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Research Article

Existence of Optimal Control for a Nonlinear-Viscous Fluid Model

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We consider the optimal control problem for a mathematical model describing steady flows of a nonlinear-viscous incompressible fluid in a bounded three-dimensional (or a two-dimensional) domain with impermeable solid walls. The control parameter is the surface force at a given part of the flow domain boundary. For a given bounded set of admissible controls, we construct generalized (weak) solutions that minimize a given cost functional.

1. Introduction

The control and optimization problems in hydrodynamics have been the focus of attention of the control theory specialists for a long time. Flow boundary control problems have attracted increasing interest in recent years (see, e.g., [1–7]). Such problems are of interest from a theoretical perspective and are beneficial to applications as boundary control is easy to implement in practice.

In this paper, we study the optimal boundary control problem for a mathematical model describing steady flows of a nonlinear-viscous incompressible fluid in a bounded domain of space $\mathbb{R}^n$, $n = 2, 3$, with impermeable solid walls. A distinguishing feature of the problem under consideration is that the surface force at the flow domain boundary is used as a control parameter instead of the nonhomogeneous Dirichlet boundary condition for the velocity field. Such an approach makes it possible to consider the case of flow control in a domain with impermeable solid walls without using external body forces as control parameters.

It should be mentioned at this point that a lot of studies have been conducted towards mathematical models of nonlinear-viscous fluids (see monograph [8] and [9–12]). Nevertheless, there are very few results on the existence and properties of solutions of control problems for nonlinear-viscous fluid flows. To the best of our knowledge, some results have only been obtained for the two-dimensional case (see [13, 14]).

Also, we would mention that there are many mathematical results concerning optimal control problems for the classical Navier-Stokes equations (see [15–17] and the references therein).

The aim of this paper is to prove the solvability of the optimal control problem, which is discussed above. More precisely, for a given bounded set of admissible boundary controls, we will construct generalized (weak) solutions that minimize a given lower weakly semicontinuous cost functional.

2. Problem Formulation and Main Result

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n = 2$ or 3) with boundary $\Gamma \in \mathscr{C}^2$. Consider the following optimal boundary control problem:

$$\mathbf{v} \cdot \nabla \mathbf{v} - \operatorname{div} \mathbf{S} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{S} = \psi\left(I_2(\mathbf{v})\right) \mathbf{D}(\mathbf{v}) \quad \text{in } \Omega, \tag{3}$$

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{4}$$

$$\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma \setminus \Gamma_c, \tag{5}$$

$$[\mathbf{S}\mathbf{n}]_\tau = \mathbf{u} \quad \text{on } \Gamma_c, \tag{6}$$

$$\mathbf{u} \in \mathbf{U}, \tag{7}$$

$$J(\mathbf{v}, \mathbf{S}, \mathbf{u}) \longrightarrow \inf, \tag{8}$$

where $\mathbf{v}$ is the velocity field, p is the pressure function, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{f}$ is the body force, the symbol ∇ denotes the gradient with respect to the spatial variables $x_1, \ldots, x_n$, the divergence $\operatorname{div} \mathbf{S}$ is the vector with coordinates

$$(\operatorname{div} \mathbf{S})_i = \sum_{j=1}^{n} \frac{\partial S_{ji}}{\partial x_j}, \tag{9}$$

$\mathbf{D}(\mathbf{v})$ is the rate of deformation tensor,

$$\mathbf{D}_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{10}$$

$I_2(\mathbf{v})$ is the second invariant of $\mathbf{D}(\mathbf{v})$,

$$I_2(\mathbf{v}) = \sum_{i,j=1}^{n} \left(\mathbf{D}_{ij}(\mathbf{v})\right)^2, \tag{11}$$

ψ is a given function, $\mathbf{n}$ is the unit vector of the outer normal to Γ, $\mathbf{u}$ is the control, $\mathbf{v} \cdot \mathbf{n}$ is the scalar product of the vectors $\mathbf{v}$ and $\mathbf{n}$ in space $\mathbb{R}^n$, the symbol $[\cdot]_\tau$ denotes the tangential component of a vector, that is,

$$[\mathbf{w}]_\tau = \mathbf{w} - (\mathbf{w} \cdot \mathbf{n})\,\mathbf{n}, \tag{12}$$

Γ_c is a part of Γ from which the control is realized, $\mathbf{U}$ is the set of admissible controls, and J is a given cost functional.

From here on, the following notations will be used. $\mathbb{M}_s^{n\times n}$ denotes the space of symmetric $n \times n$-matrices with the norm

$$\|\mathbf{A}\|_{\mathbb{M}_s^{n\times n}} = \left(\sum_{i,j=1}^{n} A_{ij}^2\right)^{1/2}. \tag{13}$$

We use the standard notations $\mathbf{L}^q(\Omega, \mathbf{E})$ and $\mathbf{W}^{m,q}(\Omega, \mathbf{E})$ for the Lebesgue and Sobolev spaces of vector functions defined on Ω with values in a finite-dimensional space $\mathbf{E}$ (for details, see [18]). The scalar product in the space $\mathbf{L}^2(\Omega, \mathbf{E})$ is denoted by $(\cdot, \cdot)$.

