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*Research Article*

# Homogeneous-Like Generalized Cubic Systems

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We consider properties and center conditions for plane polynomial systems of the forms $\dot{x} = -y - p_1(x,y) - p_2(x,y)$, $\dot{y} = x + q_1(x,y) + q_2(x,y)$ where $p_1, q_1$ and $p_2, q_2$ are polynomials of degrees $n$ and $2n-1$, respectively, for integers $n \geq 2$. We restrict our attention to those systems for which $yp_2(x,y) + xq_2(x,y) = 0$. In this case the system can be transformed to a trigonometric Abel equation which is similar in form to the one obtained for homogeneous systems ($p_2 = q_2 = 0$). From this we show that any center condition of a homogeneous system for a given $n$ can be transformed to a center condition of the corresponding generalized cubic system and we use a similar idea to obtain center conditions for several other related systems. As in the case of the homogeneous system, these systems can also be transformed to Abel equations having rational coefficients and we briefly discuss an application of this to a particular Abel equation.

## 1. Introduction

In this work we consider differential polynomial systems in the plane having the form of a linear center perturbed by homogeneous polynomials of degrees $n$ and $2n-1$ where $n \geq 2$ is an integer. We refer to these as *generalized cubic systems* since they contain the cubic system ($n = 2$) as a particular case. Specifically, we assume

$$\frac{dx}{dt} = -y - p_1(x,y) - p_2(x,y), \quad \frac{dy}{dt} = x + q_1(x,y) + q_2(x,y), \tag{1}$$

where $p_1$, $q_1$ and $p_2$, $q_2$ are homogeneous polynomials of degrees $n$ and $2n-1$, respectively. We will also have occasion to consider the reduced (homogeneous) problem

$$\frac{dx}{dt} = -y - p_1(x,y), \quad \frac{dy}{dt} = x + q_1(x,y), \tag{2}$$

in which the perturbation consists of a single polynomial. Corresponding to (1), (2) are the first-order differential equations:

$$\frac{dy}{dx} = -\frac{x + q_1(x,y) + q_2(x,y)}{y + p_1(x,y) + p_2(x,y)}, \tag{3}$$

$$\frac{dy}{dx} = -\frac{x + q_1(x,y)}{y + p_1(x,y)}. \tag{4}$$

In his original work [1] Poincaré developed a method for determining if the origin is a center by seeking an analytic solution to the equation $y' = Q/P$ where $P, Q$ are polynomials satisfying $P(0,0), Q(0,0) = 0$. For (4) it takes the form

$$U(x,y) = \frac{1}{2}\left(x^2 + y^2\right) + \sum_{k=2}^{\infty} U_{\alpha_k}(x,y), \tag{5}$$

where $U_{\alpha_k}(x,y)$ is a homogeneous polynomial of degree $\alpha_k = k(n-1) + 2$. This solution is required to satisfy the condition

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} = \sum_{k=1}^{\infty} V_{\alpha_k}\left(x^2 + y^2\right)^{\alpha_k} \equiv \sum_{\ell=2}^{\infty} \widetilde{V}_{2\ell}\left(x^2 + y^2\right)^{\ell}. \tag{6}$$

Here, $V_{\alpha_k}$, $\widetilde{V}_{2\ell}$ are called *Lyapunov coefficients* and they are homogeneous polynomials in the coefficients of the system.

Most of the known center conditions for systems of type (1) are for cubic systems and for these there is an extensive literature. We note particularly the works of Lloyd et al. [2–9], Żołądek [10], Alwash [11], Cherkas and Romanovski [12], and the many references therein. A great number of these results were obtained by an exhaustive analysis of the Lyapunov coefficients. In contrast the main results in this work were found by studying the differential equations for these systems and determining what we can learn from them. Our study will consider systems which satisfy the condition

$$y p_2(x,y) + x q_2(x,y) = 0. \tag{7}$$

The only complete set of center conditions [3] known for systems of this type is for the cubic system. We will show that all generalized cubic systems which satisfy (7) can be transformed to Abel differential equations of various types and from these we will deduce a probable set (at this time we do not consider the completeness of the center conditions because it would involve an extensive analysis of the Lyapunov coefficients) of complete center conditions for the quintic system and rederive those for the cubic system. Complete sets of center conditions are known for these two cases for systems which also satisfy $y p_1(x,y) + x q_1(x,y) = 0$. These are discussed in [11, 13] where it is shown that the only center conditions possible for these systems are symmetric centers. In Section 3 we use trigonometric forms for the Lyapunov coefficients to obtain a simple rederivation of these conditions.

For the purposes of this work an integrating factor $\mu$ of (3) is a function such that

$$\frac{\partial}{\partial y}\left(\mu(x,y)\left(x + q_1(x,y) + q_2(x,y)\right)\right) - \frac{\partial}{\partial x}\left(\mu(x,y)\left(y + p_1(x,y) + p_2(x,y)\right)\right) = 0. \tag{8}$$

We make use of the fact by Reeb [14] that if $\mu$ is analytic and nonzero on a neighborhood of the critical point $(0,0)$ then the corresponding system is a center of (1) and (2). The only integrating factors that we mention in this paper are of this type.

The Cartesian forms provided by (3) and (4) are difficult to work with so we consider various transformations of these equations. In particular it is well known that (4) can be transformed to an Abel equation of the first kind in which the coefficients are trigonometric polynomials in $\cos\theta$, $\sin\theta$. We show that the particular subcase (7) of (3) that we consider is also transformable to an Abel equation having similar form. By this equation we can easily relate the homogeneous form with its corresponding generalized cubic form and this leads directly to many of the general results that we will establish for these systems. We use it to show that any center condition for a homogeneous system can be transformed to a center condition of the corresponding generalized cubic system.

In Section 3 we present most of the main results and in Section 4 we apply these to various forms of generalized cubic systems to obtain several new center conditions for these systems. Sections 5 and 6 are devoted primarily to a brief discussion of simple closed invariant curves which can occur in these systems and by considering particular examples determine whether or not they are limit cycles. In the final section we present an example of a solvable Abel differential equation which cannot currently be solved by the Computer Algebra System Maple.

## 2. Derivation of Related Equations and Aspects of the Abel Differential Equation

Here we obtain the equations related to (3) and (4) which we will use in this work. Since this development is intended for both odd and even values of $n$, we need to pay close attention to the forms of the trigonometric polynomials which arise and to forms which various substitutions can take.

Using a polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$ in (3) and (4), respectively, we obtain

$$\frac{dr}{d\theta} = \frac{\xi_1(\theta) r^n + \xi_2(\theta) r^{2n-1}}{1 + \eta_1(\theta) r^{n-1} + \eta_2(\theta) r^{2n-2}}, \tag{9}$$

$$\frac{dr}{d\theta} = \frac{\xi_1(\theta) r^n}{1 + \eta_1(\theta) r^{n-1}}, \tag{10}$$

where

$$\begin{aligned} \xi_i(\theta) &= \sin\theta\, q_i(\cos\theta,\sin\theta) - \cos\theta\, p_i(\cos\theta,\sin\theta), \\ \eta_i(\theta) &= \sin\theta\, p_i(\cos\theta,\sin\theta) + \cos\theta\, q_i(\cos\theta,\sin\theta) \end{aligned} \tag{11}$$

for $i = 1, 2$. Here $\xi_1, \eta_1$ and $\xi_2, \eta_2$ are homogeneous trigonometric polynomials of degrees $n+1$ and $2n$, respectively, in $\cos\theta$, $\sin\theta$. If center conditions are defined in terms of these trigonometric polynomials, as is frequently the case, these equations can be used to define the corresponding coefficient functions $p_1, \ldots, q_2$. Inverting expressions (11) for the case $i = 1$ we obtain

$$\begin{aligned} p_1(x,y) &= -r^n\cos\theta\,\xi_1(\theta) + r^n\sin\theta\,\eta_1(\theta) \\ &= r^{n-1}\left(-x\xi_1(\theta) + y\eta_1(\theta)\right). \end{aligned} \tag{12}$$

The degrees of $\xi_1, \eta_1$ are such that this expression might not be a polynomial, although only the highest degree terms $n+1$ will contribute to this possibility. Writing $\xi_1 = \xi_{1H} + \widetilde{\xi_1}$, $\eta_1 = \eta_{1H} + \widetilde{\eta_1}$ where $\xi_{1H}, \eta_{1H}$ are the degree $n+1$ terms and $\widetilde{\xi_1}, \widetilde{\eta_1}$ contain all other terms, we see that this highest degree term can be written as

$$\begin{aligned} p_{1H}(x,y) &= r^{n-1}\left(-x\xi_{1H}(\theta) + y\eta_{1H}(\theta)\right) \\ &= r^{n-1}\left(-x\left(a\cos((n+1)\theta) + b\sin((n+1)\theta)\right)\right) \\ &\quad + r^{n-1}\left(y\left(c\cos((n+1)\theta) + d\sin((n+1)\theta)\right)\right), \end{aligned} \tag{13}$$

where $a, b, c, d$ are constants. For $n = 2$ this is

$$p_{1H}(x,y) = -\frac{ax^4 + (3b-c)x^3y - 3(a+d)x^2y^2 - (b-3c)xy^3 + dy^4}{x^2+y^2}. \tag{14}$$

