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Research Article

Hyperplanes That Intersect Each Ray of a Cone Once and a Banach Space Counterexample

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Suppose C is a cone contained in real vector space V. When does V contain a hyperplane H that intersects each of the 0-rays in $C \setminus \{0\}$ exactly once? We build on results found in Aliprantis, Tourky, and Klee Jr.'s work to give a partial answer to this question. We also present an example of a salient, closed Banach space cone C for which there does not exist a hyperplane that intersects each 0-ray in $C \setminus \{0\}$ exactly once.

1. Introduction

Let V be a vector space of finite or infinite dimension over the reals. A 0-ray $\subset V$ is an open ray whose source is the origin. We consider the origin of V, $\{0\}$, to be a 0-ray. For us, a cone $C \subset V$ is any union of 0-rays (precise definitions for 0-rays and cones are given in Section 2). Many results involving cones, especially convex cones, require the existence of a hyperplane H which intersects each 0-ray of $C \setminus \{0\}$ exactly once. For example, see Garrett Birkhoff's original proof of his Projective Contraction Mapping Theorem [1], which is discussed in detail in [2]. There seems to be a relatively small amount of literature on the existence of such hyperplanes. Perhaps the most accessible source is Aliprantis and Tourky's, "Cones and Duality" [3]. On the other hand, there is a large body of literature on closely related topics: on the separation of convex bodies by hyperplanes (the various separation versions of the Hann-Banach Theorem) and on the support of cones and convex sets by hyperplanes: Aliprantis and Border [4] or Klee Jr. [5].

Sections 2 and 3 consist of definitions and lemmas which lead to our main results, which are found in Section 4. In Theorem 21 we show that there exists a linear functional L such that $L > 0$ on $C \setminus \{0\}$ if and only if there exists a hyperplane H which intersects each 0-ray exactly once. Our theorem is a slight generalization of a similar result for cone bases (cone bases are defined in Definition 16), given by Aliprantis and Tourky: Theorem 1.47, page 40 of [3], which we present as Theorem 20. Aliprantis and Tourky's results, which involve cone bases, require convexity, whereas ours, which replace cone bases with hyperplanes, do not. We explore the relationship of hyperplanes to cone bases, Lemmas 14 and 15, as well as Corollary 22. Theorem 24 gives an alternative proof of Aliprantis and Tourky's previously cited Theorem 1.47. These results, combined with results of Klee Jr. [5, 6], on linear functionals, give a partial answer to when one should expect such a cone-intersecting hyperplane to exist. We end our short paper in Section 5 with an interesting counterexample involving Banach space cones. Before we prove our results, we include the following illustrative example.

Example 1. Let $V = \mathbb{R}^n$. Let $C = \mathbb{R}^n_{\geq 0}$ be the cone of nonnegative vectors:

$$\mathbb{R}^n_{\geq 0} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ \forall i = 1, 2, \ldots, n\}. \tag{1}$$

Let $L : \mathbb{R}^n \to \mathbb{R}$ be the linear functional defined as follows:

$$L(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i. \tag{2}$$

So, $L > 0$ on $\mathbb{R}^n_{\geq 0} \setminus \{0\}$. Let $H = \{x \in \mathbb{R}^n \mid L(x) = 1\}$. It is easy to see that the hyperplane H intersects each 0-ray

$$[x] = \{\lambda x \mid \lambda > 0\} \tag{3}$$

of $\mathbb{R}^n_{\geq 0} \setminus \{0\}$ exactly once, at the point $x/L(x)$.

The following is clear. If $x \in \mathbb{R}^n_{\geq 0} \setminus \{0\}$ and we let $x^H = x/L(x)$, then $x = L(x)x^H$.

Notes. $H \cap \mathbb{R}^n_{\geq 0}$ has a nice geometric interpretation; it is standard $n-1$ simplex; in probability theory $H \cap \mathbb{R}^n_{\geq 0}$ is the space of all possible probability distributions for processes with n possible outcomes.

2. Cones, 0-Rays, and Hyperplanes

Let V be a real linear space, that is, any real vector space of finite or infinite dimension. Geometrically speaking, a cone is a subset of V which can be represented as a union of rays emanating from a single source point. If that source point is considered to be part of the cone, we say the cone is pointed. In this paper we will always assume that the source of the cone is the null vector (the origin 0) of V. This assumption leads to the concise algebraic definition.

Definition 2. A subset C of a vector space V is called a cone if it is closed under positive scaling, that is, if $\lambda C \subset C$ whenever $\lambda > 0$.

