Research Article

On the Existence of Infinitely Many Solutions for Nonlocal Systems with Critical Exponents

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We study a class of semilinear nonlocal elliptic systems posed on settings without compact Sobolev embedding. By employing critical point theory and concentration estimates, we prove the existence of infinitely many solutions for values of the dimension N, where $N > 6s$, provided $0 < s < 1$.

1. Introduction and Main Result

Nonlocal problems and operators have enjoyed much attention from mathematicians in recent years probably due to their interesting analytical structure and their numerous physical applications in many fields such as flame propagation, conservation laws, ultrarelativistic limits of quantum mechanics, quasi-geostrophic flows, and the thin obstacle problem (see [1–3]). In this paper, we are concerned with the study of the infinitely many solutions for the following system:

$$(-\Delta)^s u = \frac{2\alpha}{\alpha+\beta} |u|^{\alpha-2} u |v|^{\beta} + \lambda u \quad \text{in } \Omega,$$

$$(-\Delta)^s v = \frac{2\beta}{\alpha+\beta} |u|^{\alpha} |v|^{\beta-2} v + \mu v \quad \text{in } \Omega,$$

$$u = v = 0 \quad \text{on } \partial\Omega,$$

(1)

where Ω is an open bounded domain in $\mathbb{R}^N$, $\lambda, \mu > 0$ and $\alpha > 1$, $\beta > 1$ satisfy $\alpha + \beta = 2_s^* = 2N/(N-2s)$, $s \in (0,1)$ and $N > 2s$. When $\alpha = \beta$, $\lambda = \mu$, and $u = v$, problem (1)

reduces to the Brézis-Nirenberg type problem with fractional Laplacian:

$$(-\Delta)^s u = |u|^{2_s^*-2} u + \lambda u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega.$$

(2)

In [4], Brézis and Nirenberg considered the existence of positive solutions for problem (2) with $s = 1$. Such a problem involves the critical Sobolev exponent $2^* = 2N/(N-2)$ for $N \geq 3$, and it is well known that the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact even if Ω is bounded. Hence, the associated functional of problem (2) does not satisfy the Palais-Smale condition, and critical point theory can not be applied directly to find solutions of the problem. However, it is found in [4] that the functional satisfies the $(\mathrm{PS})_c$ condition for $c \in (0, (1/N)S^{N/2})$, where S is the best Sobolev constant and $(1/N)S^{N/2}$ is the least energy level at which the Palais-Smale condition fails. So a positive solution can be found if the mountain pass value corresponding to problem (2) is strictly less than $(1/N)S^{N/2}$. In [5], a concentration-compactness principle was developed to treat noncompact critical variational problems. In the study of the existence of multiple solutions for critical problems, to retain

the compactness, it is necessary to have a full description of energy levels at which the associated functional does not satisfy the Palais-Smale condition. A global compactness result is found in [6], which describes precisely the obstacles of the compactness for critical semilinear elliptic problems. This compactness result shows that above a certain energy level it is impossible to prove the Palais-Smale condition. For this reason, to obtain infinitely many solutions for the critical problem, it is essential to find a condition that can replace the standard Palais-Smale condition. In [7], Devillanova and Solimini considered (2) with $s = 1$ and obtained infinitely many solutions for every $\lambda > 0$ if $N \geq 7$. They proved this latter result by employing the concentration estimates and lower bound of augmented Morse index on min–max points (see [8]), which seem unapplicable to the case of (1) directly. This work was extended to an analogous problem involving p-Laplacian for $1 < p < \infty$ by Cao et al. [9]. They proved that if $N > p^2 + p$, the following problem,

$$-\Delta_p u = |u|^{p^*-2} u + \lambda |u|^{p-2} u \quad \text{in } \Omega, \ u = 0 \text{ on } \partial\Omega, \quad (3)$$

where $\mu > 0$ and $p^* = pN/(N - p)$, has infinitely many solutions. Recently, Yan et al. extended the result in [10] to problem (2) and obtained infinitely many solutions for every $\lambda > 0$ if $N > 6s$, where $0 < s < 1$. Similar problems to (1) for the Laplacian operator have been studied extensively in recent years; see [6, 11–14] and the references therein. In particular, Alves et al. [11] studied the p-Laplacian system with critical growth and obtained the existence of infinitely many solutions if $N > p^2 + p$. An important cornerstone for these works has been laid out in a remarkable paper [15], where Caffarelli and Silvestre gave a new formulation of the fractional Laplacian through Dirichlet-Neumann maps. This is extensively used in the recent literature since it allows transforming nonlocal problems to local ones, which permits the use of variational methods. This will come in handy for this work. A more general form of the fractional operator has been studied and a multiplicity of solutions has been shown in several cases. For further reading, we refer the reader to these recently published papers on the fractional Laplacian [16–22].