The remainder of this with respect to $x$ is $4(b-c)xy^3 - 4(a+d)y^4$ which will be 0 if $d = -a$, $b = c$. The same conditions ensure that $q_{1H}$ is a polynomial and also holds for the case $n = 3$. We can now show in an inductive fashion that these conditions are sufficient for all $n \geq 2$. Suppose it is true for all $n \leq N$ where $N \geq 3$ is an integer. Then for $n = N + 1$ we have

$$\begin{aligned} p_{1H}(x, y) &= r^N(-x(a\cos((N+2)\theta) + b\sin((N+2)\theta))) \\ &\quad + r^N(y(c\cos((N+2)\theta) + d\sin((N+2)\theta))). \end{aligned} \tag{15}$$

Since the trigonometric polynomials are degree $N + 2$, the denominator is of order $r^{N+2}$ when expressed in Cartesian form. We can express this in terms of $\xi_{1H}$ for $n = N$ by expanding the trigonometric functions and setting $c = b$, $d = -a$. We obtain

$$\begin{aligned} p_{1H}(x, y) &= -r^N(x\cos\theta + y\sin\theta) \\ &\quad \cdot (a\cos((n+1)\theta) + b\sin((n+1)\theta)), \end{aligned} \tag{16}$$

which is a homogeneous polynomial of degree $N + 1$ when evaluated at $\cos\theta = x/r$, $\sin\theta = y/r$. The same consideration also hold for $q_{1H}$ and by obvious extension to the nonlinearities of degree $2n - 1$. Since $(n+1)\xi_{1H} + \eta'_{1H} = (n+1)((a+d)\cos((n+1)\theta) + (b-c)\sin((n+1)\theta))$ these conditions can be obtained by the vanishing of this expression. In general, the highest order terms must satisfy $((n-1)i+2)\xi_{iH} + \eta'_{iH} = 0$ for $i = 1, 2$ This basic restriction will apply to several of our results and it is sometimes satisfied by requiring that the highest order terms vanish. In the following we will simply indicate the highest allowable degrees of the trigonometric polynomials based on this condition. We will also drop the subscript 1 for the homogeneous system and simply refer to $p, q, \xi, \eta$.

Equations (9) and (10) (with $\eta_1 = \eta$) can be further transformed in a number of ways. We continue by using the transformation given in [15]

$$\rho(\theta) = \frac{r(\theta)^{n-1}}{1 + \eta(\theta) r(\theta)^{n-1}}. \tag{17}$$

With this (9) and (10), respectively, become

$$\frac{d\rho}{d\theta} = \frac{\psi_4(\theta)\rho^4 + \psi_3(\theta)\rho^3 + \psi_2(\theta)\rho^2}{1 - \eta_1(\theta)\rho + \eta_2(\theta)\rho^2}, \tag{18}$$

$$\begin{aligned} \frac{d\rho}{d\theta} &= -(n-1)\xi(\theta)\eta(\theta)\rho^3 \\ &\quad + ((n-1)\xi(\theta) - \eta'(\theta))\rho^2, \end{aligned} \tag{19}$$

where in (18)

$$\begin{aligned} \psi_2(\theta) &= (n-1)\xi_1(\theta) - \eta'_1(\theta), \\ \psi_3(\theta) &= (n-1)(\xi_2(\theta) - 2\xi_1(\theta)\eta_1(\theta)) \\ &\quad + \eta_1(\theta)\eta'_1(\theta), \\ \psi_4(\theta) &= (n-1)(\xi_1(\theta)\eta_1^2(\theta) - \xi_2(\theta)\eta_1(\theta)) \\ &\quad - \eta_2(\theta)\eta'_1(\theta). \end{aligned} \tag{20}$$

Condition (7) we are considering is equivalent to $\eta_2 = 0$. In this case we can show that $\psi_4\rho^2 + \psi_3\rho + \psi_2 = (1 - \eta_1\rho)((n-1)(\xi_2 - \xi_1\eta_1)\rho + \psi_2)$ so (18) reduces to the trigonometric Abel equation

$$\begin{aligned} \frac{d\rho}{d\theta} &= (n-1)(\xi_2(\theta) - \xi_1(\theta)\eta_1(\theta))\rho^3 \\ &\quad + ((n-1)\xi_1(\theta) - \eta'_1(\theta))\rho^2. \end{aligned} \tag{21}$$

This equation is very interesting in the fact that it shares many of the same analytical properties as (19) but describes a much larger class of systems (1). It is the equation upon which most of the results obtained in this paper are based. Since $\eta_2 = 0$, the maximum degree of $\xi_2$ in (21) is $2n-2$ whereas both $\xi_1, \eta_1$ can have maximal degree $n+1$. If we further set $\eta_1 = 0$ which is the case discussed in [11, 13] for $n = 2, 3$ we obtain

$$\frac{d\rho}{d\theta} = (n-1)\xi_2(\theta)\rho^3 + (n-1)\xi_1(\theta)\rho^2 \tag{22}$$

which is essentially the same equation as the corresponding form of (9).

Since several of the classes of solutions which we obtain are based on (21), we briefly review certain properties of Abel equations. In this work we will consider both types of Abel equations. An Abel equation of the first kind has the form

$$\frac{dy}{dx} = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x) \tag{23}$$

and an Abel equation of the second kind has the form

$$\frac{dy}{dx} = \frac{f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x)}{g_1(x)y + g_0(x)}, \tag{24}$$

where the coefficient functions are assumed to be suitably differentiable functions of $x$. Form (24) can always be transformed to an Abel equation of the first kind by the variable change

$$y(x) = \frac{1}{g_1(x)u(x)} - \frac{g_0(x)}{g_1(x)}. \tag{25}$$

For a general Abel equation of the first kind it is possible to define recursively an infinite sequence of relative invariants by [16]

$$s_3(x) = f_0(x) f_3^2(x) + \frac{2}{27} f_2^3(x) + \frac{1}{3}\Big(f_3(x) f_2'(x) - f_2(x) f_3'(x) - f_1(x) f_2(x) f_3(x)\Big), \tag{26}$$

$$s_{2k+1}(x) = f_3(x) s_{2k-1}'(x) + (2k-1)\left(\frac{1}{3} f_2^2(x) - f_3'(x) - f_1(x) f_3(x)\right) s_{2k-1}(x) \tag{27}$$

for $k \geq 2$. From these, a sequence of absolute invariants can be formed. If the first invariant $I_1 = s_5^3/s_3^5$ is constant, the Abel equation can be transformed to a separable equation. This is the only general class of Abel equations which is integrable by quadrature. We note that if $f_0 = f_2 = 0$ then $s_3 = 0$ and the Abel equation is a Bernoulli equation. In cases more general than that just indicated, the vanishing of $s_3$ means the Abel equation is transformable to a Bernoulli equation.

A sufficient condition that an Abel equation of form (23) with $f_0 = 0$ has a constant first invariant $I_1$ is that the coefficient functions satisfy a relation of the form

$$f_2(x)\left(\frac{f_3(x)}{f_2(x)}\right)' + f_1(x) f_3(x) = C f_2^2(x), \tag{28}$$

where $C$ is a constant. This gives

$$I_1 = \frac{729(1-3C)^3}{(9C-2)^2}. \tag{29}$$

In [17] we showed that a homogeneous system could be transformed to an Abel equation of the first kind having rational coefficients. This is also true for systems (4) which satisfy (7). Setting $y = vx$, interchanging the roles of $x$ and $v$, and letting $u = x^{n-1}$, we obtain an Abel equation of the second kind

$$\frac{du}{dv} = -(n-1)\frac{p_2(1,v)u^3 + p_1(1,v)u^2 + vu}{R(v)u + v^2 + 1}, \tag{30}$$

where $R(v) = q_1(1,v) + v p_1(1,v)$. An Abel equation of the first kind is obtained by using (25). Renaming the variables this gives form (23) where

$$\begin{aligned} f_3(x) &= -(n-1)\left(x^2+1\right)^3 \frac{p_2(1,x)}{R^2(x)} + (n-1)\left(x^2+1\right)^2 \frac{p_1(1,x)}{R(x)} - (n-1)x\left(x^2+1\right), \\ f_2(x) &= 3(n-1)\left(x^2+1\right)^2 \frac{p_2(1,x)}{R^2(x)} - 2(n-1)\left(x^2+1\right)\frac{p_1(1,x)}{R(x)} + \left(x^2+1\right)\frac{R'(x)}{R(x)} + (n-3)x, \\ f_1(x) &= -3(n-1)\left(x^2+1\right)\frac{p_2(1,x)}{R^2(x)} + (n-1)\frac{p_1(1,x)}{R(x)} - \frac{R'(x)}{R(x)}, \\ f_0(x) &= (n-1)\frac{p_2(1,x)}{R^2(x)}. \end{aligned} \tag{31}$$

In the following we will have cause to refer to arbitrary trigonometric polynomials. The basic form for these is

$$\alpha_0 + \sum_{k=1}^{N}\left(\alpha_{2k}\cos 2k\theta + \beta_{2k}\sin 2k\theta\right) \tag{32}$$

if $n$ is odd and

$$\sum_{k=1}^{N}\left(\alpha_{2k-1}\cos(2k-1)\theta + \beta_{2k-1}\sin(2k-1)\theta\right) \tag{33}$$

if $n$ is even. The value of $N$ depends upon $n$ and will vary according to the circumstance.

## 3. Basic Results for Generalized Cubic Systems

In this section we develop some of the basic results which are standard for generalized cubic systems. We begin with the main result which shows that any center condition for a homogeneous system of degree $n$ can be transformed into a center condition of the generalized cubic system having the same value of $n$. In this way we can truly think of the homogeneous systems as being nontrivial particular cases $(p_2, q_2 \neq 0)$ of the corresponding generalized cubic systems. We present several applications of the theorem and some of the following propositions in the next section.