Definition 3. The ray emanating from the origin and passing through the point $v \in V$ is denoted by $[v]$. As a point set,

$$[v] = \{\lambda v \mid \lambda > 0\}. \tag{4}$$

Such a ray will be called a 0-ray to emphasize its source.

The reason for the strict inequality in Definition 3 is to make 0-rays into equivalence classes.

Proposition 4. *Let V be a vector space of any dimension.*

(1) *The 0-rays partition V into equivalence classes.*

(2) *If c is an element of cone $C \subset V$, then $[c] \subset C$. Hence, any cone is partitioned by its 0-rays.*

(3) *Suppose $v, w \in V$. Then, $[v] = [w]$ if and only if $v = \lambda w$ for some $\lambda > 0$.*

(4) *0-ray $[0] \subset V$ consists of single point 0.*

Proof.

Proof of (1). Let $v, w \in V$. Since $v \in [v]$ it follows that $\bigcup_{v \in V}[v] = V$. If $[v] \cap [w] \neq \emptyset$, then $\exists x \in [v] \cap [w]$. But then $x = \lambda_v v = \lambda_w w$ for some $\lambda_v, \lambda_w > 0$. This allows us to write $v = (\lambda_w/\lambda_v)w$. But then

$$[v] = \left\{\lambda \frac{\lambda_w}{\lambda_v} w \mid \lambda > 0\right\} = [w]. \tag{5}$$

Proof of (2) and (3). These two results are a trivial consequence of part (1) and the definitions of cones and 0-rays.

Proof of (4). This result is a trivial consequence of the definition of a 0-ray. □

Definition 5. The cone opposite to C is denoted by $-C$, algebraically: $-C = \{-c \mid c \in C\}$.

Definition 6. The cone C is salient (or bounded) if $C \bigcap -C \subset \{0\}$.

Definition 7. One will say that the cone C is closed if it is a closed subset in V's topology.

Definition 8. If $S \subset V$, then Span(S) is the smallest vector space in V containing S. Alternatively, Span(S) is the set of all finite linear combinations of vectors from S.

The following proposition indicates why salient cones are sometimes called bounded.

Proposition 9. *If C is a salient and closed cone contained in normed vector space V, then C contains no lines.*

Proof. Let v_0 and v_1 be any two distinct points on a line L contained in C. Then $L = \{v_0 + (v_1 - v_0)t \mid t \in \mathbb{R}\}$. Since C is closed under positive scaling the following two sequences

$$\begin{gathered} \left\{\frac{v_0 + (v_1 - v_0)\,n}{\|v_0 + (v_1 - v_0)\,n\|}\right\}_{n=1}^{\infty}, \\ \left\{\frac{v_0 + (v_1 - v_0)(-n)}{\|v_0 + (v_1 - v_0)(-n)\|}\right\}_{n=1}^{\infty} \end{gathered} \tag{6}$$

are contained in C. They also are contained in Span($\{v_0, v_1\}$), which, being a finite dimensional, is complete. Since Span($\{v_0, v_1\}$) is complete and C is closed, these two sequences converge, respectively, to the following two points in C:

$$\begin{gathered} \frac{v_1 - v_0}{\|v_1 - v_0\|}, \\ -\frac{v_1 - v_0}{\|v_1 - v_0\|}. \end{gathered} \tag{7}$$

This contradicts C being salient. □

In Proposition 9 the requirement "C is closed" is necessary as the following example shows.

Example 10. Consider the open upper half plane: $\mathscr{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. $\mathscr{H}$ is closed under positive scaling so $\mathscr{H}$ is a cone. Since $(x, y) \in \mathscr{H} \Leftrightarrow y > 0$ it follows that $-(x, y) = (-x, -y) \notin \mathscr{H}$ and so $\mathscr{H} \cap -\mathscr{H} = \emptyset$. Thus $\mathscr{H}$ is a salient cone contained in $\mathbb{R}^2$, a Banach space. However, cone $\mathscr{H}$ contains every line $y = k$ for each $k > 0$. $\mathscr{H}$ is not a counterexample to Proposition 9 since $\mathscr{H}$ is not topologically closed.

The topologically closed upper half plane, $\overline{\mathscr{H}} = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, is a Banach space cone which contains every line $y = k$ for each $k \geq 0$. $\overline{\mathscr{H}}$ is not a counterexample to Proposition 9 since $\overline{\mathscr{H}}$ is not salient: $(x, 0)$ and $-(x, 0) \in \overline{\mathscr{H}}$ for each $x \in \mathbb{R}$.