In this paper, we prove that (1) has infinitely many solutions under the following conditions: $N > 6s$, where $0 < s < 1$, by studying system (1) in the critical case $\alpha + \beta = 2_s^*$. We adapt the original idea used in [7] which was described in the above paragraph (see also [9–11, 13]). After perturbing problem (1) into a subcritical case with a (PS) functional in all energy levels, estimates on the set of the solution sequences to the subcritical case of (1), a global compactness argument, and a (local) Pohozaev identity are used to establish the strong convergence and finally use min–max theorems on a genus homotopic class to produce infinitely many critical values. The paper is organized as follows: in Section 2, some notations and preliminary results are established. In Section 3, we establish a local Pohozaev identity which allows us to prove the $E_0^s(\mathscr{C}_\Omega)$-strong convergence of solutions for the subcritical case of problem (18), and, finally, we show how this technique allows the application of classical min–max

arguments to (1) and to prove, in this way, the existence of infinitely many solutions. Our main result is the following.

Theorem 1. *Let $0 < s < 1$ and $\lambda \geq 0$, $\mu \geq 0$ with $\lambda + \mu > 0$. Then problem (1) has infinitely many solutions when $N > 6s$.*

2. Notation and Preliminaries

The powers $(-\Delta)^s$ of the positive Laplacian operator $-\Delta$, in Ω, with zero Dirichlet boundary conditions are defined via its spectral decomposition, namely,

$$(-\Delta)^s u(x) := \sum_{j=1}^{k} a_j \rho_j^s \varphi_j(x), \quad (4)$$

where (ρ_j, φ_j) is the sequence of eigenvalues and eigenfunctions of the operator $(-\Delta)$ in Ω under zero Dirichlet boundary data and a_j are the coefficients of u for the base $\{\varphi_j\}_{j=1}^{\infty}$ in $L^2(\Omega)$. In fact, the fractional Laplacian $(-\Delta)^s$ is well defined in the space of functions

$$H_0^s(\Omega) := \left\{ u = \sum_{j=1}^{k} a_j \varphi_j \in L^2(\Omega) : \|u\|_{H_0^s} \right.$$

$$\left. = \left(\sum_{j=1}^{k} a_j^2 \rho_j^s \right)^{1/2} < \infty \right\}, \quad (5)$$

and $\|u\|_{H_0^s} = \|(-\Delta)^{s/2} u\|_{L^2(\Omega)}$.

A pair of functions $(u, v) \in H_0^s(\Omega) \times H_0^s(\Omega)$ is said to be a weak solution of problem (1) if

$$\int_\Omega \left((-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi_1 + (-\Delta)^{s/2} v (-\Delta)^{s/2} \varphi_2 \right) dx$$

$$- \int_\Omega \left(\lambda u \varphi_1 + \mu v \varphi_2 \right) dx$$

$$- \frac{2\alpha}{\alpha + \beta} \int_\Omega |u|^{\alpha-2} u |v|^\beta \varphi_1 dx \quad (6)$$

$$- \frac{2\beta}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^{\beta-2} v \varphi_2 dx = 0,$$

for all $(\varphi_1, \varphi_2) \in H_0^s(\Omega) \times H_0^s(\Omega)$. Solutions to problem (1) will be obtained as critical points of the corresponding energy functional

$$J_{\lambda,\mu}(u, v) := \frac{1}{2} \int_\Omega \left(\left| (-\Delta)^{s/2} u \right|^2 + \left| (-\Delta)^{s/2} v \right|^2 \right) dx$$

$$- \frac{1}{2} \int_\Omega \left(\lambda |u|^2 + \mu |v|^2 \right) dx \quad (7)$$

$$- \frac{2}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^\beta dx.$$

The functional is well defined in $H_0^s(\Omega) \times H_0^s(\Omega)$, and, moreover, the critical points of the functional $J_{\lambda,\mu}$ correspond to solutions of (1). We now conclude the main ingredients

of a recently developed technique use in order to deal with fractional powers of the Laplacian operator. Motivated by the work of Caffarelli and Silvestre [15], several authors have considered an equivalent definition of the operator $(-\Delta)^{s/2}$ in a bounded domain with zero Dirichlet boundary data means of auxiliary variable, see [2, 23]. Associated with the bounded domain Ω, let us consider the cylinder $\mathscr{C}_\Omega := \Omega \times (0, +\infty) \subset \mathbb{R}^{N+1}_+$. Now, for a function $u \in H^s_0$, we define the s-harmonic extension $w = E_s(u)$ to the cylinder $\mathscr{C}_\Omega$ as the solution of the problem:

$$\operatorname{div}\left(y^{1-2s}\nabla w\right) = 0 \quad \text{in } \mathscr{C}_\Omega$$

$$w = 0 \quad \text{on } \partial_L \Omega$$

$$w = u \quad \text{on } \Omega \times \{0\}, \tag{8}$$

$$(-\Delta)^s u(x) = -k_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y),$$

where $k_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$ is a normalization constant. The extension function $w(x, y)$ belongs to the space

$$H^1_{0,L}(\mathscr{C}_\Omega) := \left\{ w \in L^2(\mathscr{C}_\Omega) : w \right.$$

$$\left. = 0 \text{ on } \partial_L \Omega, \int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla w|^2 \, dx \, dy < \infty \right\} \tag{9}$$

endowed with the norm

$$\|z\|_{H^1_{0,L}(\mathscr{C}_\Omega)} := \left(\int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla z|^2 \, dx \, dy \right)^{1/2}. \tag{10}$$