**Theorem 1.** *Let $\xi$, $\eta$ given by (11) define a center of (2) and (4) for some integer $n \geq 2$. Then this condition can be transformed to a center condition for the same $n$ of the generalized cubic system defined by $\xi_1$, $\eta_1$, $\xi_2$ with $\eta_2 = 0$.*

*Proof.* We will show that there exist generalized cubic systems which satisfy the differential equation (21) which is the same as (19) for the homogeneous system. The two equations will be the same if $(n-1)\xi_1 - \eta_1' = (n-1)\xi - \eta'$ and $\xi_2 - \xi_1\eta_1 = -\xi\eta$. Set $\xi_1 = \xi + \tilde{\xi}$ and $\eta_1 = \eta + \tilde{\eta}$ where $\tilde{\xi}$, $\tilde{\eta}$ are trigonometric polynomials of degree not greater than $n-1$ defined by either

(32) or (33) according to $n$ being odd or even. The required identities will be satisfied if we take

$$\begin{aligned}
\xi_1(\theta) &= \xi(\theta) + \tilde{\xi}(\theta) = \xi(\theta) + \frac{1}{n-1}\tilde{\eta}'(\theta), \\
\eta_1(\theta) &= \eta(\theta) + \tilde{\eta}(\theta), \\
\xi_2(\theta) &= \xi(\theta)\tilde{\eta}(\theta) + \frac{1}{n-1}\tilde{\eta}'(\theta)\eta(\theta) \\
&\quad + \frac{1}{n-1}\tilde{\eta}'(\theta)\tilde{\eta}(\theta).
\end{aligned} \tag{34}$$

In this $\tilde{\eta}$ can be arbitrarily chosen. □

A couple of points are worth mentioning regarding these results. Since $\xi$, $\eta$ are assumed to define a homogeneous polynomial system, the manner in which $\xi_1$, $\eta_1$ are defined ensures that they too define polynomials. Also, the maximum degree of $\xi_2$ is $2n-2$ but the terms $\xi\tilde{\eta}$, $\tilde{\eta}'\eta$ could be of degree $2n$ if the degree of $\xi, \eta$ is $n+1$. However, in accordance with the definition of $\tilde{\xi}, \tilde{\eta}$ we can show that these highest degree terms will cancel leaving an expression for $\xi_2$ having degree $2n-2$.

Another significant consequence of Theorem 1 concerns the integrability of the systems involved. It follows directly that if the original homogeneous system is integrable, then so is the resulting generalized cubic system. This is because (19) and (21) are the same and (19) must be solvable if the homogeneous system is integrable.

We now present several more results which help characterize the nature of certain generalized cubic systems. The first two are valid for general systems and the remainder for the case $\eta_2 = 0$.

**Proposition 2.** *Let $\psi_2$, $\psi_3$, $\psi_4$ and $\eta_1$, $\eta_2$ be odd and even trigonometric polynomials, respectively, defined by (11) and (20) (or there exists a translation $\theta \to \theta + \theta_0$ for which it is true). Then the solution of (18) is an even function of $\theta$ and the origin is a center for system (1) and (3).*

*Proof.* It is straightforward to show that $\rho(\theta)$ and $\rho(-\theta)$ satisfy the same differential equation and the evenness of the solution gives $\rho(-\pi) = \rho(\pi)$. □

The conditions of Proposition 2 are clearly satisfied if $\xi_1$, $\xi_2$ are odd and $\eta_1$, $\eta_2$ are even. In this case the solution $r(\theta)$ of (9) is also an even function. These conditions define *symmetric* or *time-reversible* centers because the $x$-axis is a line of symmetry for the phase portrait.

The next result shows that a center condition of a generalized cubic system for a given $n$ is also a center condition for any generalized cubic system with $n_1 = n + 2k$ where $k \geq 1$ is an integer.

**Proposition 3.** *Let $n_1 \geq 2$ be an integer and $n_2 > n_1$ an integer having the same parity as $n_1$. Suppose that $\xi_{1,n_1}$, $\eta_{1,n_1}$, $\xi_{2,n_1}$, $\eta_{2,n_1}$ defined by (11) define a center for (1) and (3) for $n_1$. Then $\xi_{1,n_2} = (n_1-1)/(n_2-1)\xi_{1,n_1}$, $\xi_{2,n_2} = (n_1-1)/(n_2-1)\xi_{2,n_1}$, $\eta_{1,n_2} = \eta_{1,n_1}$, $\eta_{2,n_2} = \eta_{2,n_1}$ define a center for $n_2$.*

*Proof.* Since $n_1$ gives a center the solution of (9) is $2\pi$-periodic. Setting $R = r^{n-1}$ in (9), we have for $n = n_2$ form of it

$$\begin{aligned}
(n_2-1)\xi_{1,n_2}(\theta) &= (n_2-1)\frac{n_1-1}{n_2-1}\xi_{1,n_1}(\theta) \\
&= (n_1-1)\xi_{1,n_1}(\theta)
\end{aligned} \tag{35}$$

with a similar result for $\xi_{2,n_2}$. Thus the transformed equation is unchanged by the substitution and $n_2$ system is also a center. □

The following is similar to Theorem 1 in structure but instead relates to generalized cubic systems having a particular form. It is established in the same manner as Theorem 1.

**Proposition 4.** *Let $n \geq 2$ be an integer and let $\xi_1$, $\eta_1$, $\xi_2$ with $\eta_2 = 0$ define a center condition of (1) and (21). Then the system defined by $\xi_1^*$, $\eta_1^*$, $\xi_2^*$ with $\eta_2^* = 0$, where*

$$\begin{aligned}
\xi_1^*(\theta) &= \xi_1(\theta) + \tilde{\xi}(\theta) = \xi_1(\theta) + \frac{1}{n-1}\tilde{\eta}'(\theta), \\
\eta_1^*(\theta) &= \eta_1(\theta) + \tilde{\eta}(\theta), \\
\xi_2^*(\theta) &= \xi_2(\theta) + \xi_1(\theta)\tilde{\eta}(\theta) + \frac{1}{n-1}\tilde{\eta}'(\theta)\eta_1(\theta) \\
&\quad + \frac{1}{n-1}\tilde{\eta}'(\theta)\tilde{\eta}(\theta),
\end{aligned} \tag{36}$$

*satisfies the same equation (21) as the original system and defines a center for (1). In this $\tilde{\eta}(\theta)$ is a trigonometric polynomial of degree not greater than $n-1$ defined by either (32) or (33) according to $n$ being odd or even.*

We will refer to any pair of systems which satisfy the conditions of Proposition 4 as *conjugate systems.* Conjugate systems also exist for the homogeneous case but it is much more difficult to find them because we do not have the freedom allowed in generalized cubic systems by the simple definition of $\xi_2^*$. We gave such a pair of homogeneous systems in [18], but unfortunately one of the systems was presented incorrectly. They should be

$$\begin{aligned}
p(x,y) &= \left(a + \frac{2}{9(n-1)}\frac{b^2}{a}\right)x^{n-1}y, \\
q(x,y) &= ax^n + bx^{n-1}y \\
&\quad - \left((n-1)a - \frac{2(n-2)}{9(n-1)}\frac{b^2}{a}\right)x^{n-2}y^2 \\
&\quad - \frac{1}{3}(n-2)x^{n-3}y^3, \\
p(x,y) &= \frac{b}{3(n-1)}x^n - \left(a - \frac{2}{9(n-1)}\frac{b^2}{a}\right)x^{n-1}y \\
&\quad - \frac{(2n-3)b}{3(n-1)}x^{n-2}y^2,
\end{aligned}$$

$$q(x,y) = \frac{(2n-3)b}{3(n-1)}x^{n-1}y - (n-2)\left(a - \frac{2}{9(n-1)}\frac{b^2}{a}\right)x^{n-2}y^2 - \frac{(n-2)^2 b}{3(n-1)}x^{n-3}y^3 \quad (37)$$

for $n \geq 2$ and for arbitrary parameters $a \neq 0$ and $b$. These systems can be mapped to center conditions for the corresponding generalized cubic systems which will lead to solvable Abel equations (30) and (31), although we do not pursue this at this time. An example of such a calculation is given in the final section of the paper. It is also possible to find other conjugate homogeneous systems and, of those we know, they produce centers of generalized cubic systems characterized by part (2) of Proposition 8 for the particular case of quintic systems.

The case where $\eta_1 = 0$ as well has been extensively studied and complete sets of center conditions [11, 13] are known for $n = 2, 3$ systems. In the following result we consider the general case of these systems and relate them to certain homogeneous systems.

**Proposition 5.** *Let $n \geq 4$ be an integer and suppose that a generalized cubic system satisfies $\eta_1 = \eta_2 = 0$. Then any center condition of the homogeneous system of degree $n-2$ is a center condition of this system.*

*Proof.* In this case the system satisfies (22) where $\xi_1$, $\xi_2$ are arbitrary and have degrees $n-1$ and $2(n-1)$, respectively. The result is established by noting that we can simply select $\xi_1 = ((n-3)\xi - \eta')/(n-1)$ and $\xi_2 = -(n-3)\xi\eta/(n-1)$, where $\xi$, $\eta$ define a center of the homogeneous system.