By definition, put

$$\mathbf{L}_\tau^2(\Gamma_c, \mathbb{R}^n) = \left\{\mathbf{w} \in \mathbf{L}^2(\Gamma_c, \mathbb{R}^n) : \ \mathbf{w} \cdot \mathbf{n} = 0\right\}. \tag{14}$$

Moreover, we introduce the space

$$\mathbf{X}(\Omega, \mathbb{R}^n) = \left\{\mathbf{v} \in \mathbf{W}^{1,2}(\Omega, \mathbb{R}^n) : \nabla \cdot \mathbf{v} = 0, \ \mathbf{v}|_\Gamma \cdot \mathbf{n} = 0, \ \mathbf{v}|_{\Gamma\setminus\Gamma_c} = \mathbf{0}\right\} \tag{15}$$

with the following norm:

$$\|\mathbf{v}\|_{\mathbf{X}(\Omega,\mathbb{R}^n)} = \|\mathbf{D}(\mathbf{v})\|_{\mathbf{L}^2(\Omega,\mathbb{M}_s^{n\times n})}. \tag{16}$$

In the right-hand side of (15), the restriction of a vector function $\mathbf{v} : \Omega \to \mathbb{R}^n$ to Γ is defined by the formula

$$\mathbf{v}|_\Gamma = \gamma_0 \mathbf{v}, \tag{17}$$

where $\gamma_0 : \mathbf{W}^{1,2}(\Omega, \mathbb{R}^n) \to \mathbf{L}^2(\Gamma, \mathbb{R}^n)$ is the trace operator.

It follows from Korn's inequality (see [8]) that the norm $\|\cdot\|_{\mathbf{X}(\Omega,\mathbb{R}^n)}$ is equivalent to the norm induced from $\mathbf{W}^{1,2}(\Omega, \mathbb{R}^n)$. Furthermore, we have the following estimates:

$$\begin{gathered} \|\mathbf{v}\|_{\mathbf{L}^2(\Gamma,\mathbb{R}^n)} \le C_1 \|\mathbf{v}\|_{\mathbf{X}(\Omega,\mathbb{R}^n)}, \\ \|\mathbf{v}\|_{\mathbf{L}^2(\Omega,\mathbb{R}^n)} \le C_2 \|\mathbf{v}\|_{\mathbf{X}(\Omega,\mathbb{R}^n)}, \end{gathered} \tag{18}$$

where C_1 and C_2 are positive constants.

Suppose the following:

(i) the function ψ is measurable and there exist constants a_1 and a_2 such that

$$0 < a_1 \le \psi(t) \le a_2, \quad t \in [0, +\infty), \tag{19}$$

(ii) for any $\mathbf{A}, \mathbf{B} \in \mathbb{M}_s^{n\times n}$, we have

$$\sum_{i,j=1}^{n} \left(\psi\left(\|\mathbf{A}\|_{\mathbb{M}_s^{n\times n}}^2\right) A_{ij} - \psi\left(\|\mathbf{B}\|_{\mathbb{M}_s^{n\times n}}^2\right) B_{ij}\right)\left(A_{ij} - B_{ij}\right) \ge 0, \tag{20}$$

(iii) the set $\mathbf{U}$ is bounded and sequentially weakly closed in $\mathbf{L}_\tau^2(\Gamma_c, \mathbb{R}^n)$,

(iv) the functional $J : \mathbf{X}(\Omega, \mathbb{R}^n) \times \mathbf{L}^2(\Omega, \mathbb{M}_s^{n\times n}) \times \mathbf{L}_\tau^2(\Gamma_c, \mathbb{R}^n) \to \mathbb{R}$ is lower weakly semicontinuous; that is, for any sequence $\{(\mathbf{v}^k, \mathbf{S}^k, \mathbf{u}^k)\}_{k=1}^\infty$ such that $\mathbf{v}^k \to \mathbf{v}$ weakly in $\mathbf{X}(\Omega, \mathbb{R}^n)$, $\mathbf{S}^k \to \mathbf{S}$ weakly in $\mathbf{L}^2(\Omega, \mathbb{M}_s^{n\times n})$, and $\mathbf{u}^k \to \mathbf{u}$ weakly in $\mathbf{L}_\tau^2(\Gamma_c, \mathbb{R}^n)$, we have

$$J(\mathbf{v}, \mathbf{S}, \mathbf{u}) \le \liminf_{k\to\infty} J\left(\mathbf{v}^k, \mathbf{S}^k, \mathbf{u}^k\right). \tag{21}$$