The extension operator is an isometry between $H^s_0(\Omega)$ and $H^1_{0,L}(\mathscr{C}_\Omega)$; namely,

$$\|u\|_{H^s_0(\Omega)} = \|E_s(u)\|_{H^1_{0,L}(\mathscr{C}_\Omega)}, \quad \forall u \in H^s_0(\Omega). \tag{11}$$

With this extension, we can reformulate (1) as the following local problem:

$$-\operatorname{div}\left(y^{1-2s}\nabla w_1\right) = 0,$$

$$-\operatorname{div}\left(y^{1-2s}\nabla w_2\right) = 0$$

$$\text{in } \mathscr{C}_\Omega,$$

$$w_1 = w_2 = 0, \quad \text{on } \partial_L \Omega,$$

$$y^{1-2s}\frac{\partial w_1}{\partial \nu}$$

$$= \frac{2\alpha}{\alpha + \beta} |w_1(x,0)|^{\alpha-2} w_1(x,0) |w_2(x,0)|^\beta \tag{12}$$

$$+ \lambda w_1(x,0) \quad \text{on } \Omega \times \{0\},$$

$$y^{1-2s}\frac{\partial w_2}{\partial \nu}$$

$$= \frac{2\beta}{\alpha + \beta} |w_1(x,0)|^\alpha |w_2(x,0)|^{\beta-2} w_2(x,0)$$

$$+ \mu w_2(x,0) \quad \text{on } \Omega \times \{0\},$$

where

$$\frac{\partial w_i}{\partial \nu} := -k_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial w_i}{\partial y}, \quad i = 1, 2 \tag{13}$$

and $w_1, w_2 \in H^1_{0,L}(\mathscr{C}_\Omega)$ are the s-harmonic extension of $u, v \in H^s_0(\Omega)$, respectively. Let

$$E^s_0(\mathscr{C}_\Omega) := H^1_{0,L}(\mathscr{C}_\Omega) \times H^1_{0,L}(\mathscr{C}_\Omega) \tag{14}$$

be equipped with the norm

$$\|(w_1, w_2)\|_{E^s_0(\mathscr{C}_\Omega)} = \|w_1\|_{H^1_{0,L}(\mathscr{C}_\Omega)} + \|w_2\|_{H^1_{0,L}(\mathscr{C}_\Omega)}. \tag{15}$$

An energy solution to this problem is a function $(w_1, w_2) \in E^s_0(\mathscr{C}_\Omega)$ satisfying

$$\int_{\mathscr{C}_\Omega} y^{1-2s}\nabla w_1 \nabla \varphi_1 dx \, dy + \int_{\mathscr{C}_\Omega} y^{1-2s}\nabla w_2 \nabla \varphi_2 dx \, dy$$

$$= \lambda \int_\Omega w_1 \varphi_1 dx + \frac{2\alpha}{\alpha+\beta} \int_\Omega |w_1|^{\alpha-2} w_1 |w_2|^\beta \varphi_1 dx$$

$$+ \mu \int_\Omega w_2 \varphi_2 dx \tag{16}$$

$$+ \frac{2\beta}{\alpha+\beta} \int_\Omega |w_1|^\alpha |w_2|^{\beta-2} w_2 \varphi_2 dx$$

for all $(\varphi_1, \varphi_2) \in E^s_0(\mathscr{C}_\Omega)$. If $(w_1, w_2) \in E^s_0(\mathscr{C}_\Omega)$ satisfies (12), then $(u, v) = (w_1(\cdot, 0), w_2(\cdot, 0))$, defined in the sense of traces, belongs to the space $H^s_0(\Omega) \times H^s_0(\Omega)$ and it is a solution of the original problem (1). The associated energy functional to problem (12) is denoted by

$$I_{\lambda,\mu}(w) := I_{\lambda,\mu}(w_1, w_2)$$

$$= \frac{1}{2} \int_{\mathscr{C}_\Omega} y^{1-2s}\left(|\nabla w_1|^2 + |\nabla w_2|^2\right) dx \, dy$$

$$- \frac{1}{2} \int_\Omega \left(\lambda |w_1|^2 + \mu |w_2|^2\right) dx \tag{17}$$

$$- \frac{2}{\alpha+\beta} \int_\Omega |w_1|^\alpha |w_2|^\beta dx.$$

Critical points of $I_{\lambda,\mu}$ in $E^s_0(\mathscr{C}_\Omega)$ correspond to the critical points of $J_{\lambda,\mu} : H^s_0(\Omega) \times H^s_0(\Omega) \to \mathbb{R}$. Since the problem is critical, the functional $I_{\lambda,\mu}$ does not satisfy the Palais-Smale condition. Thus the min–max theorems can not be applied directly to obtain infinitely many solutions for (1). Following

the original idea in [7], employed in a closer setting in [10], we deal first with the following perturbed problem:

