpha \in (0,1)$. We note that $\mathrm{Fix}(f) = \{(t,t),(t,-t) \mid t \in \mathbb{R}\}$ which is not convex but still contractible.

Moreover, with a slight modification of f, we obtain

$$g(x,y) = \begin{cases} (x, \sqrt{x}y); & \sqrt{x} \le y \le 1, \\ (x, |x|); & x \le y \le \sqrt{x}. \end{cases} \quad (59)$$

Then $g : [-1,1]^2 \to [-1,1]^2$ is a continuous quasi-nonexpansive mean-type mapping and $\mathrm{Fix}(g) = \mathrm{Fix}(f) \cap [-1,1]^2$. We note that g is not nonexpansive because

$$\begin{aligned} \left\|g(0,1) - g\left(\frac{1}{4},1\right)\right\| &= \left\|(0,0) - \left(\frac{1}{4},\frac{1}{2}\right)\right\| = \frac{1}{2} > \frac{1}{4} \\ &= \left\|(0,1) - \left(\frac{1}{4},1\right)\right\|. \end{aligned} \quad (60)$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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