Definition 11. If $v \in V$ and W is a vector subspace of V, then one calls $v + W$ a hyperplane.

The following standard result, regarding the smallest hyperplane generated by a subset $\mathcal{S}$, of a vector space V, is useful.

Lemma 12. *Let $\mathcal{S}$ be any subset of vector space V. Let s_0 be any fixed element of $\mathcal{S}$ and let*

$$W = \operatorname{Span}\{s_1 - s_2 \mid s_1, s_2 \in \mathcal{S}\}. \tag{8}$$

Then $s_0 + W$ is the smallest hyperplane containing $\mathcal{S}$.

Proof. W is a vector subspace of V. So $s_0 + W$ is a hyperplane. Let $s \in \mathcal{S}$. Then, $s = s_0 + (s - s_0) \in s_0 + W$. So $\mathcal{S} \subset s_0 + W$. Now, let H' be any hyperplane that contains $\mathcal{S}$. Since $\mathcal{S} \subset H'$, we can write $H' = s_0 + W'$, for some vector subspace $W' \subset V$. But then, $-s_0 + H' = W'$. This implies $-s_0 + S \subset W'$. If $s_1, s_2 \in \mathcal{S}$, then

$$(-s_0 + s_1) - (-s_0 + s_2) = s_1 - s_2 \in W'. \tag{9}$$

So $W \subset W'$. Hence, $s_0 + W \subset s_0 + W' = H'$. □

The following standard result about convex sets is stated without proof. See Klee [7] or Lay [8] for details.

Proposition 13. *Suppose $\mathcal{S}$ is a convex set contained in V. If $s_1, s_2, \ldots, s_n \in \mathcal{S}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive real numbers, then*

$$\frac{\sum_{i=1}^{n} \alpha_i s_i}{\sum_{i=1}^{n} \alpha_i} \in \mathcal{S}. \tag{10}$$

3. Intersection Lemmas

Lemma 14. *Suppose that C is a cone contained in V, a vector space over the reals of finite or infinite dimension, and that there exists linear functional L from V to the reals such that $0 < L(c)$ for each $c \in C \setminus \{0\}$. Let $H = \{x \in V \mid L(x) = 1\}$. For each $c \in C \setminus \{0\}$, let*

$$\frac{c}{L(c)} = c^H \tag{11}$$

be the central projection of c onto H. Then

(1) *C is salient.*

Suppose $c, c' \in C \setminus \{0\}$; then

(2) *$c^H = (c')^H$ if and only if $[c] = [c']$;*

(3) *$(c^H)^H = c^H$;*

(4) *$[c^H] = [c]$;*

(5) *H intersects each 0-ray $[c]$ in $C \setminus \{0\}$ once and only once; in particular, $[c] \cap H = \{c^H\}$;*

(6) *$H = L^{-1}(1) = c^H + \ker(L)$, so that H is a hyperplane in V.*

Proof. (1) If $x \in C \bigcap -C \setminus \{0\}$, then $-x \in C \bigcap -C \setminus \{0\}$. But then $L(x) > 0$ and $L(-x) = -L(x) > 0$, which is impossible. So $C \bigcap -C \subset \{0\}$, implying C is salient.

(2) Let $c, c' \in C \setminus \{0\}$. Then $L(c), L(c') > 0$. If $c^H = (c')^H$, then $c/L(c) = c'/L(c')$, which implies $[c'] \cap [c] \neq \emptyset$. By Proposition 4, part (1), the 0-rays are equivalence classes. Hence, $[c] = [c']$.

On the other hand, if $[c'] = [c]$, then there exists $\lambda > 0$ such that $c' = \lambda c$. But then

$$(c')^H = \frac{c'}{L(c')} = \frac{\lambda c}{L(\lambda c)} = \frac{\lambda c}{\lambda L(c)} = \frac{c}{L(c)} = c^H. \tag{12}$$

(3)

$$\begin{aligned}(c^H)^H &= \left(\frac{c}{L(c)}\right)^H = \frac{c/L(c)}{L(c/L(c))} = \frac{c/L(c)}{(L(c)/L(c))} \\ &= \frac{c}{L(c)} = c^H.\end{aligned} \tag{13}$$

(4) $c^H = c/L(c)$, so $[c^H] \cap [c] \neq \emptyset$. The 0-rays are equivalence classes, by Proposition 4, part (1), so $[c^H] = [c]$.