$$-\operatorname{div}\left(y^{1-2s}\nabla w_1\right) = 0,$$

$$-\operatorname{div}\left(y^{1-2s}\nabla w_2\right) = 0$$

$$\text{in } \mathscr{C}_\Omega$$

$$w_1 = w_2 = 0, \quad \text{on } \partial_L \Omega$$

$$y^{1-2s}\frac{\partial \omega_1}{\partial \nu}$$

$$= \frac{2(\alpha - \varepsilon)}{\alpha + \beta - 2\varepsilon}\left|w_1(x,0)\right|^{\alpha-2-\varepsilon} w_1(x,0)\left|w_2(x,0)\right|^{\beta-\varepsilon} \quad (18)$$

$$+ \lambda w_1(x,0) \quad \text{on } \Omega \times \{0\}$$

$$y^{1-2s}\frac{\partial \omega_2}{\partial \nu}$$

$$= \frac{2(\beta - \varepsilon)}{\alpha + \beta - 2\varepsilon}\left|w_1(x,0)\right|^{\alpha-\varepsilon}\left|w_2(x,0)\right|^{\beta-2-\varepsilon} w_2(x,0)$$

$$+ \mu w_2(x,0) \quad \text{on } \Omega \times \{0\},$$

where $(w_1, w_2) \in E_0^s(\mathscr{C}_\Omega)$ and $0 < \varepsilon < \min\{\alpha-1, \beta-1, 2_s^*-1\}$.

The functional corresponding to (18) becomes

$$I_{\lambda,\mu}^\varepsilon(w,\varphi) = \frac{1}{2}\int_{\mathscr{C}_\Omega} y^{1-2s}\left(\left|\nabla w\right|^2 + \left|\nabla \varphi\right|^2\right)dx\,dy$$

$$- \frac{1}{2}\int_\Omega \left(\lambda \left|w\right|^2 + \mu \left|\varphi\right|^2\right)dx \quad (19)$$

$$- \frac{2}{\alpha + \beta - 2\varepsilon}\int_\Omega \left|w\right|^{\alpha-\varepsilon}\left|\varphi\right|^{\beta-\varepsilon}dx,$$

where $(w, \varphi) \in E_0^s(\mathscr{C}_\Omega)$. Now $I_{\lambda,\mu}^\varepsilon$ is even and satisfies the Palais-Smale condition at all energy levels. It follows from the symmetric mountain pass lemma [24] that (18) has infinitely many solutions. More precisely, it follows from [25, Theorem 6.1] that there are positive numbers $c_{\varepsilon,l}$, $l = 1, 2, \ldots$, with $c_{\varepsilon,l} \to +\infty$, as $l \to +\infty$ and a solution $(w_{\varepsilon,l}, \varphi_{\varepsilon,l})$ for (18), satisfying $I^\varepsilon(w_{\varepsilon,l}, \varphi_{\varepsilon,l}) = c_{\varepsilon,l}$. To obtain the existence of infinitely many solutions for (1), the first step is to investigate whether $(w_{\varepsilon,l}, \varphi_{\varepsilon,l})$ converges strongly in $E_0^s(\mathscr{C}_\Omega)$ as $\varepsilon \to 0$. That is, we need to study the compactness of the set of solutions for (18), with $\varepsilon > 0$ small enough.

3. $E_0^s(\mathscr{C}_\Omega)$-Strong Convergence of Solutions for the Subcritical Case of Problem (18)

In this section, we establish the strong convergence of solutions for the perturbed problem (18). Throughout this paper, we denote the norms of $E_0^s(\mathscr{C}_\Omega)$, $L^l(\Omega)$ ($1 \le l < \infty$) by $\|(w_1, w_2)\|_{E_0^s(\mathscr{C}_\Omega)} = \|w_1\|_{H^1_{0,L}(\mathscr{C}_\Omega)} + \|w_2\|_{H^1_{0,L}(\mathscr{C}_\Omega)}$, $\|u\|_{L^l(\Omega)} = \left(\int_\Omega |u|^l dx\right)^{1/l}$, respectively, and positive constants (possibly different) by C.

Proposition 2. *Suppose that $N > 6s$ with $0 < s < 1$ and $\lambda, \mu \ge 0$ satisfy $\lambda + \mu > 0$. Then for any sequence (u_n, v_n) ($n = 1, 2, \ldots$) which is a solution of (18) with $\varepsilon = \varepsilon_n \to 0$, and for (u_n, v_n) satisfies $\|(u_n, v_n)\|_{E_0^s(\mathscr{C}_\Omega)} \le C$ for some constant $C > 0$ independent of n, then $(u_n, v_n)_{n \ge 1}$ has a subsequence which converges strongly in $E_0^s(\mathscr{C}_\Omega)$ as $n \to +\infty$.*

Before giving the proof of Proposition 2, we give some estimates for (u_n, v_n). Let D be a bounded domain such that $\Omega \Subset D$ and $(u_n(x,0), v_n(x,0)) = (0,0)$ in $D \setminus \Omega$. We choose $A > 0$ large enough so that, for all $x, y \in \mathbb{R}$, the following is verified:

$$\left|\frac{2\alpha_n}{\alpha_n + \beta_n}\left|x\right|^{\alpha_n-2}x\left|y\right|^{\beta_n} + \lambda x\right| \le 2\left(\left|x\right| + \left|y\right|\right)^{2_s^*-1} + A,$$

$$\left|\frac{2\beta_n}{\alpha_n + \beta_n}\left|x\right|^{\alpha_n}\left|y\right|^{\beta_n-2}y + \mu y\right| \le 2\left(\left|x\right| + \left|y\right|\right)^{2_s^*-1} + A,$$