Example 1. Let us consider the following cost functionals:

$$\begin{aligned} J_1(\mathbf{v}, \mathbf{S}, \mathbf{u}) &= \lambda_1 \|\mathbf{v} - \tilde{\mathbf{v}}\|_{\mathbf{L}^2(\Omega,\mathbb{R}^n)}^2 + \lambda_2 \left\|\mathbf{S} - \tilde{\mathbf{S}}\right\|_{\mathbf{L}^2(\Omega,\mathbb{M}_s^{n\times n})}^2 \\ &\quad + \lambda_3 \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\mathbf{L}_\tau^2(\Gamma_c,\mathbb{R}^n)}^2, \\ J_2(\mathbf{v}, \mathbf{S}, \mathbf{u}) &= -\lambda_1 \|\mathbf{v} - \widetilde{\mathbf{w}}\|_{\mathbf{L}^2(\Omega,\mathbb{R}^n)}^2 \\ &\quad + \lambda_2 \left\|\mathbf{S} - \tilde{\mathbf{S}}\right\|_{\mathbf{L}^2(\Omega,\mathbb{M}_s^{n\times n})}^2 \\ &\quad + \lambda_3 \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\mathbf{L}_\tau^2(\Gamma_c,\mathbb{R}^n)}^2, \end{aligned} \tag{22}$$

where $\tilde{\mathbf{v}}$ is a favorable velocity field; $\widetilde{\mathbf{w}}$ is an unfavorable velocity field, that is, a velocity field whose appearance is undesirable; $\tilde{\mathbf{S}}$ is a favorable extra-stress tensor; $\tilde{\mathbf{u}}$ is a favorable surface force at Γ_c; and λ_1, λ_2, and λ_3 are positive cost parameters. It is obvious that condition (iv) holds for the functionals $J = J_i$, $i = 1, 2$.

Remark 2. We do not assume that the set of admissible controls is convex. As is known, the convexity condition is widely used in studying of optimal control problems (see, e.g., [17]). However, this condition does not always hold in applications. Obviously, condition (iii) is weaker than the convexity condition. For example, (iii) is satisfied if the set $\mathbf{U}$ can be represented as the union of finite number of convex closed sets in the space $\mathbf{L}^2_\tau(\Gamma_c, \mathbb{R}^n)$.

Now we introduce the concept of admissible triplets of (1)–(8) by analogy with the definition of generalized (weak) solutions to hydrodynamic models with slip boundary conditions (see, e.g., [8, 19, 20]).

Let $\mathbf{f} \in \mathbf{L}^2(\Omega, \mathbb{R}^n)$.

Definition 3. One says that a triplet $(\mathbf{v}, \mathbf{S}, \mathbf{u}) \in \mathbf{X}(\Omega, \mathbb{R}^n) \times \mathbf{L}^2(\Omega, \mathbb{M}^{n\times n}_s) \times \mathbf{L}^2_\tau(\Gamma_c, \mathbb{R}^n)$ is an *admissible triplet* of control system (1)–(8) if the equality

$$-\sum_{i=1}^{n}\left(v_i\mathbf{v}, \frac{\partial \boldsymbol{\varphi}}{\partial x_i}\right) + \left(\psi\left(I_2(\mathbf{v})\right)\mathbf{D}(\mathbf{v}), \mathbf{D}(\boldsymbol{\varphi})\right) = (\mathbf{f}, \boldsymbol{\varphi}) + \int_{\Gamma_c} \mathbf{u}\cdot\boldsymbol{\varphi}\, d\Gamma_c \tag{23}$$

holds for any $\boldsymbol{\varphi} \in \mathbf{X}(\Omega, \mathbb{R}^n)$ and if conditions (3) and (7) hold.

Remark 4. Equation (23) appears for the following reasons. Let us assume that $(\mathbf{v}, \mathbf{S}, p, \mathbf{u})$ is a classical solution of (1)–(7). We take the $\mathbf{L}^2$-scalar product of (1) with $\boldsymbol{\varphi} \in \mathbf{X}(\Omega, \mathbb{R}^n)$. By integrating by parts, we obtain

$$-\sum_{i=1}^{n}\left(v_i\mathbf{v}, \frac{\partial \boldsymbol{\varphi}}{\partial x_i}\right) + \left(\mathbf{S}, \mathbf{D}(\boldsymbol{\varphi})\right) - \int_{\Gamma_c} (\mathbf{S}\mathbf{n})\cdot\boldsymbol{\varphi}\, d\Gamma_c = (\mathbf{f}, \boldsymbol{\varphi}). \tag{24}$$

Combining this with (3) and (6), we get (23).

On the other hand, it is not difficult to prove that if an admissible triplet $(\mathbf{v}, \mathbf{S}, \mathbf{u})$ is sufficiently smooth, then there exists a function p such that $(\mathbf{v}, \mathbf{S}, p, \mathbf{u})$ is a classical solution to (1)–(7).

Let $\mathbf{M}$ be the set of admissible triplets to problem (1)–(8).

Definition 5. A triplet $(\mathbf{v}_*, \mathbf{S}_*, \mathbf{u}_*) \in \mathbf{M}$ is called a *solution* of optimization problem (1)–(8) if the equality

$$J(\mathbf{v}_*, \mathbf{S}_*, \mathbf{u}_*) = \inf_{(\mathbf{v},\mathbf{S},\mathbf{u})\in\mathbf{M}} J(\mathbf{v}, \mathbf{S}, \mathbf{u}) \tag{25}$$

holds.

Our main result provides existence of solutions to (1)–(8).

Theorem 6. *If conditions (i), (ii), (iii), and (iv) hold, then optimization problem (1)–(8) has at least one solution.*

3. Proof of Theorem 6

The proof of Theorem 6 is based on the Galerkin method and monotonicity methods [21], as well as the following lemma.