(5) $c^H \in [c]$ by part (4) of this lemma. $c^H \in H$ because

$$L(c^H) = L\left(\frac{c}{L(c)}\right) = \frac{L(c)}{L(c)} = 1. \tag{14}$$

So $\{c^H\} \subset [c] \cap H$. The following shows $[c] \cap H \subset \{c^H\}$. Let $c' \in [c] \cap H$. By part (4) of this lemma, $[c] = [c^H]$. So, $c' = \lambda c^H$ for some $\lambda > 0$. c' is in H. So, $L(c') = L(\lambda c^H) = \lambda L(c^H) = 1$. By (14), $L(c^H) = 1$, so $\lambda = 1$ and $c' = c^H$.

(6) We can use the following elementary, standard result: if L is any linear functional (not identically zero) from V to $\mathbb{R}$, then $L^{-1}(r)$ is a hyperplane $\forall r \in \mathbb{R}$. Moreover, $L^{-1}(r)$ can always be written in form $v_r + \ker(L)$, where v_r is any element of $L^{-1}(r)$. To prove this, simply use the linearity of L and note that L being nonidentically zero implies $L^{-1}(r) \neq \emptyset$. To prove (6), note that $H = L^{-1}(1)$ and $c^H \in L^{-1}(1)$. □

Lemma 15. *Suppose that C is a cone contained in V, an arbitrary vector space of finite or infinite dimension; that C contains at least one nonzero vector; and that there exists hyperplane H which intersects every 0-ray in $C \setminus \{0\}$ exactly once.*

(1) *Then there exists linear functional L mapping V to the reals such that $L > 0$ on $C \setminus \{0\}$.*

One can sharpen this result. For each $d \in C \setminus \{0\}$, let d^H be the unique intersection of 0-ray $[d]$ and hyperplane H. In other words, $[d] \cap H = \{d^H\}$.

(2) *Linear functional L, mentioned in part (1), can be chosen so that $\forall d \in C \setminus \{0\}$; we have $d = L(d)d^H$. With this choice of L, we have $L > 0$ on $C \setminus \{0\}$ and $L = 1$ on $C \cap H$.*

Proof. The representative Figure 1 serves to illustrate some of the arguments detailed in this proof. In Figure 1 the vector space V is represented as $\mathbb{R}^2$; the cone C as $\mathbb{R}^2_{\geq 0}$; the

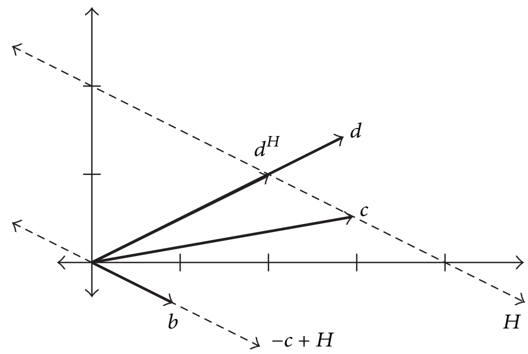

Figure 1

hyperplane H as the solution to $x + 2y = 4$; the nonzero element of $C \setminus \{0\}$ as c; and the vector subspace $-c + H$ as the solution to $x + 2y = 0$. In our proof the basis for subspace $-c+H$ will be denoted by $\mathscr{B}_{-c+H}$; in the figure it is represented by $\{b\}$. In our proof the basis for V will be denoted by $\mathscr{B}$; in Figure 1 it is represented by $\{b, c\}$. The linear functional L, mentioned above (in part (2) of this lemma), is given by $L(x, y) = (x + 2y)/4$. □

Let $c \in H \cap C \setminus \{0\}$, such a c exists by this lemma's main assumption.

Claim 1. $-c + H$ is a vector subspace of V. Proof is as follows: since H is hyperplane there exists vector $v \in V$ and vector subspace $W \subset V$ such that $H = v+W$. But then, $c = v+w_c$ for some $w_c \in W$. Hence $-c+H = -(v+w_c)+v+W = w_c+W = W$.

Claim 2. $c \notin -c + H$. Proof is as follows: if $c \in -c + H$ then $c = -c + h_c$ for some $h_c \in H$, but then $2c = h_c \in [c]$. That means $c, 2c \in H \cap [c]$. This contradicts with the fact that each ray intersects H exactly once. So claim 2 is proven.

Let $\mathscr{B}_{-c+H}$ be a basis for $-c + H$. Since $c \notin -c + H$, which is a vector subspace of V, the set $\mathscr{B}_{-c+H} \cup \{c\}$ forms a basis for $\mathrm{Span}(-c + H, c)$. Since $H \subset \mathrm{Span}(-c + H, c)$ it follows that $C \subset \mathrm{Span}(-c + H, c)$.