(20)

for all $x, y \in \mathbb{R}$. Let (ω_n) ($n = 1, 2, \ldots$) with $\omega_n \ge 0$ be a solution of

$$-\operatorname{div}\left(y^{1-2s}\nabla\omega\right) = 0, \quad \text{in } \mathscr{C}_D,$$

$$\omega = 0, \quad \text{on } \partial_L D,$$

$$y^{1-2s}\frac{\partial \omega}{\partial \nu} = 2\left(\left|u_n(x,0)\right| + \left|v_n(x,0)\right|\right)^{2_s^*-1} + A$$

(21)

$$\text{on } D \times \{0\}.$$

By choosing D and A, we find

$$-\operatorname{div}\left(y^{1-2s}\nabla\left(\omega_n \pm u_n\right)\right) = 0, \quad \text{in } \mathscr{C}_\Omega,$$

$$\omega_n \pm u_n = 0, \quad \text{on } \partial_L \Omega, \quad (22)$$

$$y^{1-2s}\frac{\partial\left(\omega_n \pm u_n\right)}{\partial \nu} \ge 0 \quad \text{on } \Omega \times \{0\}.$$

Multiplying (22) by $\left(\omega_n \pm u_n\right)^-$ and integrating by part, we see that

$$\left|u_n\right| \le \omega_n. \quad (23)$$

Similarly, for v_n, we have

$$\left|v_n\right| \le \omega_n. \quad (24)$$

Then by (23) and (24) we have

$$\frac{1}{2}\left(\left|u_n\right| + \left|v_n\right|\right) \le \omega_n. \quad (25)$$

So in the following sections we can consider just the estimates of w_n in $\mathscr{C}_\Omega$.

Let

$$\mathscr{S}_{s,\alpha,\beta}$$

$$:= \inf_{(w_1,w_2)\in E_0^s(\mathscr{C}_\Omega)} \frac{\int_{\mathscr{C}_\Omega} y^{1-2s}\left(\left|\nabla w_1\right|^2 + \left|\nabla w_2\right|^2\right)dx\,dy}{\left(\int_\Omega \left|w_1\right|^\alpha \left|w_2\right|^\beta dx\right)^{2/2_s^*}}. \quad (26)$$

Then we have (see Lemma 2.5 [23])

$$
\mathcal{S}_{s,\alpha,\beta} = \left[\left(\frac{\alpha}{\beta} \right)^{\beta/2_s^*} + \left(\frac{\beta}{\alpha} \right)^{\alpha/2_s^*} \right] \mathcal{S}(s, N), \qquad (27)
$$

where $\mathcal{S}(s, N)$ is the best Sobolev constant defined by

$$
\mathcal{S}(s, N) = \inf_{z \in H^1_{0,L}(\mathscr{C}_\Omega) \setminus \{0\}} \frac{\int_{\mathscr{C}_\Omega} y^{1-2s} |\nabla z(x, y)|^2 \, dx \, dy}{\left(\int_\Omega |z(x, 0)|^{2_s^*} \, dx \right)^{2/2_s^*}}, \quad (28)
$$

which is achieved if and only if $\Omega = \mathbb{R}^N$ by functions w_ε are the s-harmonic extensions of

$$
u_\varepsilon(x) := \frac{\varepsilon^{(N-2s)/2}}{(\varepsilon^2 + |x|^2)^{(N-2s)/2}}, \quad \varepsilon > 0, \; x \in \mathbb{R}^N. \qquad (29)
$$

Now we introduce the "problem at infinity":

$$
-\mathrm{div}\left(y^{1-2s} \nabla u \right) = 0,
$$
$$
-\mathrm{div}\left(y^{1-2s} \nabla v \right) = 0
$$
$$
\text{in } \mathbb{R}^{N+1}_+,
$$
$$
y^{1-2s} \frac{\partial u}{\partial \nu} = \frac{2\alpha}{\alpha + \beta} u^{\alpha-1}(x, 0) v^\beta(x, 0)
$$
$$
\text{on } \mathbb{R}^N \times \{0\},
$$
$$
y^{1-2s} \frac{\partial v}{\partial \nu} = \frac{2\beta}{\alpha + \beta} u^\alpha(x, 0) v^{\beta-1}(x, 0)
$$
$$
\text{on } \mathbb{R}^N \times \{0\},
$$
$$
U > 0,
$$
$$
V > 0
$$
$$
\text{in } \mathbb{R}^N \times \{0\}. \qquad (30)
$$

Here $\alpha > 1$, $\beta > 1$, and $\alpha + \beta = 2_s^*$. Set

$$
U = \left(\frac{2\alpha}{\alpha + \beta} \right)^{1/(\alpha+\beta-2)} \left(\frac{\beta}{\alpha} \right)^{\beta/2(\alpha+\beta-2)} u,
$$
$$
V = \left(\frac{2\beta}{\alpha + \beta} \right)^{1/(\alpha+\beta-2)} \left(\frac{\alpha}{\beta} \right)^{\alpha/2(\alpha+\beta-2)} v. \qquad (31)
$$