$n = 2, 3$ cases are not covered by this result; however, we can easily show that they can have only symmetric centers. For $n = 3$ the most general form is given by (32) with $\xi_1(\theta) = \alpha_2 \cos 2\theta + \beta_2 \sin 2\theta + \alpha_0$ and $\xi_2(\theta) = \gamma_4 \cos 4\theta + \delta_4 \sin 4\theta + \gamma_2 \cos 2\theta + \delta_2 \sin 2\theta + \gamma_0$. Since there always exists a transformation $\theta \to \theta + \theta_0$ such that $\alpha_2 \cos 2(\theta + \theta_0) + \beta_2 \sin 2(\theta + \theta_0)$ is odd, we can take $\alpha_2 = 0$. In [19] we used trigonometric integrals to calculate the first five Lyapunov coefficients of a certain homogeneous system. Adapting them to this system ($\xi - \eta'/(n-1) \to \xi_1, -\xi\eta \to \xi_2$) we see that the first four Lyapunov coefficients will be zero if

$$\int_0^{2\pi} \xi_1(\theta)\,d\theta = \int_0^{2\pi} \xi_2(\theta)\,d\theta = \int_0^{2\pi} \bar{\xi}_1(\theta)\,\xi_2(\theta)\,d\theta = \int_0^{2\pi} \bar{\xi}_1(\theta)^2\,\xi_2(\theta)\,d\theta = 0, \quad (38)$$

where $\bar{\xi}_1' = \xi_1$. Evaluating these integrals and assuming $\xi_1 \neq 0$ (otherwise it is just a homogeneous system) we find sequentially that $\alpha_0 = \gamma_0 = \gamma_2 = \gamma_4 = 0$. This leaves $\xi_1(\theta) = \beta_2 \sin 2\theta$ and $\xi_2(\theta) = \delta_4 \sin 4\theta + \delta_2 \sin 2\theta$ and in view of Proposition 2 these are symmetric centers. The proof for $n = 2$ case can be carried out in exactly the same fashion. We will say more about $n \geq 4$ cases in the next section. □

## 4. Center Conditions for Generalized Cubic Systems Such That $\eta_2=0$

It is generally accepted that two of the independent center conditions for homogeneous systems are the Hamiltonian and symmetric systems. On the basis of Theorem 1 we will determine how each of these conditions transform to center conditions of generalized cubic systems which satisfy $\eta_2 = 0$. The Hamiltonian condition for homogeneous systems is most easily given in terms of polar representation as $\xi = -\eta'/(n+1)$. Applying the results of Theorem 1 and rescaling $\rho$ transforms (21) to

$$\frac{d\rho}{d\eta} = \frac{n^2-1}{4n^2}\eta\rho^3 + \rho^2 \quad (39)$$

which satisfies (28) with a value $C = (n^2-1)/(4n^2)$. Thus we always obtain a constant invariant equation with $I_1 = 729(n^2+3)^3/(4n^2(n^2-9)^2)$ except in the case $n = 3$ when $C = 2/9$. In this last case we obtain a Bernoulli rather than constant invariant form.

For homogeneous systems the basic condition for symmetric centers is given by $\xi$, $\eta$ being, respectively, odd and even. In this case the coefficients of $\rho^3$, $\rho^2$ in (19) are both odd and the solutions of both (10) and (19) are even functions. The condition $r(-\theta) = r(\theta)$ shows that these solutions are symmetric with respect to $x$-axis. Application of Theorem 1 to this system gives a generalized cubic system in which the coefficient functions $\xi_1$, $\eta_1$, $\xi_2$ are defined in terms of an arbitrary trigonometric polynomial of type (32) or (33) having maximum degree $n-1$. With $\xi$, $\eta$ having the stated parity, even for $n = 2$ or $n = 3$ cases there is no reason to expect any of the coefficient functions of the generalized cubic system to be either even or odd. By construction (19) and (21) are still the same equation and have an even solution $\rho(\theta)$ which is the condition which guarantees that they are centers. On the other hand there is no reason to expect (9) to have an even solution (or a translated $r(\theta + \theta_0)$ even solution), so these are not generally symmetric centers. A similar situation also exists for homogeneous systems and we gave examples of these in [20].

We now apply Theorem 1 to the two homogeneous systems for which the center conditions are fully known, that is, $n = 2, 3$ cases. The general quadratic system can be written as (the Kapteyn form [21])

$$p(x,y) = a_0x^2 + a_1xy + a_2y^2, \quad q(x,y) = b_0x^2 + b_1xy - b_0y^2 \quad (40)$$

and it is known that there are 4 independent center conditions for this system.

**Theorem 6.** *The homogeneous quadratic system written in Kapteyn form has a center at the origin if and only if one of the following conditions is satisfied:*

(1) $b_1 - 2a_0 = a_1 + 2b_0 = 0$, *the Hamiltonian system;*

(2) $a_1 = b_0 = 0$, *the reversible or symmetric case;*

(3) $a_1 + 2b_0 = b_1 + 3a_0 + 5a_2 = b_0^2 + 2a_2^2 + a_0a_2 = 0$, *the critical case;*

(4) $a_0 + a_2 = 0$, *the Lotka-Volterra case.*

The case of center conditions for the cubic system satisfying $\eta_2 = 0$ was considered in both [3, 22]. In each case 4 separate conditions are found and we can easily show that they correspond exactly to the systems obtained by applying Theorem 1 and Proposition 4 to the known conditions given in Theorem 6. In this case there is a one-to-one correspondence between the center conditions for the two systems and it gives rise to the question if this is true for all $n$. It is not directly true as there are several more parameters in the cubic system than in the homogeneous system. However, if we eliminate some of these parameters from the conditions for the cubic system we can recover the above conditions. It may be possible that some type of restricted converse of this type is true, but we also doubt that it will ever be proved. On the other hand we think it is a real possibility that we can use results from the generalized cubic system to find as yet unknown center conditions for the corresponding homogeneous problem. We also believe that it is quite likely that application of Theorem 1 and Proposition 4 to the homogeneous degree 3 system will produce a complete set of conditions for the quintic ($n = 3$) system.

Lunkevich, Sibiriskii, and Malkin [23, 24] have shown that the center conditions for the homogeneous system of degree 3 have three independent components.

**Theorem 7.** *The general homogeneous cubic system can be written as*

$$p(x, y) = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3,$$
$$q(x, y) = b_0x^3 + b_1x^2y + a_1xy^2 + b_3y^3. \quad (41)$$

*This system has a center at the origin if and only if one of the following conditions is satisfied:*

(1) $3a_1 - b_1 = a_2 - 3b_3 = 0$, *the Hamiltonian system;*

(2) $a_2 - b_1 = a_3 - b_0 = a_0 - b_3 = 0$, *the reversible or symmetric case;*

(3) $a_2-b_1+3(a_0-b_3) = a_3-b_0 = 5a_0+2a_2-b_3 = a_1+3a_3 = 0$ *and* $(3a_2 + b_3)(a_2 + 7b_3) + 100a_3^2 = 0$, *the critical case.*

As with the case of the quadratic system, each of these conditions transforms directly to a center condition for $n = 3$ quintic system. We are not aware of these having been previously given, so we formally present them below in Proposition 8. As we have seen the Hamiltonian case transforms to a Bernoulli equation with $s_3 = 0$ as defined by (26) and the symmetric case transforms to a system in which the coefficients of (21) are odd. Since these ideas were developed with respect to a polar representation we will continue to present them in this manner. In order to do so we need to express each of the homogeneous systems in polar form. We can make the conditions of the third case solvable without radicals by setting $3a_2 + b_3 = -Ka_3$ and $a_2 + 7b_3 = 100a_3/K$. From this we find

$$\xi(\theta) = a_3 \sin 4\theta - \frac{\left(K^2 - 100\right)a_3}{20K}\cos 4\theta - \frac{\left(K^2 + 100\right)a_3}{10K}\cos 2\theta,$$
$$\eta(\theta) = a_3 \cos 4\theta + \frac{\left(K^2 - 100\right)a_3}{20K}\sin 4\theta - \frac{\left(K^2 + 100\right)a_3}{20K}\sin 2\theta. \quad (42)$$

If $a_3 = 0$ we can replace these forms by one of

$$\xi(\theta) = -a_2 \cos 4\theta + 2a_2 \cos 2\theta,$$
$$\eta(\theta) = a_2 \sin 4\theta + a_2 \sin 2\theta \quad (43)$$

or

$$\xi(\theta) = -b_3 \cos 4\theta - 2b_3 \cos 2\theta,$$
$$\eta(\theta) = b_3 \sin 4\theta - b_3 \sin 2\theta. \quad (44)$$

**Proposition 8.** *If any of the following conditions are satisfied, then the corresponding system is a center of $n = 3$ (quintic) generalized cubic system defined by $\xi_1$, $\eta_1$, $\xi_2$ with $\eta_2 = 0$. In these $\xi_1 = \xi + \tilde{\xi} = \xi + \tilde{\eta}'/2$, $\eta_1 = \eta + \tilde{\eta}$, where $\tilde{\eta}$ is an arbitrary trigonometric polynomial having maximum degree 2 as defined by (32) and $\xi$, $\eta$ define a center of the homogeneous system. The systems given by (1), (2)(a), and (3) are based on Theorem 1.*

(1) *Transformation of Hamiltonian condition is as follows: let $\eta$ be an arbitrary trigonometric polynomial of degree 4 given by (32). Define $\xi = -\eta'/4$ and set*

$$\xi_2(\theta) = -\frac{1}{4}\eta'(\theta)\tilde{\eta}(\theta) + \frac{1}{2}\tilde{\eta}'(\theta)\eta(\theta) + \frac{1}{2}\tilde{\eta}'(\theta)\tilde{\eta}(\theta). \quad (45)$$