Lemma 7. *Let $\mathscr{B}_R = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_{\mathbb{R}^m} \le R\}$ be a closed ball. Suppose the continuous mapping $\mathbf{F} : \mathscr{B}_R \times [0,1] \to \mathbb{R}^m$ satisfies the following conditions:*

(a) $\mathbf{F}(\mathbf{x}, \lambda) \ne \mathbf{0}$ *for any* $(\mathbf{x}, \lambda) \in \partial\mathscr{B}_R \times [0,1]$,

(b) $\mathbf{F}(\mathbf{x}, 0) = \mathbf{A}\mathbf{x}$ *for any* $\mathbf{x} \in \mathscr{B}_R$,

where $\mathbf{A} : \mathbb{R}^m \to \mathbb{R}^m$ *is an isomorphism;*

then for any $\lambda \in [0,1]$ the equation $\mathbf{F}(\mathbf{x}, \lambda) = \mathbf{0}$ has at least one solution $\mathbf{x}_\lambda \in \mathscr{B}_R$.

Lemma 7 can be proved by methods of topological degree theory (see, e.g., [22]).

Proof of Theorem 6. First we show that the set of admissible triplets is nonempty. Let us fix an element $\mathbf{u}^0 = (u^0_1, \ldots, u^0_n) \in \mathbf{U}$. Suppose $\{\boldsymbol{\varphi}^j\}_{j=1}^{\infty}$ is an orthonormal basis of the space $\mathbf{X}(\Omega, \mathbb{R}^n)$.

For an arbitrary fixed number $m \in \mathbb{N}$, we consider the following auxiliary problem.

Find a vector $(\alpha_{m1}, \ldots, \alpha_{mm}) \in \mathbb{R}^m$ such that

$$-\lambda\sum_{i=1}^{n}\left(v^m_i\mathbf{v}^m, \frac{\partial \boldsymbol{\varphi}^j}{\partial x_i}\right) + \left(\psi\left(\lambda I_2(\mathbf{v}^m)\right)\mathbf{D}(\mathbf{v}^m), \mathbf{D}(\boldsymbol{\varphi}^j)\right) = \lambda\left(\mathbf{f}, \boldsymbol{\varphi}^j\right) + \lambda\int_{\Gamma_c} \mathbf{u}^0\cdot\boldsymbol{\varphi}^j\, d\Gamma_c, \quad j = 1, \ldots m, \tag{26}$$

$$\mathbf{v}^m = \sum_{j=1}^{m}\alpha_{mj}\boldsymbol{\varphi}^j, \tag{27}$$

where λ is a parameter, $\lambda \in [0,1]$.

First we prove some a priori estimates of solutions to problem (26) and (27). Let $(\alpha_{m1}, \ldots, \alpha_{mm})$ be a solution of system (26) and (27) with a fixed parameter $\lambda \in [0,1]$. We multiply (26) by α_{mj} and add the corresponding equalities for $j = 1, \ldots, m$. Taking into account the equality

$$\sum_{i=1}^{n}\left(v^m_i\mathbf{v}^m, \frac{\partial \mathbf{v}^m}{\partial x_i}\right) = 0, \tag{28}$$

we obtain

$$\left(\psi\left(\lambda I_2(\mathbf{v}^m)\right)\mathbf{D}(\mathbf{v}^m), \mathbf{D}(\mathbf{v}^m)\right) = \lambda\left(\mathbf{f}, \mathbf{v}^m\right) + \lambda\int_{\Gamma_c} \mathbf{u}^0\cdot\mathbf{v}^m\, d\Gamma_c. \tag{29}$$

Using (18) and (19), from (29) we obtain the estimate

$$\begin{aligned} a_1\left\|\mathbf{v}^m\right\|^2_{\mathbf{X}(\Omega,\mathbb{R}^n)} &\le \|\mathbf{f}\|_{\mathbf{L}^2(\Omega,\mathbb{R}^n)}\left\|\mathbf{v}^m\right\|_{\mathbf{L}^2(\Omega,\mathbb{R}^n)} + \left\|\mathbf{u}^0\right\|_{\mathbf{L}^2_\tau(\Gamma_c,\mathbb{R}^n)}\left\|\mathbf{v}^m\right\|_{\mathbf{L}^2_\tau(\Gamma_c,\mathbb{R}^n)} \\ &\le \left(C_2\|\mathbf{f}\|_{\mathbf{L}^2(\Omega,\mathbb{R}^n)} + C_1\left\|\mathbf{u}^0\right\|_{\mathbf{L}^2_\tau(\Gamma_c,\mathbb{R}^n)}\right)\left\|\mathbf{v}^m\right\|_{\mathbf{X}(\Omega,\mathbb{R}^n)}. \end{aligned} \tag{30}$$

This yields that

$$\|\mathbf{v}^m\|_{\mathbf{X}(\Omega,\mathbb{R}^n)} \le a_1^{-1}\left(C_2\|\mathbf{f}\|_{\mathbf{L}^2(\Omega,\mathbb{R}^n)} + C_1\left\|\mathbf{u}^0\right\|_{\mathbf{L}^2_\tau(\Gamma_c,\mathbb{R}^n)}\right). \quad (31)$$

Applying Lemma 7 to system (26) and (27), we see that problem (26) and (27) is solvable for any $\lambda \in [0,1]$ and $m \in \mathbb{N}$.