Then problem (30) can be rewritten as

$$
-\mathrm{div}\left(y^{1-2s} \nabla U \right) = 0,
$$
$$
-\mathrm{div}\left(y^{1-2s} \nabla V \right) = 0
$$
$$
\text{in } \mathbb{R}^{N+1}_+,
$$
$$
y^{1-2s} \frac{\partial U}{\partial \nu} = U^{\alpha-1}(x, 0) V^\beta(x, 0)
$$
$$
\text{on } \mathbb{R}^N \times \{0\},
$$
$$
y^{1-2s} \frac{\partial V}{\partial \nu} = U^\alpha(x, 0) V^{\beta-1}(x, 0)
$$
$$
\text{on } \mathbb{R}^N \times \{0\},
$$
$$
U > 0,
$$
$$
V > 0
$$
$$
\text{in } \mathbb{R}^N \times \{0\}. \qquad (32)
$$

Lemma 3. *Let (U, V) be a solution of (32). Then $U(x, 0) = V(x, 0)$ for all $x \in \mathbb{R}^N$.*

Proof. Let (U, V) be a solution of (32). Then

$$
-\mathrm{div}\left(y^{1-2s} \nabla(U - V) \right) = 0, \quad \text{in } \mathbb{R}^{N+1}_+,
$$
$$
y^{1-2s} \frac{\partial(U - V)}{\partial \nu} = U^{\alpha-1}(x, 0) V^{\beta-1}(x, 0)(V - U)
$$
$$
\text{on } \mathbb{R}^N \times \{0\}, \quad (33)
$$
$$
U > 0,
$$
$$
V > 0
$$
$$
\text{in } \mathbb{R}^N \times \{0\}.
$$

Multiplying (33) by $(U - V)$ and integrating by part, we obtain

$$
\int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla(U - V)|^2 \, dx \, dy
$$
$$
= -\int_{\mathbb{R}^N} U^{\alpha-1}(x, 0) V^{\beta-1}(x, 0)(U - V)^2 \, dx \le 0, \qquad (34)
$$

which implies that

$$
U(x, y) - V(x, y) = \text{const}. \qquad (35)
$$

Note that $U > 0$, $V > 0$, in $\mathbb{R}^N$. It follows from (30) that $U^{\alpha-1}(x,0)V^\beta(x,0) = y^{1-2s}(\partial U/\partial \nu) = U^\alpha(x,0)V^{\beta-1}(x,0)$ in $\mathbb{R}^N \times \{0\}$. Hence $U(x, 0) = V(x, 0)$. $\qquad \square$

From Proposition 6 in the Appendix, we have the global compactness result on (1). Let (u_n, v_n) $(n = 1, 2, \ldots)$ be a solution of problem (18) with $\varepsilon = \varepsilon_n \to 0$ as $n \to +\infty$, satisfying $\|(u_n, v_n)\|_{E^s_0(\mathscr{C}_\Omega)} \le C$ for some constant independent of n. Then replacing the solution if necessary with a subsequence, there exist k sequences mutually diverging scaling σ_n^i with respective concentration points x_n^i such that, as $n \to +\infty$,

$$
u_n = u_0 + a_0^{-1} \sum_{i=1}^k \left(\sigma_n^i \right)^{(N-2s)/2} U\left(\sigma_n^i \left(x - x_n^i \right), 0 \right)
$$
$$
+ \omega_n^1, \qquad (36)
$$
$$
v_n = v_0 + b_0^{-1} \sum_{i=1}^k \left(\sigma_n^i \right)^{(N-2s)/2} U\left(\sigma_n^i \left(x - x_n^i \right), 0 \right) + \omega_n^2,
$$

where $(u_0, v_0) \in E_0^s(\mathscr{C}_\Omega)$ is a weak solution of problem (12), $\|(\omega_n^1, \omega_n^2)\|_{E_0^s(\mathscr{C}_\Omega)} \to 0$ as $n \to +\infty$, U achieves the constant $\mathscr{S}(s, N)$, which is given in (28), and

$$
a_0 = \left(\left(\frac{2\beta}{\alpha + \beta} \right)^2 \left(\frac{\beta}{\alpha} \right)^{\beta - 2} \right)^{1/2(\alpha + \beta - 2)},
$$

$$
b_0 = \left(\left(\frac{2\alpha}{\alpha + \beta} \right)^2 \left(\frac{\alpha}{\beta} \right)^{\alpha - 2} \right)^{1/2(\alpha + \beta - 2)}.
\tag{37}
$$