If $\mathrm{Span}(-c + H, c) = V$, let $\mathscr{B} = \mathscr{B}_{-c+H} \cup \{c\}$. If $\mathrm{Span}(-c+H, c)$ is a proper vector subspace of V we can extend the basis $\mathscr{B}_{-c+H} \cup \{c\}$ to a basis $\mathscr{B}$ for V. See Theorem 4.72 in [9] regarding extensions of bases.

For each $x \in V$ we define $L(x)$ as follows. We can write x uniquely as a finite linear combination of basis elements from $\mathscr{B}$:

$$x = \alpha_x c + \sum_{i=1}^{n_x} \alpha_i b_i, \tag{15}$$

where $c, b_i \in \mathscr{B}$. Define $L(x) = \alpha_x$.

If $d \in C \setminus \{0\}$ then, by this lemma's assumption, the set $[d] \cap H$ contains a single element, which we will call d^H. So $\{d^H\} = [d] \cap H$. Since $d^H \in H = c + (-c + H)$ we must have $d^H = c + \sum_{i=1}^{n_d} \alpha_i b_i$ with $b_i \in \mathscr{B}_{-c+H}$. By Proposition 4, part (1), $[d] = [d^H]$, so there exists $\alpha_d > 0$ such that $d = \alpha_d d^H$. So $d = \alpha_d d^H = \alpha_d c + \alpha_d \sum_{i=1}^{n_d} \alpha_i b_i$. So $L(d) = \alpha_d > 0$ and $d = L(d) d^H$. Hence, if $d \in C \cap H$, then $d^H = d$ and $L(d) = 1$.

The following definition comes from Aliprantis and Tourky [3].

Definition 16. $\mathscr{B}$ is a base for the cone C if $\mathscr{B}$ is a convex subset of $C \setminus \{0\}$ and if for each $c \in C \setminus \{0\}$ there exists a unique $b \in B$ and a unique $\lambda > 0$ such that $c = \lambda b$.

Proposition 17. *If $\mathscr{B}$ is a base for cone C and $c \in C \setminus \{0\}$, then $\mathscr{B}$ intersects 0-ray $[c]$ in exactly one point.*

Proof. By the definition of a cone base, $c = \lambda b$ for a unique $b \in \mathscr{B}$ and a unique $\lambda > 0$. But then $b = c/\lambda \in [c]$. If $b' \in \mathscr{B} \cap [c]$, then since $[c]$ is a 0-ray, $b' = \lambda' c$ for some $\lambda' > 0$. But then $b'/\lambda' = c$. Uniqueness forces $b' = b$. □

Lemma 18. *Suppose $\mathscr{B}$ is a base for cone $C \subset V$. Let $H_{\mathscr{B}}$ be the smallest hyperplane containing $\mathscr{B}$. Then $H_{\mathscr{B}}$ intersects each 0-ray in $C \setminus \{0\}$ exactly once. Moreover, $H_{\mathscr{B}} \cap C = \mathscr{B}$.*

Proof. Suppose $[c]$ is a 0-ray in $C \setminus \{0\}$. Since $\mathscr{B}$ is a cone base $c = \lambda_c b$ for a unique $\lambda_c > 0$ and a unique $b \in \mathscr{B}$. So we can write $[c]$ in terms of b: $[c] = \{\lambda b \mid \lambda > 0\}$. This means every element in $H_{\mathscr{B}} \cap [c]$ can be written in the form λb for some $\lambda > 0$. We will show that if $\lambda b \in H_{\mathscr{B}} \cap [c]$, then $\lambda = 1$, which will prove the lemma.

By Lemma 12, we can write $H_{\mathscr{B}}$ in the form $H_{\mathscr{B}} = b + W$, where $W = \mathrm{Span}\{b_1 - b_2 \mid b_1, b_2 \in \mathscr{B}\}$. So, if $\lambda b \in H_{\mathscr{B}} \cap [c]$, we can write

$$\lambda b = b + \sum_{i-1}^{n} \alpha_i \left(b_{i,1} - b_{i,2}\right) \tag{16}$$

with $\alpha_i \in \mathbb{R}$, $\alpha_i \neq 0$, and $b_{i,j} \in \mathscr{B}$, $i = 1, 2, \ldots, n$ and $j = 1, 2$. A little algebra transforms (16) into

$$b = \sum_{i=1}^{n} \frac{\alpha_i}{\lambda - 1} \left(b_{i,1} - b_{i,2}\right) \tag{17}$$

provided $\lambda \neq 1$. We can rewrite (17) in the following form:

$$b = \sum_{i=1}^{n} \beta_i \left(b_{i,1} - b_{i,2}\right), \tag{18}$$

where all coefficients β_i are positive by letting

$$\beta_i = \left| \frac{\alpha_i}{\lambda - 1} \right| \tag{19}$$

and by relabeling $b_{i,j}$ if necessary, that is, by switching $b_{i,1}$ with $b_{i,2}$ when $\alpha_i/(\lambda - 1) < 0$.