(2) *Transformation of symmetric condition is as follows:*

(a) *Basic form: let $\xi$, $\eta$ be respectively odd and even arbitrary trigonometric polynomials of degree 4 defined by (32). Define $\xi_2$ by*

$$\xi_2(\theta) = \xi(\theta)\tilde{\eta}(\theta) + \frac{1}{2}\tilde{\eta}'(\theta)\eta(\theta) + \frac{1}{2}\tilde{\eta}'(\theta)\tilde{\eta}(\theta). \quad (46)$$

*Then the coefficient functions $-2\xi\eta = 2(\xi_2-\xi_1\eta_1)$ and $2\xi - \eta' = 2\xi_1 - \eta_1'$ of (19) and (21) are odd* *functions and the solutions of these equations are even functions such that $\rho(-\pi) = \rho(\pi)$. These forms do not generally define symmetric centers unless further conditions are imposed on $\xi_1$, $\eta_1$.*

(b) *Alternate form: let $\xi$, $\eta$ be respectively odd and even arbitrary trigonometric polynomials of degree 4 defined by (32). Define $\xi_2$ to be any trigonometric polynomial given by (32) of degree 4 such that its odd part $(\xi_2)_o \neq 0$ and its even part $(\xi_2)_e$ is equal to the even part of $\xi_1\eta_1$. That is, $(\xi_2)_e = (\xi_1\eta_1)_e$. Then the coefficient functions $-2\xi\eta$, $2(\xi_2 - \xi_1\eta_1)$ and $2\xi - \eta' = 2\xi_1 - \eta_1'$ of (19) and (21) are once again odd functions and the subsequent analysis is the same as in the previous part.*

(3) *Transformation of critical case is as follows: let $\xi$, $\eta$ be given by any one of (42)–(44) and define $\xi_2$ as in part (a) of (2).*

We give Cartesian forms of the systems defined by (1) and (2)(a). For the Bernoulli case we rederived the result in a Cartesian format in order to obtain a simpler form in which some of the constants have been suitably redefined. The transformation of the Hamiltonian case for $n = 3$ gives the following quintic system:

$$
\begin{aligned}
p_1(x,y) &= a_0x^3 + a_1x^2y + 3b_3xy^2 + a_3y^3, \\
q_1(x,y) &= b_0x^3 + 3a_0x^2y + b_2xy^2 + b_3y^3, \\
p_2(x,y) &= c_0x^5 + c_1x^4y + c_2x^3y^2 + c_3x^2y^3 + c_4xy^4
\end{aligned}
\tag{47}
$$

and $q_2(x,y) = -yp_2(x,y)/x$, where

$$
\begin{aligned}
c_1 &= \frac{1}{3}(a_1 - b_2)b_0 + \frac{1}{3}(2a_1 - 3b_0 + b_2)\frac{c_0}{a_0}, \\
c_2 &= (a_1 - b_2)a_0 - 3c_0 + 3b_3\frac{c_0}{a_0}, \\
c_3 &= \frac{1}{9}(a_1^2 + a_1b_2 - 2b_2^2) - \frac{1}{3}(a_1 - 3a_3 + 2b_2)\frac{c_0}{a_0}, \\
c_4 &= \frac{1}{3}(a_1 - b_2)b_3 - b_3\frac{c_0}{a_0}.
\end{aligned}
\tag{48}
$$

The symmetric form is developed from the forms $\xi(\theta) = c_4\sin 4\theta + b_2\sin 2\theta$ and $\eta(\theta) = c_4\cos 4\theta + c_2\cos 2\theta + c_0$ with $\tilde{\eta}$ given by (32) with $N = 1$. It can be expressed as

$$
\begin{aligned}
p_1(x,y) &= \beta_2x^3 + a_1x^2y - 3\beta_2xy^2 + a_3y^3, \\
q_1(x,y) &= (a_3 + 2\alpha_2 + 2c_2)x^3 - 3\beta_2x^2y \\
&\quad + (a_1 - 6\alpha_2 + 4b_2 - 2c_2)xy^2 + \beta_2y^3, \\
p_2(x,y) &= \beta_2(a_3 + 2\alpha_2 + 2c_2)x^5 + c_1x^4y \\
&\quad - 2\beta_2(3\alpha_2 - 2b_2 + c_2)x^3y^2 + c_3x^2y^3 \\
&\quad - \beta_2a_3xy^4
\end{aligned}
\tag{49}
$$

and $q_2(x,y) = -yp_2(x,y)/x$, where

$$
\begin{aligned}
a_1 &= \alpha_0 + 3\alpha_2 - 2b_2 + c_0 + c_2 - 3c_4, \\
a_3 &= \alpha_0 - \alpha_2 + c_0 - c_2 + c_4, \\
c_1 &= 2\alpha_0\alpha_2 - 2(b_2 + 2c_4) + 2\alpha_2^2 - 2(b_2 - c_0 - c_2 + c_4) \\
&\quad - 2\beta_2^2, \\
c_3 &= 2\alpha_0\alpha_2 - 2(b_2 - 2c_4) - 2\alpha_2^2 + 2(b_2 + c_0 - c_2 - c_4) \\
&\quad + 2\beta_2^2.
\end{aligned}
\tag{50}
$$

For $K \neq 0$ the homogeneous system upon which case (3) of Proposition 8 is based has an integrating factor given by

$$
\mu(x,y) = \frac{A\left[a_3^3T_1^6 - 30a_3^2KT_1^3T_2 + 150K^2a_3\left(A\left(x^2+y^2\right) + 40Kxy\right)\right] - 20000K^4}{\left(a_3^2T_1^4 - 20Ka_3T_1T_2 + 100K^2\right)^4},
\tag{51}
$$

where $T_1 = 10x + Ky$, $T_2 = Kx + 10y$, $A = K^2 + 100$. The corresponding integrating factor for the quintic system is a massive expression having the same basic form but with an additional factor which clearly arises from $\tilde{\eta}$. It can be given as $\mu(x,y) = [(\alpha_0 + \alpha_2)x^2 + 2\beta_2xy + (\alpha_0 - \alpha_2)y^2 + 1]^2\mathscr{P}(x,y)/\mathscr{Q}^4(x,y)$ where $\mathscr{P}$, $\mathscr{Q}$ are polynomials having degrees 6, 4, respectively, such that $\mathscr{P}(0,0), \mathscr{Q}(0,0) \neq 0$.

The conditions provided by Theorems 6 and 7 can also be applied directly to systems which satisfy $\eta_1 = 0$ as well. By Theorem 1 and Proposition 5, Theorem 6 provides a set of 4 center conditions for the septic ($n = 4$) system and Theorem 7 gives a set of 3 conditions for the nonic ($n = 5$) system. Once again we think these conditions are probably complete, but we do not attempt to establish that herein. For systems satisfying (22) a type of converse can be found. That is, for general values of $n$ and using arbitrary (i.e., not necessarily center producing) forms for the coefficient functions of (22), we can transform it to an equation of the form (21) in which

$\xi_1, \eta_1, \xi_2$ have the proper degrees $n-1$, $n-1$, and $2n-6$, respectively. However, we are not able to satisfy the remaining conditions on $\xi_1, \eta_1$ such that they would define a polynomial system, so this form of (21) would instead produce a rational system. These additional conditions would depend upon the actual forms of the coefficient functions of (22) and in this regard it is clear that they must exist for any of the center cases which are a consequence of Theorem 1.

## 5. Constant Invariant Solutions and Limit Cycles

The origin of the modified system

$$\frac{dx}{dt} = \lambda x - y - p_1(x,y) - p_2(x,y),$$
$$\frac{dy}{dt} = x + \lambda y + q_1(x,y) + q_2(x,y) \tag{52}$$

will be a focus if $\lambda \neq 0$ or a center or focus if $\lambda = 0$. In [25] Giné and Llibre considered the problem for the case when $\eta_1 = \eta_2 = 0$ for general values of $n$. With the help of (28) they constructed systems with either a center or a focus and from this determined the existence of certain limit cycles. In this and the following section we carry out the same type of analysis for our $\eta_2 = 0$ systems and also extend it to certain more general cases for which $\eta_2 \neq 0$ as well. It is straightforward to show that (52) can be transformed to

$$\frac{dr}{d\theta} = \frac{\lambda r + \xi_1(\theta) r^n + \xi_2(\theta) r^{2n-1}}{1 + \eta_1(\theta) r^{n-1} + \eta_2(\theta) r^{2n-2}} \tag{53}$$

and if $\eta_2 = 0$, the counterpart of (21) is

$$\begin{aligned}\frac{d\rho}{d\theta} &= (n-1)\left(\xi_2(\theta) - \xi_1(\theta)\eta_1(\theta) + \lambda\eta_1^2(\theta)\right)\rho^3 \\ &\quad + \left((n-1)\xi_1(\theta) - 2\lambda(n-1)\eta_1(\theta) - \eta_1'(\theta)\right)\rho^2 \\ &\quad + \lambda(n-1)\rho.\end{aligned} \tag{54}$$

The form of (54) having $\eta_1 = 0$ was used in [25] and it was shown that this equation has constant invariant solutions. We show that the general form of (54) has similar solutions which for particular values of $\lambda$ and $C$ subsume those in [25]. All the systems that we obtain in this section are *Darboux* integrable since they lead to constant invariant Abel equations.