As in [7, 9, 10] we shall introduce the following facts which are essentials to prove the strong convergence of $\{(u_n, v_n)\}$ in $E_0^s(\mathscr{C}_\Omega)$. Among all the bubbles in (28), we can choose the slowest concentration rate, denoted by σ_n, which concentrates in x_n the slowest rate. That is, the corresponding σ is the lowest order infinity among the ones appearing in the bubbles. Note that the number of the bubbles in (u_n, v_n) is finite; we can always choose a constant $\overline{C} > 0$, independent of n, such that the region,

$$
\mathscr{A}_n^1 := \left\{ Z = (x, y) : Z \right.
$$
$$
\left. \in \left(B_{(\overline{C}+5)\sigma_n^{-1/2}}(x_n, 0) \setminus B_{\overline{C}\sigma_n^{-1/2}}(x_n, 0) \right) \cap \mathscr{C}_\Omega \right\},
\tag{38}
$$

does not contain any concentration point of (u_n, v_n) for every n. We set two thinner subsets as follows:

$$
\mathscr{A}_n^2 := \left\{ Z = (x, y) : Z \right.
$$
$$
\left. \in \left(B_{(\overline{C}+4)\sigma_n^{-1/2}}(x_n, 0) \setminus B_{(\overline{C}+1)\sigma_n^{-1/2}}(x_n, 0) \right) \cap \mathscr{C}_\Omega \right\},
$$
$$
\mathscr{A}_n^3 := \left\{ Z = (x, y) : Z \right.
$$
$$
\left. \in \left(B_{(\overline{C}+3)\sigma_n^{-1/2}}(x_n, 0) \setminus B_{(\overline{C}+2)\sigma_n^{-1/2}}(x_n, 0) \right) \cap \mathscr{C}_\Omega \right\}.
\tag{39}
$$

Then the following integral estimates hold (see Propositions 4.1 and 4.2 in [10]).

Lemma 4. *Let $(u_n, v_n)_{n \geq 1}$ be solution of (18) with $\varepsilon = \varepsilon_n \to 0$. Then, there exists a constant $C > 0$ independent of n, such that, for all $p \geq 1$, one has*

$$
\int_{\mathscr{A}_n^2} y^{1-2s} \left(|u_n|^p + |v_n|^p \right) dx \, dy \leq C \sigma_n^{-(N + 2_s^*)/2_s^*},
$$

$$
\int_{\mathscr{A}_n^2 \cap \{y=0\}} \left(|u_n|^p + |v_n|^p \right) dx \leq C \sigma_n^{-N/2}.
\tag{40}
$$

Moreover, one has

$$
\int_{\mathscr{A}_n^3} y^{1-2s} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) dx \, dy \leq C \sigma_n^{-N/2_s^*}.
\tag{41}
$$

Proof. For the proof of this lemma, we use inequality (25) and we refer to [10, Propositions 4.1, 4.2]. $\square$

Note that $X = (x, y) \in \mathbb{R}_+^{N+1}$, and we have the following local Pohozaev identity.

Lemma 5. *Let $\alpha_n = \alpha - \varepsilon_n$, $\beta_n = \beta - \varepsilon_n$, and $\varepsilon = \varepsilon_n \to 0$ as $n \to +\infty$ and B_n be any bounded set in $\mathscr{C}_\Omega$. Then one has the local Pohozaev identity on $B_n \subset \mathscr{C}_\Omega$ associated for equations of (18):*

$$
\frac{\varepsilon_n (N - 2s)^2}{N - \varepsilon_n (N - 2s)} \int_{B_n \cap \{y=0\}} |u_n|^{\alpha_n} |v_n|^{\beta_n} dx
$$
$$
+ s \int_{B_n \cap \{y=0\}} (\lambda u_n^2 + \mu v_n^2) dx
$$
$$
= \int_{\partial(B_n \cap \{y=0\})} \frac{2}{\alpha_n + \beta_n} |u_n|^{\alpha_n} |v_n|^{\beta_n} ((x - x_0)
$$
$$
\cdot \nu) \, d\sigma + \frac{1}{2} \int_{\partial(B_n \cap \{y=0\})} (\lambda u_n^2 + \mu v_n^2) ((x - x_0)
$$
$$
\cdot \nu) \, d\sigma + \frac{N - 2s}{2} \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left(u_n \frac{\partial u_n}{\partial \nu} \right.
$$
$$
\left. + v_n \frac{\partial v_n}{\partial \nu} \right) d\sigma - \frac{1}{2}
$$
$$
\cdot \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) ((X - z_0)
$$
$$
\cdot \nu) \, d\sigma
$$
$$
+ \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left((\nabla u_n \cdot (X - z_0)) \frac{\partial u_n}{\partial \nu} \right.
$$
$$
\left. + (\nabla v_n \cdot (X - z_0)) \frac{\partial v_n}{\partial \nu} \right) d\sigma,
\tag{42}
$$

where ν is the outward unit normal to ∂B_n and $z_0 = (x_0, 0)$, with x_0 being a point in $\mathbb{R}^N$.