We can simplify (18) by letting $\beta = \sum_{i=1}^{n} \beta_i$ and rearranging terms:

$$\begin{aligned} b &= \frac{\beta \sum_{i=1}^{n} \beta_i \left(b_{i,1} - b_{i,2}\right)}{\sum_{i=1}^{n} \beta_i} \\ &= \beta \left(\frac{\sum_{i=1}^{n} \beta_i b_{i,1}}{\sum_{i=1}^{n} \beta_i} - \frac{\sum_{i=1}^{n} \beta_i b_{i,2}}{\sum_{i=1}^{n} \beta_i} \right) = \beta \left(b_1 - b_2\right), \end{aligned} \tag{20}$$

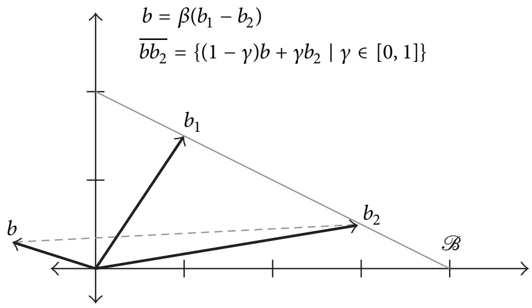

Figure 2

where

$$b_1 = \frac{\sum_{i=1}^{n} \beta_i b_{i,1}}{\sum_{i=1}^{n} \beta_i},$$
$$b_2 = \frac{\sum_{i=1}^{n} \beta_i b_{i,2}}{\sum_{i=1}^{n} \beta_i}. \tag{21}$$

$\mathscr{B}$ is convex. So Proposition 13 implies $b_1, b_2 \in \mathscr{B}$. Since $b, b_2 \in \mathscr{B}$, if $\gamma \in [0,1]$, the convexity of $\mathscr{B}$ implies

$$(1-\gamma) b + \gamma b_2 \in \mathscr{B}. \tag{22}$$

We expand (22) using (20):

$$\begin{aligned}(1-\gamma) b + \gamma b_2 &= (1-\gamma) \underbrace{\beta (b_1 - b_2)}_{b} + \gamma b_2 \\ &= (1-\gamma) \beta b_1 - (1-\gamma) \beta b_2 + \gamma b_2 \\ &= (1-\gamma) \beta b_1 + (\gamma - (1-\gamma) \beta) b_2 \\ &\in \mathscr{B}.\end{aligned} \tag{23}$$

We find a $\gamma \in (0,1)$ such that the coefficient of b_2 in the third line of (23) is zero. That is, we find the intersection of line segment $\overline{b\, b_2} = \{(1-\gamma)b + \gamma b_2 \mid \gamma \in [0,1]\}$, which is contained in $\mathscr{B}$ by convexity, with the 0-ray $[b_1]$. See Figure 2.

Here is the algebra:

$$\begin{aligned}\gamma - (1-\gamma)\beta &= 0, \\ \gamma - \beta + \gamma\beta &= 0, \\ \gamma(1+\beta) - \beta &= 0, \\ \gamma &= \frac{\beta}{1+\beta} \in (0,1).\end{aligned} \tag{24}$$

So, with this choice of γ, (23) becomes

$$\begin{aligned}(1-\gamma)\beta b_1 &= \left(1 - \frac{\beta}{1+\beta}\right) \beta b_1 \\ &= \left(\frac{1+\beta}{1+\beta} - \frac{\beta}{1+\beta}\right) \beta b_1 = \frac{\beta}{1+\beta} b_1 \in \mathscr{B}.\end{aligned} \tag{25}$$

But $\beta > 0$, which implies $\beta/(1+\beta) \in (0,1)$. So $b_1 \neq (\beta/(1+\beta))b_1$, which means we can write $b_1 \in \mathscr{B} \subset C$ as multiples of two distinct elements from $\mathscr{B}$, namely, b_1 and $(\beta/(1+\beta))b_1$. That is, $b_1 = 1b_1$ and $b_1 = ((1+\beta)/\beta)((\beta/(1+\beta))b_1)$. This contradicts $\mathscr{B}$ being a cone base. Hence, our assumption that $\lambda \neq 1$ must have been wrong. □