Let $\{\mathbf{v}^m\}_{m=1}^\infty$ be a sequence of vector functions that satisfy (26) and (27) with $\lambda = 1$. It is clear that

$$-\sum_{i=1}^n\left(v_i^m\mathbf{v}^m, \frac{\partial\boldsymbol{\varphi}^j}{\partial x_i}\right) + \left(\psi\left(I_2\left(\mathbf{v}^m\right)\right)\mathbf{D}\left(\mathbf{v}^m\right), \mathbf{D}\left(\boldsymbol{\varphi}^j\right)\right) = \left(\mathbf{f},\boldsymbol{\varphi}^j\right) + \int_{\Gamma_c}\mathbf{u}^0\cdot\boldsymbol{\varphi}^j\,d\Gamma_c, \quad j = 1,\dots,m. \quad (32)$$

Note that estimate (31) is independent of m. This shows the existence of a vector function $\mathbf{v}^0 \in \mathbf{X}(\Omega,\mathbb{R}^n)$ and a subsequence $m' \to \infty$ such that $\mathbf{v}^{m'} \to \mathbf{v}^0$ weakly in $\mathbf{X}(\Omega,\mathbb{R}^n)$. For the sake of simplicity, we assume that

$$\mathbf{v}^m \longrightarrow \mathbf{v}^0 \quad \text{weakly in } \mathbf{X}\left(\Omega,\mathbb{R}^n\right) \text{ as } m \longrightarrow \infty. \quad (33)$$

Moreover, by the Sobolev embedding theorems, we have

$$\mathbf{v}^m \longrightarrow \mathbf{v}^0 \quad \text{strongly in } \mathbf{L}^4\left(\Omega,\mathbb{R}^n\right) \text{ as } m \longrightarrow \infty. \quad (34)$$

Using (34), we get

$$\sum_{i=1}^n\left(v_i^m\mathbf{v}^m, \frac{\partial\boldsymbol{\varphi}^j}{\partial x_i}\right) \longrightarrow \sum_{i=1}^n\left(v_i^0\mathbf{v}^0, \frac{\partial\boldsymbol{\varphi}^j}{\partial x_i}\right) \quad \text{as } m \longrightarrow \infty. \quad (35)$$

Therefore we can pass to the limit $m \to \infty$ in equality (32) and obtain

$$\lim_{m\to\infty}\left(\psi\left(I_2\left(\mathbf{v}^m\right)\right)\mathbf{D}\left(\mathbf{v}^m\right), \mathbf{D}\left(\boldsymbol{\varphi}^j\right)\right) = \sum_{i=1}^n\left(v_i^0\mathbf{v}^0, \frac{\partial\boldsymbol{\varphi}^j}{\partial x_i}\right) + \left(\mathbf{f},\boldsymbol{\varphi}^j\right) + \int_{\Gamma_c}\mathbf{u}^0\cdot\boldsymbol{\varphi}^j\,d\Gamma_c \quad (36)$$

for any $j \in \mathbb{N}$. Since $\{\boldsymbol{\varphi}^j\}_{j=1}^\infty$ is a basis of the space $\mathbf{X}(\Omega,\mathbb{R}^n)$, it follows that equality (36) remains valid if we replace $\boldsymbol{\varphi}^j$ by an arbitrary vector function $\boldsymbol{\varphi} \in \mathbf{X}(\Omega,\mathbb{R}^n)$:

$$\lim_{m\to\infty}\left(\psi\left(I_2\left(\mathbf{v}^m\right)\right)\mathbf{D}\left(\mathbf{v}^m\right), \mathbf{D}\left(\boldsymbol{\varphi}\right)\right) = \sum_{i=1}^n\left(v_i^0\mathbf{v}^0, \frac{\partial\boldsymbol{\varphi}}{\partial x_i}\right) + \left(\mathbf{f},\boldsymbol{\varphi}\right) + \int_{\Gamma_c}\mathbf{u}^0\cdot\boldsymbol{\varphi}\,d\Gamma_c. \quad (37)$$

Now we multiply (32) by α_{mj} and add the corresponding equalities for $j = 1,\dots,m$. The result is

$$\left(\psi\left(I_2\left(\mathbf{v}^m\right)\right)\mathbf{D}\left(\mathbf{v}^m\right), \mathbf{D}\left(\mathbf{v}^m\right)\right) = \left(\mathbf{f},\mathbf{v}^m\right) + \int_{\Gamma_c}\mathbf{u}^0\cdot\mathbf{v}^m\,d\Gamma_c. \quad (38)$$

Hence we find in the limit

$$\lim_{m\to\infty}\left(\psi\left(I_2\left(\mathbf{v}^m\right)\right)\mathbf{D}\left(\mathbf{v}^m\right), \mathbf{D}\left(\mathbf{v}^m\right)\right) = \left(\mathbf{f},\mathbf{v}^0\right) + \int_{\Gamma_c}\mathbf{u}^0\cdot\mathbf{v}^0\,d\Gamma_c. \quad (39)$$