Using the coefficient functions of (54) we can write (28) as

$$\frac{du}{d\theta} + \lambda(n-1)u = C\psi(\theta), \tag{55}$$

where $u = f_3/f_2$ and $\psi = (n-1)\xi_1 - 2\lambda(n-1)\eta_1 - \eta_1'$. Solving this equation, substituting for $u$, and isolating $\xi_2$ give

$$\begin{aligned}\xi_2(\theta) &= \frac{Ce^{-\lambda(n-1)\theta}}{n-1}\left(\int e^{\lambda(n-1)\theta}\psi(\theta)\,d\theta + A\right)\psi(\theta) \\ &\quad - \lambda\eta_1^2(\theta) + \xi_1(\theta)\eta_1(\theta).\end{aligned} \tag{56}$$

The integration constant $A$ is nonzero only in the case for which $\lambda = 0$ and $n$ is odd. The result does produce a trigonometric polynomial and it does give constant invariant forms for (56); however, the only way to satisfy the maximum degree requirement of $\xi_2$ being $2n-2$ is to restrict the maximum degrees of $\xi_1, \eta_1$ to $n-1$.

In [18] we showed that a constant invariant Abel equation (23) always has particular solutions of the form

$$y(x) = \frac{1}{f_3(x)}\left(K' s_3(x)^{(1/3)} - \frac{f_2(x)}{3}\right), \tag{57}$$

where $s_3$ is given by (26) and $K'$ is a constant. In the case where $f_0 = 0$ and (28) is satisfied we can write $f_3' f_2 - f_3 f_2' + f_1 f_2 f_3 = C f_2^3$ and $s_3$ becomes

$$s_3(x) = \left(\frac{2}{27} - \frac{C}{3}\right) f_2^3(x). \tag{58}$$

With this the particular solution is just

$$y(x) = K\frac{f_2(x)}{f_3(x)} \tag{59}$$

for some constant $K$. For (54) $K$ is defined by the equation $K^2 + K + C = 0$ which only has real solutions if $C \leq 1/4$. We can use the coefficient functions of (54) to find the particular solutions, but it is easier to use the equation obtained from it by using the usual rationalizing substitution $z = \tan\theta$. There are no useful results when $n$ is even, so we will restrict our attention to odd values by considering the case $n = 3$. Also, the expressions obtained become very large so we will further restrict our attention to those systems in which $C = -2$ and $\lambda = 1$. There is little loss of generality by making these specific choices, except for the values $C = 0, 1/4$ which must be dealt with separately. We will briefly consider these cases at the end of the section. Also, we do not consider $\lambda = 0$ because these systems are centers. For the indicated choices there are two separate solutions arising from (59) given by

$$\rho(z) = \frac{-4\left(z^2+1\right)K}{C\left(\left(2a_2 + 8a_3 + B - b_3\right)z^2 + \left(8a_2 + 2B + 10b_3\right)z + 2a_2 + 7b_0 - 3b_1 + b_2 + b_3\right)}, \tag{60}$$

where $B = b_0 - b_1 - b_2$ and $K/C = -1/2, 1$. Substituting for $z$ will then give the appropriate forms for (54).

The Cartesian form of the system which is obtained from (56) can be given as

$$\begin{aligned} p_1(x,y) &= (a_2 - b_1 + b_3)x^3 + (a_3 + b_0 - b_2)x^2y \\ &\quad + a_2xy^2 + a_3y^3, \\ q_1(x,y) &= b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3, \\ p_2(x,y) &= c_0x^5 + c_1x^4y + c_2x^3y^2 + c_3x^2y^3 + c_4xy^4 \end{aligned} \tag{61}$$

and $q_2(x,y) = -yp_2(x,y)/x$, where

$$\begin{aligned} c_0 &= \frac{1}{4}(3a_2 - 2b_1 + 3b_3)(2a_2 - 3b_1 + b_2 + b_3) + 8b_0^2 \\ &\quad + \frac{1}{4}(33a_2 - 30b_1 + 4b_2 + 29b_3)b_0, \\ c_1 &= \frac{1}{2}(18a_2 - 19b_1 - 3b_2 + 37b_3)a_2 + b_1^2 + \frac{5}{2}b_1b_2 \\ &\quad - \frac{21}{2}b_1b_3 - \frac{1}{2}b_2^2 - b_2b_3 + \frac{19}{2}b_3^2 - \frac{3}{2}b_0^2 + 8a_3b_0 \\ &\quad + \frac{1}{2}(35a_2 - 3b_1 - 14b_2 + 40b_3)b_0 \\ &\quad + (2a_2 - 3b_1 + b_2 + b_3)a_3, \\ c_2 &= (10a_2 - 3b_1 - 8b_2 + 19b_3)a_2 + \frac{1}{2}b_1^2 + \frac{3}{2}b_1b_2 + b_2^2 \\ &\quad - \frac{19}{2}b_2b_3 + \frac{19}{2}b_3^2 + 19a_3b_0 \\ &\quad - \frac{1}{2}(2a_2 + b_1 + 2b_2 + 19b_3)b_0 \\ &\quad + (18a_2 - 10b_1 - b_2 + 19b_3)a_3, \\ c_3 &= -\frac{1}{2}(b_1 + 3b_2 + 17b_3)a_2 + \frac{1}{2}b_1b_2 + \frac{1}{2}b_1b_3 + \frac{1}{2}b_2^2 \\ &\quad + b_2b_3 - \frac{19}{2}b_3^2 + 9a_3^2 - \frac{1}{2}b_0^2 - a_3b_0 \\ &\quad - \frac{1}{2}(a_2 - b_1)b_0 \\ &\quad + (20a_2 - 3b_1 - 8b_2 + 19b_3)a_3, \\ c_4 &= -\frac{1}{4}(a_2 + 3b_3)(2a_2 + b_0 - b_1 - b_2 - b_3) + 9a_3^2 \\ &\quad + a_3b_0 - (b_1 + b_2 + 8b_3)a_3. \end{aligned} \tag{62}$$

Reversing the transformations which led to (54) and its equivalent form in terms of $z$ gives us two invariant curves of the original system. From (60) and (61) we obtain a pair of conics $\mathscr{C}_1, \mathscr{C}_2$ having general form $A_kx^2 + B_kxy + C_ky^2 + D_k = 0$ for $k = 1, 2$. The coefficients of these are linear expressions given in terms of the six arbitrary parameters $a_2, a_3, b_0, b_1, b_2, b_3$ of the system. We have

$$\begin{aligned} &(2a_2 + 5b_0 - 3b_1 + b_2 + b_3)x^2 \\ &\quad + 2(3a_2 + b_0 - b_1 - b_2 + 4b_3)xy \\ &\quad + (2a_2 + 6a_3 + b_0 - b_1 - b_2 - b_3)y^2 - 2 = 0, \\ &(2a_2 + 11b_0 - 3b_1 + b_2 + b_3)x^2 \\ &\quad + 2(6a_2 + b_0 - b_1 - b_2 + 7b_3)xy \\ &\quad + (2a_2 + 12a_3 + b_0 - b_1 - b_2 - b_3)y^2 + 4 = 0. \end{aligned} \tag{63}$$

Since the determinant of the coefficient matrix of the linear system defined by the coefficients of these relations is nonzero, we can uniquely solve for any set of values $\{A_1, B_1, C_1, A_2, B_2, C_2\}$ in terms of the system parameters. That is, any combination of hyperbolae, pairs of lines, circles, ellipses, or null conics is possible in this system. The situation is similar for any other choice of $\lambda, C$ except for those values explicitly mentioned, so nothing is lost by considering the specific values $\lambda = 1, C = -2$. Also, in determining the form of (63) we find that the transformation from (60) shows that $x^2 + y^2 = 0$ is another invariant of these systems.

A necessary condition that a curve be a limit cycle is that it is a closed curve encircling the critical point. For the invariant curves defined by (63) this means that we are dealing with ellipses or circles which have their centers at the origin. However, not all such curves are limit cycles because they are dependent upon the particular configuration. If one invariant curve (circle or ellipse) lies entirely inside another, then they are both limit cycles provided that no critical points of the system lie on either of the curves. However, if they intersect the flow pattern is markedly different with the points of intersection being saddle points (see (64) below and the results in the next section). In the following we give examples of both of these situations. The system

$$\begin{aligned} \frac{dx}{dt} &= x - y - \frac{49}{144}x^3 + \frac{1}{9}x^2y - \frac{37}{144}xy^2 + \frac{1}{9}y^3 \\ &\quad + \frac{13}{576}x^5 + \frac{1}{144}x^4y + \frac{1}{32}x^3y^2 + \frac{1}{144}x^2y^3 \\ &\quad + \frac{5}{576}xy^4, \\ \frac{dy}{dt} &= x + y - \frac{5}{36}x^3 - \frac{53}{144}x^2y - \frac{5}{36}xy^2 - \frac{41}{144}y^3 \\ &\quad + \frac{13}{576}x^4y + \frac{1}{144}x^3y^2 + \frac{1}{32}x^2y^3 + \frac{1}{144}xy^4 \\ &\quad + \frac{5}{576}y^5 \end{aligned} \tag{64}$$

has a solution given by $\mathscr{U}(x, y) = \mathscr{C}$, where

$$\begin{aligned}\mathscr{U}(x, y) = 3\ln\left(x^2 + y^2\right) + 6\arctan\left(\frac{x}{y}\right) \\ - \ln\left(2x^2 + xy + y^2 - 24\right) \\ - 2\ln\left(x^2 + y^2 - 4\right).\end{aligned} \quad (65)$$