Lemma 19. *Suppose that C is a convex cone and H is a hyperplane which intersects each 0-ray in $C \setminus \{0\}$ exactly once. Then $C \cap H$ is cone base for C.*

Proof. C is convex by assumption. Hyperplanes are always convex. Since the intersection of convex sets is convex, $H \cap C$ is convex. If $c \in C \setminus \{0\}$ then set $H \cap [c]$ contains a single, unique point, say h_c. Since $h_c \in [c]$, there exists $\lambda_1 > 0$ such that $h_c = \lambda_1 c$. Since $c \neq 0$, λ_1 is unique. Let $\lambda = 1/\lambda_1$. Then $c = \lambda h_c$ with h_c being unique with respect to $H \cap C$ and $\lambda > 0$ being unique with respect to h_c. □

4. Intersection Theorems

Theorem 20, below, can be found in Aliprantis and Tourky [3]. It relies on convexity.

Theorem 20. *Suppose that convex cone C is a subset of V, an arbitrary vector space of finite or infinite dimension, and that C contains at least one nonzero vector. Then the following are equivalent.*

(1) *There exists base $\mathscr{B}$ for cone C.*

(2) *There exists a linear functional on V which is strictly positive on $C \setminus \{0\}$.*

Proof.

Sketch. The linear functional is defined as follows. Suppose $c \in C \setminus \{0\}$. Since $\mathscr{B}$ is a cone base for C there exists a unique $\lambda_c > 0$ and a unique $b_c \in \mathscr{B}$ such that $c = \lambda_c b_c$. Define $f(c) = \lambda_c$. Aliprantis and Tourky use the convexity of $\mathscr{B}$ to show that functional f is linear. For details and the rest of the proof, see Aliprantis and Tourky [3], Theorem 1.47, page 40. □

Theorem 21, below, is a slight generalization of Theorem 20. Our version of the theorem does not require convexity and it replaces the cone base by an intersecting hyperplane.

Theorem 21. *Suppose that cone C is a subset of V, an arbitrary vector space of finite or infinite dimension, and that C contains at least one nonzero vector. Then the following are equivalent.*

(1) *There exists a linear functional on V which is strictly positive on $C \setminus \{0\}$.*

(2) *There exists hyperplane H such that H intersects each 0-ray in $C \setminus \{0\}$ exactly once.*

Proof. (1) is equivalent to (2) is proven in Lemmas 14 and 15. □

Corollary 22. *If cone C has a cone base, then both of the following are true.*

(1) *There exists a linear functional on V which is strictly positive on $C \setminus \{0\}$.*

(2) *There exists hyperplane H such that H intersects each 0-ray in $C \setminus \{0\}$ exactly once.*

Proof. Lemma 18 combined with Theorem 21 proves Corollary 22. □

The converse of Corollary 22 is not true as the following example shows.

Example 23. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Let C be the nonconvex cone $[e_1] \cup [e_2] \subset \mathbb{R}^2$. Hyperplane $H = \{e_1 + \alpha(e_2 - e_1) \mid \alpha \in \mathbb{R}\}$ intersects each 0-ray in $C \setminus \{0\}$ exactly once. However no cone base exists for C.

If the cone C is convex, then the converse of Corollary 22 is true. See Theorem 24.

Theorem 24. *Suppose that convex cone C is a subset of V, an arbitrary vector space of finite or infinite dimension, and that C contains at least one nonzero vector. Then the following are equivalent.*

(1) *There exists base $\mathscr{B}$ for cone C.*

(2) *There exists a linear functional on V which is strictly positive on $C \setminus \{0\}$.*

(3) *There exists hyperplane H such that H intersects each 0-ray in $C \setminus \{0\}$ exactly once.*

Suppose (1), (2), or (3) holds. Let $H_{\mathscr{B}}$ be the smallest hyperplane containing base $\mathscr{B}$. Then

(i) $H_{\mathscr{B}} \cap C = \mathscr{B}$;

(ii) *$H_{\mathscr{B}}$ will intersect each 0-ray in $C \setminus \{0\}$ exactly once.*

Proof. (1) is equivalent to (2) is proven in Lemmas 18 and 19. For an alternative proof that (1) is equivalent to (2) see Aliprantis and Tourky [3], Theorem 1.47, page 40. (2) is equivalent to (3) as proven in Lemmas 14 and 15. (i) and (ii) are proven in Lemma 18. □