Taking into account (20), (33), (37), and (39), we obtain the estimate

$$\begin{aligned}
&-\mu\sum_{i=1}^n\left(v_i^0\mathbf{v}^0, \frac{\partial\boldsymbol{\varphi}^j}{\partial x_i}\right) + \mu\left(\psi\left(I_2\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right)\right)\mathbf{D}\left(\mathbf{v}^0\right.\right.\\
&\left.\left.- \mu\boldsymbol{\varphi}^j\right), \mathbf{D}\left(\boldsymbol{\varphi}^j\right)\right) - \mu\left(\mathbf{f},\boldsymbol{\varphi}^j\right) - \mu\int_{\Gamma_c}\mathbf{u}^0\cdot\boldsymbol{\varphi}^j\,d\Gamma_c\\
&= -\left(\mathbf{f},\mathbf{v}^0\right) - \int_{\Gamma_c}\mathbf{u}^0\cdot\mathbf{v}^0\,d\Gamma_c\\
&+ \sum_{i=1}^n\left(v_i^0\mathbf{v}^0, \frac{\partial\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right)}{\partial x_i}\right) + \left(\mathbf{f},\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right)\\
&+ \int_{\Gamma_c}\mathbf{u}^0\cdot\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right)d\Gamma_c\\
&+ \mu\left(\psi\left(I_2\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right)\right)\mathbf{D}\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right), \mathbf{D}\left(\boldsymbol{\varphi}^j\right)\right)\\
&= -\lim_{m\to\infty}\left(\psi\left(I_2\left(\mathbf{v}^m\right)\right)\mathbf{D}\left(\mathbf{v}^m\right), \mathbf{D}\left(\mathbf{v}^m\right)\right)\\
&+ \lim_{m\to\infty}\left(\psi\left(I_2\left(\mathbf{v}^m\right)\right)\mathbf{D}\left(\mathbf{v}^m\right), \mathbf{D}\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right)\right)\\
&+ \lim_{m\to\infty}\left(\psi\left(I_2\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right)\right)\mathbf{D}\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right), \mathbf{D}\left(\mathbf{v}^m\right)\right.\\
&\left.- \mathbf{D}\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right)\right) = -\lim_{m\to\infty}\left(\psi\left(I_2\left(\mathbf{v}^m\right)\right)\mathbf{D}\left(\mathbf{v}^m\right)\right.\\
&- \psi\left(I_2\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right)\right)\mathbf{D}\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right), \mathbf{D}\left(\mathbf{v}^m\right)\\
&\left.- \mathbf{D}\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right)\right) \le 0
\end{aligned} \quad (40)$$

for any number $\mu > 0$. Multiplying the obtained inequality by μ^{-1}, we get

$$-\sum_{i=1}^n\left(v_i^0\mathbf{v}^0, \frac{\partial\boldsymbol{\varphi}^j}{\partial x_i}\right) + \left(\psi\left(I_2\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right)\right)\mathbf{D}\left(\mathbf{v}^0 - \mu\boldsymbol{\varphi}^j\right), \mathbf{D}\left(\boldsymbol{\varphi}^j\right)\right) - \left(\mathbf{f},\boldsymbol{\varphi}^j\right) - \int_{\Gamma_c}\mathbf{u}^0\cdot\boldsymbol{\varphi}^j\,d\Gamma_c \le 0 \quad (41)$$

for any $j \in \mathbb{N}$ and $\mu > 0$.

Using Krasnoselskii's theorem [22] on continuity of Nemytskii operators, we can pass to the limit $\mu \to 0$ in (41):

$$-\sum_{i=1}^n\left(v_i^0\mathbf{v}^0, \frac{\partial\boldsymbol{\varphi}^j}{\partial x_i}\right) + \left(\psi\left(I_2\left(\mathbf{v}^0\right)\right)\mathbf{D}\left(\mathbf{v}^0\right), \mathbf{D}\left(\boldsymbol{\varphi}^j\right)\right) - \left(\mathbf{f},\boldsymbol{\varphi}^j\right) - \int_{\Gamma_c}\mathbf{u}^0\cdot\boldsymbol{\varphi}^j\,d\Gamma_c \le 0. \quad (42)$$

Since $\{\boldsymbol{\varphi}^j\}_{j=1}^{\infty}$ is a basis of the space $\mathbf{X}(\Omega,\mathbb{R}^n)$, it follows that inequality (42) remains valid if we replace $\boldsymbol{\varphi}^j$ by an arbitrary vector function $\boldsymbol{\varphi} \in \mathbf{X}(\Omega,\mathbb{R}^n)$. Furthermore, since $\boldsymbol{\varphi}$ is an arbitrary vector function from the space $\mathbf{X}(\Omega,\mathbb{R}^n)$, we have

$$-\sum_{i=1}^{n}\left(v_i^0\mathbf{v}^0,\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\right)+\left(\psi\left(I_2\left(\mathbf{v}^0\right)\right)\mathbf{D}\left(\mathbf{v}^0\right),\mathbf{D}(\boldsymbol{\varphi})\right)-(\mathbf{f},\boldsymbol{\varphi})-\int_{\Gamma_c}\mathbf{u}^0\cdot\boldsymbol{\varphi}\,d\Gamma_c=0. \tag{43}$$

This implies that the triplet $(\mathbf{v}^0,\psi(I_2(\mathbf{v}^0))\mathbf{D}(\mathbf{v}^0),\mathbf{u}^0)$ is an admissible triplet of problem (1)–(7) and thus $\mathbf{M}\neq\emptyset$.