Since the circle $x^2 + y^2 = 4$ lies entirely inside the ellipse $2x^2 + xy + y^2 = 24$ and the system has no other real-valued critical points, both are limit cycles. The origin is an unstable focus ($\lambda = 1$) so the circle is stable and the ellipse is unstable as $t \to \infty$. The system

$$\begin{aligned}\frac{dx}{dt} &= x - y - \frac{5}{16}x^3 - \frac{3}{16}x^2y - \frac{5}{4}xy^2 + \frac{3}{4}y^3 + \frac{1}{64}x^5 \\ &\quad + \frac{5}{64}x^4y + \frac{17}{64}x^3y^2 + \frac{5}{64}x^2y^3 + \frac{1}{4}xy^4, \\ \frac{dy}{dt} &= x + y - \frac{1}{8}x^3 - \frac{5}{16}x^2y - \frac{17}{16}xy^2 - \frac{5}{4}y^3 \\ &\quad + \frac{1}{64}x^4y + \frac{5}{64}x^3y^2 + \frac{17}{64}x^2y^3 + \frac{5}{64}xy \\ &\quad + \frac{1}{4}y^5\end{aligned} \quad (66)$$

has the circle $x^2 + y^2 = 4$ and the ellipse $x^2 + 16y^2 = 16$ as invariant curves. It has nine real-valued critical points; one at the origin and four others at the intersections of the invariant curves. These last four are saddle points which are given by $(x^*, y^*) = (\pm 4/\sqrt{5}, \pm 2/\sqrt{5})$ in which we take all combinations of the signs. Translating each of these points to the origin we find that the eigenvalues of the linear portion of the resulting systems are given by the two sets of values $r = \pm 1, \mp 2$. The remaining four critical points are centers which are located in the four regions outside one of the curves and inside the other. They are given by $(x', y') = (-50\alpha^3 + 14\alpha, 2\alpha)$ where $\alpha$ is a root of $125x^4 - 45x^2 + 2 = 0$. Translating each to the origin, we find that the eigenvalues of the linear parts satisfy one of the two relations $r^2 + (123 \pm 13\sqrt{41})/50 = 0$. Hence they are pure imaginary and a rotation and rescaling of the time will allow us to write the systems in standard form as $\dot{X} = -Y + \cdots$, $\dot{Y} = X + \cdots$. The critical points are centers because the systems are integrable.

In presenting system (66) our original intention was to give a system in which the invariant curves intersect at some general set of points, but we found the complexity of the resulting behaviour somewhat surprising. For example, let $\mathscr{D}$ be the region containing the critical point at the origin. The boundary $\partial\mathscr{D}$ of $\mathscr{D}$ consists of four arcs, two from the ellipse and two from the circle, which meet at the four critical points $(x^*, y^*)$. Clearly, these are points where a unique tangent fails to exist on $\partial\mathscr{D}$. Any trajectory originating in $\mathscr{D}$ approaches $\partial\mathscr{D}$ in a fashion similar to that of a general limit cycle but is displaced from this boundary as it approaches one of the critical (saddle) points. Thus $\partial\mathscr{D}$ acts somewhat like a stable limit cycle for trajectories in $\mathscr{D}$ except that in this case it is not smooth. The behaviour in the four bounded regions outside $\mathscr{D}$ as well as in the unbounded region exterior to both curves is also quite interesting. We believe that it would be of interest to analyze this system (or ones like it) more fully than we can do here.

The final case we present is the simple system

$$\begin{aligned}p_1(x, y) &= \frac{1}{3R_1^2R_2^2}\left(x^2 + y^2\right) \\ &\quad \cdot \left(3\left(R_1^2 + R_2^2\right)x - \left(2R_1^2 + R_2^2\right)y\right), \\ q_1(x, y) &= -\frac{1}{3R_1^2R_2^2}\left(x^2 + y^2\right) \\ &\quad \cdot \left(\left(2R_1^2 + R_2^2\right)x + 3\left(R_1^2 + R_2^2\right)y\right), \\ p_2(x, y) &= -\frac{1}{R_1^2R_2^2}x\left(x^2 + y^2\right)^2, \\ q_2(x, y) &= \frac{1}{R_1^2R_2^2}y\left(x^2 + y^2\right)^2.\end{aligned} \quad (67)$$

It has only a single critical point at the origin if $R_2 \neq R_1$ and has two circular limit cycles $x^2 + y^2 = R_1^2$, $x^2 + y^2 = R_2^2$. For $R_1^2 = 1$ and $R_2^2 = 3$ a somewhat different system than this, based on the values $\lambda = -3$, $C = 3/16$ and the condition $\eta_1 = 0$, was given in [23]. See also the results in the next section.

The preceding discussion does not apply directly to the two cases $C = 0, 1/4$. If $C = 0$ we can see from (56) that the coefficient of $\rho^3$ in the trigonometric Abel equation (54) vanishes and the equation reduces to a Bernoulli equation. The roots of $K^2 + K + C = 0$ are $K = 0, -1$ and the main consequence of this is that there is only one conic $\mathscr{C}_1$ rather than the two which occur in the general case. For $C = 1/4$ the roots are equal and the two conics coalesce, so we again have a situation in which there is only a single conic.

## 6. Concentric Circular Invariant Curves

In [26] Llibre and Rodríguez construct a vector field having an arbitrary distribution of circular limit cycles and show that this system has a Darboux first integral. The solution of (67) is very close to the general form given in [26] but with some minor differences in the coefficients. Here we consider somewhat more general conditions for which a generalized cubic system for which $n$ being odd can have two isolated, concentric circular invariant curves by removing the condition that $\eta_2$ should be zero. This will produce systems which have (at least) two isolated closed trajectories, but in many cases these are not limit cycles. For the forms that we consider these systems are transformable to Riccati equations which frequently can be solved in terms of special functions. Clearly, the easiest way to search for such solutions is to use a polar representation. Let $\xi_1$, $\xi_2$ be given by (32) and write $\xi_2 = a + \widetilde{\xi_2}$, $\xi_1 = b + \widetilde{\xi_1}$ where $a, b$ are constants and $\widetilde{\xi_2}, \widetilde{\xi_1}$ have no constant parts. If $r = R_0 \neq 0$ is a constant solution of (53) then we must have $\lambda R_0 + R_0^n \xi_1 + R_0^{2n-1}\xi_2 = R_0(\lambda + R_0^{n-1}(b + \widetilde{\xi_1}) + R_0^{2(n-1)}(a + \widetilde{\xi_2})) = 0$. Since the trigonometric functions (as in (32)) are independent this splits into two equations:

$$\begin{gathered} aR_0^{2(n-1)} + bR_0^{n-1} + \lambda = 0, \\ \widetilde{\xi_1}(\theta) + R_0^{n-1}\widetilde{\xi_2}(\theta) = 0. \end{gathered} \tag{68}$$

If $R_1$, $R_2$ are distinct roots of the quadratic this gives $a = \lambda/(R_1R_2)^2$ and $b = -\lambda(R_1^2 + R_2^2)/(R_1R_2)^2$. Then from the second equation of (68) we have

$$\widetilde{\xi_2}(\theta) = -\frac{1}{R_1^{n-1}}\widetilde{\xi_1}(\theta) = -\frac{1}{R_2^{n-1}}\widetilde{\xi_1}(\theta) \tag{69}$$

which can be satisfied only if $\widetilde{\xi_1} = \widetilde{\xi_2} = 0$. Hence, in order to have two distinct solutions of this type we assume in the following that $\xi_2 = \lambda/(R_1R_2)^2$, $\xi_1 = -\lambda(R_1^2+R_2^2)/(R_1R_2)^2$. This can be seen to be the case for system (67). At this point we can choose $\eta_1$, $\eta_2$ arbitrarily subject to the necessary degree constraints. If we take $\eta_1 = \eta_2 = 0$ and $R_1 = 1$, $R_2 = \sqrt{3}$, $\lambda = -3$ we obtain the system given in [25] mentioned in the last section.

In what follows we will take $\eta_1(\theta) = c_2\cos 2\theta + d_2\sin 2\theta + c_0$ and $\eta_2(\theta) = \alpha_2\cos 2\theta + \beta_2\sin 2\theta + \alpha_0$. These forms can be assumed for any odd value of $n$, but we will once again restrict our attention to the particular case $n = 3$. With $r^2 = R$, (53) becomes

$$\frac{dR}{d\theta} = \frac{2\lambda}{R_1^2R_2^2}\frac{R\left(R - R_1^2\right)\left(R - R_2^2\right)}{1 + \left(c_2\cos 2\theta + d_2\sin 2\theta + c_0\right)R + \left(\alpha_2\cos 2\theta + \beta_2\sin 2\theta + \alpha_0\right)R^2}. \tag{70}$$

Eliminating the trigonometric functions by setting $z = \tan\theta$ and then interchanging the roles of $z$ and $R$ lead to the Riccati equation

$$\begin{aligned} \frac{dz}{dR} = {} & \frac{R_1^2R_2^2}{2\lambda}\frac{\left(\alpha_0 - \alpha_2\right)R^2 + \left(c_0 - c_2\right)R + 1}{R\left(R - R_1^2\right)\left(R - R_2^2\right)}z^2 \\ & + \frac{R_1^2R_2^2}{\lambda}\frac{\beta_2 R + d_2}{\left(R - R_1^2\right)\left(R - R_2^2\right)}z \\ & + \frac{R_1^2R_2^2}{2\lambda}\frac{\left(\alpha_0 + \alpha_2\right)R^2 + \left(c_0 + c_2\right)R + 1}{R\left(R - R_1^2\right)\left(R - R_2^2\right)}. \end{aligned} \tag{71}$$