Corollary 25. *If C is a convex, salient, closed cone in finite dimensional normed linear space V, then there exists hyperplane H such that H intersects each 0-ray in C exactly once. $H \cap C$ is compact.*

Proof. According to Corollary 3.8 of Aliprantis and Tourky [3], Klee Jr. [6] proved that every convex, salient, closed cone C in a finite dimensional normed linear space has a compact base, $\mathscr{B}$. But then Theorem 24 implies there exists a hyperplane, H, which intersects each 0-ray in C exactly once and $H \cap C = \mathscr{B}$. □

Remark 26. Klee Jr.'s paper, "Separation Properties of Convex Cones" [5], much referenced in the literature, shows that a closed, salient, convex cone C in a separable normed linear space will have associated to it a linear functional which is strictly positive on $C \setminus \{0\}$. However, the following example shows that given an arbitrary salient cone C in a Banach space V, we cannot always find a strictly positive linear functional on $C \setminus \{0\}$. By Theorem 24, this means we cannot always find hyperplane H which intersects every 0-ray in $C \setminus \{0\}$ exactly once.

5. Banach Space Counterexample

We cannot always find hyperplane H which intersects every 0-ray of a closed cone C exactly once, even if the underlying vector space is Banach, as the following example, based upon Problem 6, page 42 of [3], shows. By Theorem 21, the existence of such a hyperplane is equivalent to the existence of a nonzero linear functional on V which is strictly positive on $C \setminus \{0\}$.

Example 27. Let $V = B(\Omega)$ = the set of all bounded functions from Ω = an uncountable set, to $\mathbb{R}$. V equipped with the sup norm (if $\phi \in B(\Omega)$, then $\|\phi\|_\infty = \sup_{\omega\in\Omega}|\phi(\omega)|$), is an l^∞ Banach Space. Let C = all the bounded nonnegative functions from Ω to $\mathbb{R}$. For each $A \subset \Omega$ let

$$\chi_A(x) = \begin{cases} 1, & x \in A; \\ 0, & x \notin A. \end{cases} \tag{26}$$

Then $\chi_A \in C$, for each $A \subset \Omega$. Note that $\chi_\emptyset = 0$, and if $A \neq \emptyset$, then $\|\chi_A\|_\infty = 1$. If A, B are subsets of Ω, we have the following identity:

$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}. \tag{27}$$

If $A \subset B$, then we can write B as disjoint union $B = A \cup (B - A)$. Equation (27) implies

$$\chi_B = \chi_{A\cup(B-A)} = \chi_A + \chi_{B-A} - \chi_{A\cap(B-A)} = \chi_A + \chi_{B-A}. \tag{28}$$

Suppose L is any linear functional on V; that $A \subset B \subset \Omega$; and that F is any finite subset of Ω. Then (28) implies

$$L(\chi_B) = L(\chi_{B\setminus A}) + L(\chi_A), \tag{29}$$

$$L(\chi_F) = \sum_{\omega\in F} L(\chi_\omega). \tag{30}$$

Let us suppose that linear functional $L > 0$ on all of $C \setminus \{0\}$. If A is strictly contained in B, then (29) implies

$$L(\chi_B) > L(\chi_A). \tag{31}$$

Let $S_{1/n} = \{\omega \in \Omega \mid L(\chi_\omega) > 1/n\}$. If the cardinality of $S_{1/n}$ is infinite, then $L(\chi_{S_{1/n}})$ must be infinite. We can see this as follows. Let $F \subset S_{1/n}$ be a finite set. By (30) and the definition of $S_{1/n}$, $L(\chi_F) > |F|/n$, where $|F|$ is the size of F. By (31), $L(\chi_{S_{1/n}}) > L(\chi_F)$. So, if $S_{1/n}$ is infinite, then $L(\chi_{S_{1/n}})$ must be infinite. However, $L : V \to \mathbb{R}$. So $L(\chi_{S_{1/n}})$ must be real, which means $S_{1/n}$ must be finite. But then $S = \bigcup_{n=1}^{\infty} S_{1/n}$ is at most countable. This implies $\Omega \setminus S \neq \emptyset$. Suppose $\omega_0 \in \Omega \setminus S$. Then $L(\chi_{\omega_0}) = 0$, which contradicts $L > 0$ on $C \setminus \{0\}$.

Competing Interests

The author declares that there are no competing interests.

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