We will show that $\mathbf{M}$ is bounded in the space $\mathbf{X}(\Omega,\mathbb{R}^n)\times\mathbf{L}^2(\Omega,\mathbb{M}_s^{n\times n})\times\mathbf{L}_\tau^2(\Gamma_c,\mathbb{R}^n)$. Suppose $(\mathbf{v},\mathbf{S},\mathbf{u})$ is an arbitrary triplet from $\mathbf{M}$ and $\boldsymbol{\varphi}=\mathbf{v}$. It follows from (23) that

$$\left(\psi\left(I_2(\mathbf{v})\right)\mathbf{D}(\mathbf{v}),\mathbf{D}(\mathbf{v})\right)=(\mathbf{f},\mathbf{v})+\int_{\Gamma_c}\mathbf{u}\cdot\mathbf{v}\,d\Gamma_c. \tag{44}$$

This yields that

$$\|\mathbf{v}\|_{\mathbf{X}(\Omega,\mathbb{R}^n)}\le a_1^{-1}\left(C_2\|\mathbf{f}\|_{\mathbf{L}^2(\Omega,\mathbb{R}^n)}+C_1\sup_{\mathbf{w}\in\mathbf{U}}\|\mathbf{w}\|_{\mathbf{L}_\tau^2(\Gamma_c,\mathbb{R}^n)}\right). \tag{45}$$

Moreover, taking into account (19), we obtain

$$\begin{aligned}\|\mathbf{S}\|_{\mathbf{L}^2(\Omega,\mathbb{M}_s^{n\times n})}&=\left\|\psi\left(I_2(\mathbf{v})\right)\mathbf{D}(\mathbf{v})\right\|_{\mathbf{L}^2(\Omega,\mathbb{M}_s^{n\times n})}\\&\le a_2\|\mathbf{D}(\mathbf{v})\|_{\mathbf{L}^2(\Omega,\mathbb{M}_s^{n\times n})}=a_2\|\mathbf{v}\|_{\mathbf{X}(\Omega,\mathbb{R}^n)}\\&\le a_2a_1^{-1}\left(C_2\|\mathbf{f}\|_{\mathbf{L}^2(\Omega,\mathbb{R}^n)}+C_1\sup_{\mathbf{w}\in\mathbf{U}}\|\mathbf{w}\|_{\mathbf{L}_\tau^2(\Gamma_c,\mathbb{R}^n)}\right).\end{aligned} \tag{46}$$

Recall that the set $\mathbf{U}$ is bounded in $\mathbf{L}_\tau^2(\Gamma_c,\mathbb{R}^n)$. Therefore from estimates (45) and (46) it follows that the set $\mathbf{M}$ is bounded in the space $\mathbf{X}(\Omega,\mathbb{R}^n)\times\mathbf{L}^2(\Omega,\mathbb{M}_s^{n\times n})\times\mathbf{L}_\tau^2(\Gamma_c,\mathbb{R}^n)$.

Now we will show that the set $\mathbf{M}$ is sequentially weakly closed. Take a sequence $\{(\mathbf{v}^k,\mathbf{S}^k,\mathbf{u}^k)\}_{k=1}^{\infty}\subset\mathbf{M}$ such that $\mathbf{v}^k\to\widehat{\mathbf{v}}$ weakly in $\mathbf{X}(\Omega,\mathbb{R}^n)$, $\mathbf{S}^k\to\widehat{\mathbf{S}}$ weakly in $\mathbf{L}^2(\Omega,\mathbb{M}_s^{n\times n})$, and $\mathbf{u}^k\to\widehat{\mathbf{u}}$ weakly in $\mathbf{L}_\tau^2(\Gamma_c,\mathbb{R}^n)$ as $k\to\infty$. Let us check that $(\widehat{\mathbf{v}},\widehat{\mathbf{S}},\widehat{\mathbf{u}})\in\mathbf{M}$.