The usual substitution $y = -u'/(P(x)u)$ for $y' = P(x)y^2 + Q(x)y + R(x)$ will then further convert this to a linear, second-order equation. We do not give the result of this conversion except to mention that it has certain similarities to the general *Heun* differential equation [27]

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)}y = 0 \tag{72}$$

and its confluent forms. These equations are very general and include the hypergeometric cases ${}_2\mathcal{F}_1$, ${}_1\mathcal{F}_1$, ${}_0\mathcal{F}_1$ as well as other classes of functions. The solutions of (71) are frequently expressible in terms of *Heun functions* with $R = 0, R_1^2, R_2^2$ reflecting the singularities which appear in (72). In addition to the solutions given in [25], it is possible for (71) to have other elementary solutions. For example, the pair of systems given by

$$\begin{aligned} \eta_1(\theta) = {} & \pm\frac{R_1^2 - R_2^2}{R_1^2R_2^2}\cos 2\theta \pm \frac{\lambda\left(R_1^2 - R_2^2\right)}{R_1^2R_2^2}\sin 2\theta \\ & - \frac{R_1^2 + R_2^2}{R_1^2R_2^2}, \\ \eta_2(\theta) = {} & \mp\frac{R_1^4 - R_2^4}{2R_1^2R_2^2}\cos 2\theta + \frac{R_1^4 + R_2^4}{2R_1^2R_2^2} \end{aligned} \tag{73}$$

reduce the linearized form of (71) to the Euler equation $z'' + z'/R - z/(4\lambda^2R^2) = 0$.

Each of the systems defined by (71) has the property that the circles $r = R_1, R_2$ are invariant curves, although most of the solutions are non-Liouvillian. By considering certain numerical examples it can be seen that there are systems for which both, one, or neither of these circles is a limit cycle. As we saw in the cases of systems (64)–(67) the exact nature of the curves is dependent upon the existence and location of other critical points of the system.

We have not considered the remaining possibility in (68) when the quadratic has equal roots. Instead we will look at the case of the center-focus form ($\lambda = 0$) when the system has a single invariant circle $r = R_1$. Similar calculations show that a relation, $\xi_2 = -\xi_1/R_1^{n-1}$, of the same type as the second equation of (68) must be satisfied. Due to parity considerations, such an equality is only possible for odd values of $n$ and shows that circular invariant curves centered at the origin cannot exist if $n$ is even. Whether or not the circle is a limit cycle is once again dependent upon the structure of the system. The system

$$\begin{aligned} \frac{dx}{dt} = {} & -y + \frac{4}{5}x^3 - \frac{22}{5}x^2y + \frac{8}{5}xy^2 + \frac{4}{5}y^3 \\ & - \frac{4}{5}x\left(x^2 + y^2\right)\left(x^2 - 4xy - 4y^2\right), \\ \frac{dy}{dt} = {} & x + \frac{12}{5}x^3 - \frac{8}{5}x^2y - \frac{14}{5}xy^2 - \frac{4}{5}y^3 \\ & - \frac{2}{5}\left(x^2 + y^2\right)\left(3x^3 + 8x^2y - 5xy^2 - 2y^3\right) \end{aligned} \tag{74}$$

has $x^2 + y^2 = 1$ as an invariant curve and in this case it is not a limit cycle. The origin is asymptotically stable (as $t \to \infty$) and apart from this the system has eight other real-valued critical points. Four of these lie on the circle with two of these being unstable nodes and two saddle points. The saddle points are given by

$$(x,y)=\left(\pm\frac{\sqrt{38}}{13}\pm\frac{3\sqrt{14}}{26},\pm\frac{3\sqrt{38}}{26}\mp\frac{\sqrt{14}}{13}\right)\approx(\pm 0.906,\pm 0.423)\quad(75)$$

and the nodes by

$$(x,y)=\left(\pm\frac{\sqrt{38}}{13}\mp\frac{3\sqrt{14}}{26},\pm\frac{3\sqrt{38}}{26}\pm\frac{\sqrt{14}}{13}\right)\approx(\pm 0.042,\pm 0.999).\quad(76)$$

All trajectories inside the circle appear to emanate (as $t\to-\infty$) from one of the nodes or the other, but it does not seem to be possible to specify from which one a particular trajectory originates. As a trajectory approaches a neighbourhood of one of the saddle points, it is deflected away in a direction towards the origin. The remaining four critical points lie just outside the circle and in conjunction with those on the circle give rise to a fairly complex flow pattern in that region.

## 7. Application to Abel Differential Equations

Many physical systems can be converted to Abel differential equations and because of this it is always of interest when solvable Abel equations are encountered. In an earlier work [17] we showed that a homogeneous system can be transformed to such an equation (set $p_2=0, p_1=p, q_1=q$ in (30)) having rational coefficients and herein we have shown that $\eta_2=0$ generalized cubic systems can also be similarly transformed. One consequence of this is that any integrable system of this type can be transformed to a solvable Abel equation. In the papers by Cheb-Terrab and Roche [16, 28] the authors clearly demonstrate the need for a convenient method of classifying Abel equations. Two Abel equations are said to belong to the same *equivalence class* if one can be transformed into the other using a transformation of a specific type (see [28]). If this is possible then if one equation of a particular class is solvable so are all the other members of that class. Many of the ideas in these papers have been incorporated in the computer algebra system Maple. All symbolic computations in this paper were carried out in the most recent version Maple 2016 which has an excellent suite of routines for solving such equations; however, we encountered several equations which could not be solved. These include (21) and its corresponding rational form given by setting $z=\tan\theta$ as well as (30) and its first kind form defined by (31). In the following we will present an example of a system which produces Abel equations which were not solved by the software but which can be transformed to a solvable equation using a nonstandard transformation.

One simple cubic system defined from case 4 of Theorem 6 by the functions

$$\begin{aligned}\xi(\theta)&=\frac{1}{4}(2a_1-b_2)\sin 3\theta-\frac{1}{4}(a_2+2b_1)\cos 3\theta\\&\quad-\frac{1}{4}(2a_1+b_2)\sin\theta+\frac{1}{4}(a_2-2b_1)\cos\theta,\\\eta(\theta)&=\frac{1}{4}(a_2+2b_1)\sin 3\theta+\frac{1}{4}(2a_1-b_2)\cos 3\theta\\&\quad+\frac{1}{4}(a_2-2b_1)\sin\theta+\frac{1}{4}(2a_1+b_2)\cos\theta\end{aligned}\quad(77)$$

and $\tilde{\eta}(\theta)=\alpha_1\cos\theta+\beta_1\sin\theta$ with $a_1=5/2, a_2=b_1=-b_2=3, \alpha_1=5, \beta_1=9/2$ produced a system having four invariant lines. The phase plane equation (3) is

$$\frac{dy}{dx}=-\frac{4x+30x^2+48xy-30y^2+75x^2y+95xy^2-75y^3}{4y-6x^2+28xy+6y^2-75x^3-95x^2y+75xy^2}.\quad(78)$$

This is easily solved by Maple using symmetries and the resulting rational (*Darboux*) solution can be expressed as

$$\mathscr{U}(x,y)=\frac{(5y+2)^{66}(21x-2y+2)^{39}(13x+18y+2)^{77}}{(10x+9y+2)^{182}}=\mathscr{C}.\quad(79)$$

This cubic system produces an Abel equation (30) given by

$$\frac{dy}{dx}=-\frac{5(15x^2-19x-15)y^3+2(3x^2+14x-3)y^2+4xy}{2(3x^3-x^2+21x+15)y+4x^2+4}\quad(80)$$

and the corresponding homogeneous system produces the equation

$$\frac{dy}{dx}=-\frac{6(x^2+x-1)y^2-2xy}{(x-1)(2x+5)(3x+1)y-2x^2-2},\quad(81)$$

neither of which was solved by Maple. Since the transformation $y=-2u/((9x+10)u-2)$ transforms (80) into (81) these equations belong to the same equivalence class. Moreover, the equation arising from (21) has the same first absolute invariant $I_1$ (when suitably modified) as (80) and (81), so it too belongs to the same equivalence class. Thus failure of the algorithm to solve one means that the others would also not be solved. However, each of these equations is solvable since solutions can be obtained by suitably transforming (79).

If we transform (81) by setting

$$x=-\frac{3v-5u+13}{9v-2u+13},\qquad y=\frac{3}{13}v-\frac{2}{39}u+\frac{1}{3}\quad(82)$$

we obtain the Abel equation

$$\frac{dv}{du} = -\frac{2}{3}\frac{33v^2 - (29u - 91)v}{30uv - 11u^2 + 52u}. \quad (83)$$

Since (82) is not of a type which will preserve the equivalence class of the original equation (or even guarantee that the new equation is of Abel type), (83) belongs to a different class than (81). In the terminology of [28] it is of type AIA (Abel, Inverse-Abel) which means that if we interchange the roles of $u$ and $v$ we obtain a new equation having the same form, but of a different equivalence class. In the case of this example, both of these new equations are of type constant invariant and hence are solvable.

The primary reason for presenting this particular example is because of its simplicity and clarity. In this study we have encountered systems whose first integrals are much more complicated than the one just given but will still generate solvable Abel equations. At this time we have not studied the nature of these new Abel equations, although it is highly unlikely that they are as simple as (83) with regard to solvability.

## Disclosure

The author is now retired but was formerly a member of the Math, Physics and Geology Department. At present he holds the position of "Senior Scholar" which basically allows him to continue to publish through his affiliation with the university.

## Competing Interests

The author declares that there are no competing interests.

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