By definition, we have

$$-\sum_{i=1}^{n}\left(v_i^k\mathbf{v}^k,\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\right)+\left(\psi\left(I_2\left(\mathbf{v}^k\right)\right)\mathbf{D}\left(\mathbf{v}^k\right),\mathbf{D}(\boldsymbol{\varphi})\right)=(\mathbf{f},\boldsymbol{\varphi})+\int_{\Gamma_c}\mathbf{u}^k\cdot\boldsymbol{\varphi}\,d\Gamma_c \tag{47}$$

for any $\boldsymbol{\varphi}\in\mathbf{X}(\Omega,\mathbb{R}^n)$. Arguing as above, we conclude that

$$-\sum_{i=1}^{n}\left(\widehat{v}_i\widehat{\mathbf{v}},\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\right)+\left(\psi\left(I_2(\widehat{\mathbf{v}})\right)\mathbf{D}(\widehat{\mathbf{v}}),\mathbf{D}(\boldsymbol{\varphi})\right)=(\mathbf{f},\boldsymbol{\varphi})+\int_{\Gamma_c}\widehat{\mathbf{u}}\cdot\boldsymbol{\varphi}\,d\Gamma_c. \tag{48}$$

From condition (iii), we get $\widehat{\mathbf{u}}\in\mathbf{U}$. Thus, it remains to show that

$$\widehat{\mathbf{S}}=\psi\left(I_2(\widehat{\mathbf{v}})\right)\mathbf{D}(\widehat{\mathbf{v}}). \tag{49}$$

Since $\mathbf{v}^k\to\widehat{\mathbf{v}}$ weakly in $\mathbf{X}(\Omega,\mathbb{R}^n)$, we see that

$$\mathbf{D}\left(\mathbf{v}^k\right)\longrightarrow\mathbf{D}(\widehat{\mathbf{v}})\quad\text{weakly in }\mathbf{L}^2\left(\Omega,\mathbb{R}_s^{n\times n}\right)\text{ as }k\longrightarrow\infty. \tag{50}$$

Note also that

$$\left(\psi\left(I_2\left(\mathbf{v}^k\right)\right)\mathbf{D}\left(\mathbf{v}^k\right),\boldsymbol{\Phi}\right)=\left(\mathbf{S}^k,\boldsymbol{\Phi}\right)\longrightarrow\left(\widehat{\mathbf{S}},\boldsymbol{\Phi}\right)\quad\text{as }k\longrightarrow\infty, \tag{51}$$

for any $\boldsymbol{\Phi}\in\mathbf{L}^2(\Omega,\mathbb{R}_s^{n\times n})$.

Using the equality $\mathbf{S}^k=\psi(I_2(\mathbf{v}^k))\mathbf{D}(\mathbf{v}^k)$, we rewrite (47) as follows:

$$-\sum_{i=1}^{n}\left(v_i^k\mathbf{v}^k,\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\right)+\left(\mathbf{S}^k,\mathbf{D}(\boldsymbol{\varphi})\right)=(\mathbf{f},\boldsymbol{\varphi})+\int_{\Gamma_c}\mathbf{u}^k\cdot\boldsymbol{\varphi}\,d\Gamma_c. \tag{52}$$

Passing to the limit $k\to\infty$ in this equality, we obtain

$$-\sum_{i=1}^{n}\left(\widehat{v}_i\widehat{\mathbf{v}},\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\right)+\left(\widehat{\mathbf{S}},\mathbf{D}(\boldsymbol{\varphi})\right)=(\mathbf{f},\boldsymbol{\varphi})+\int_{\Gamma_c}\widehat{\mathbf{u}}\cdot\boldsymbol{\varphi}\,d\Gamma_c. \tag{53}$$

Substituting $\widehat{\mathbf{v}}$ for $\boldsymbol{\varphi}$ in (53), we get

$$\left(\widehat{\mathbf{S}},\mathbf{D}(\widehat{\mathbf{v}})\right)=(\mathbf{f},\widehat{\mathbf{v}})+\int_{\Gamma_c}\widehat{\mathbf{u}}\cdot\widehat{\mathbf{v}}\,d\Gamma_c. \tag{54}$$

Further, substituting $\mathbf{v}^k$ for $\boldsymbol{\varphi}$ in (47), we find

$$\lim_{k\to\infty}\left(\psi\left(I_2\left(\mathbf{v}^k\right)\right)\mathbf{D}\left(\mathbf{v}^k\right),\mathbf{D}\left(\mathbf{v}^k\right)\right)=(\mathbf{f},\widehat{\mathbf{v}})+\int_{\Gamma_c}\widehat{\mathbf{u}}\cdot\widehat{\mathbf{v}}\,d\Gamma_c. \tag{55}$$

Combining this with (54), we obtain

$$\lim_{k\to\infty}\left(\psi\left(I_2\left(\mathbf{v}^k\right)\right)\mathbf{D}\left(\mathbf{v}^k\right),\mathbf{D}\left(\mathbf{v}^k\right)\right)=\left(\widehat{\mathbf{S}},\mathbf{D}(\widehat{\mathbf{v}})\right). \tag{56}$$

By [21, Chapter III, Lemma 1.3] and (50), (51), and (56), we get (49).

Applying the generalized Weierstrass theorem (see [23]), we conclude that there exists an element $(\mathbf{v}_*,\mathbf{S}_*,\mathbf{u}_*)\in\mathbf{M}$ such that

$$J(\mathbf{v}_*,\mathbf{S}_*,\mathbf{u}_*)=\inf_{(\mathbf{v},\mathbf{S},\mathbf{u})\in\mathbf{M}}J(\mathbf{v},\mathbf{S},\mathbf{u}). \tag{57}$$

This proves Theorem 6. □

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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