Proof. Using the divergence theorem, we get

$$
\frac{N - 2s}{2} \int_{\partial B_n} y^{1-2s} \left(u_n \frac{\partial u_n}{\partial \nu} + v_n \frac{\partial v_n}{\partial \nu} \right) d\sigma = \frac{1}{2} \int_{\partial B_n} y^{1-2s} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) ((X - z_0) \cdot \nu) \, d\sigma
$$
$$
- \int_{\partial B_n} y^{1-2s} \left((\nabla u_n \cdot (X - z_0)) \frac{\partial u_n}{\partial \nu} + (\nabla v_n \cdot (X - z_0)) \frac{\partial v_n}{\partial \nu} \right) d\sigma,
\tag{43}
$$

where ν is the outward unit normal to ∂B_n and $z_0 = (x_0, 0)$, with x_0 being a point in $\mathbb{R}^N$.

Since (u_n, v_n) is solution of (18), we have the following equations:

$$y^{1-2s} u_n \frac{\partial u_n}{\partial \nu} = \frac{2\alpha_n}{\alpha_n + \beta_n} |u_n|^{\alpha_n} |v_n|^{\beta_n} + \lambda u_n^2$$

$$\text{on } y = 0,$$

$$y^{1-2s} v_n \frac{\partial v_n}{\partial \nu} = \frac{2\beta_n}{\alpha_n + \beta_n} |u_n|^{\alpha_n} |v_n|^{\beta_n} + \mu v_n^2$$

$$\text{on } y = 0. \tag{44}$$

We obtain, from (43) and (44),

$$\frac{N - 2s}{2} \int_{B_n \cap \{y=0\}} \left(2 |u_n|^{\alpha_n} |v_n|^{\beta_n} + \lambda u_n^2 + \mu v_n^2 \right) dx$$

$$+ \frac{N - 2s}{2} \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left(u_n \frac{\partial u_n}{\partial \nu} + v_n \frac{\partial v_n}{\partial \nu} \right) d\sigma = \frac{1}{2} \int_{B_n \cap \{y=0\}} y^{1-2s} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) ((x - x_0) \cdot \nu) \, dx$$

$$+ \frac{1}{2} \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) (X \cdot \nu) \, d\sigma - \int_{B_n \cap \{y=0\}} y^{1-2s} \left((\nabla u_n \cdot (x - x_0)) \frac{\partial u_n}{\partial \nu} + (\nabla v_n \cdot (x - x_0)) \frac{\partial v_n}{\partial \nu} \right) dx$$

$$- \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left((\nabla u_n \cdot (X - z_0)) \frac{\partial u_n}{\partial \nu} + (\nabla v_n \cdot (X - z_0)) \frac{\partial v_n}{\partial \nu} \right) d\sigma. \tag{45}$$

Noting that the dot product $((x - x_0) \cdot \nu) = 0$, on $B_n \cap \{y = 0\}$, we obtain

$$\frac{1}{2} \int_{B_n \cap \{y=0\}} y^{1-2s} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) ((x - x_0) \cdot \nu) \, dx$$

$$= 0. \tag{46}$$

Moreover, we have

$$\int_{B_n \cap \{y=0\}} y^{1-2s} \left((\nabla u_n \cdot (x - x_0)) \frac{\partial u_n}{\partial \nu} + (\nabla v_n \cdot (x - x_0)) \frac{\partial v_n}{\partial \nu} \right) dx = \int_{B_n \cap \{y=0\}} (\nabla u_n \cdot (x - x_0))$$

$$\cdot \left(\frac{2\alpha_n}{\alpha_n + \beta_n} |u_n|^{\alpha_n} |v_n|^{\beta_n} + \lambda u_n^2 \right) dx + \int_{B_n \cap \{y=0\}} (\nabla v_n \cdot (x - x_0)) \left(\frac{2\beta_n}{\alpha_n + \beta_n} |u_n|^{\alpha_n} |v_n|^{\beta_n} + \mu v_n^2 \right) dx$$

$$= \int_{B_n \cap \{y=0\}} \nabla \left(\frac{2}{\alpha_n + \beta_n} |u_n|^{\alpha_n} |v_n|^{\beta_n} + \frac{1}{2} \left(\lambda u_n^2 + \mu v_n^2 \right) \right) \cdot (x - x_0) \, dx \tag{47}$$

$$= -N \int_{B_n \cap \{y=0\}} \left(\frac{2}{\alpha_n + \beta_n} |u_n|^{\alpha_n} |v_n|^{\beta_n} + \frac{1}{2} \left(\lambda u_n^2 + \mu v_n^2 \right) \right) dx$$

$$+ \int_{\partial(B_n \cap \{y=0\})} \left(\frac{2}{\alpha_n + \beta_n} |u_n|^{\alpha_n} |v_n|^{\beta_n} + \frac{1}{2} \left(\lambda u_n^2 + \mu v_n^2 \right) \right) ((x - x_0) \cdot \nu) \, d\sigma.$$

Therefore, from (45), (46), and (47), we infer that (44) holds. □

Proof of Proposition 2. Let $(u_n, v_n)_{n \geq 1}$ be a bounded sequence in $E_0^s(\mathscr{C}_\Omega)$ composed of solutions for (18). Thus, in order to prove the $E_0^s(\mathscr{C}_\Omega)$-strong convergence in (36), we just need to show that the bubbles $(\sigma_n^i)^{(N-2s)/2} U(\sigma_n^i (x - x_n^i), 0)$ $(1 \leq i \leq k)$ in (36) will not appear in the decomposition of u_n and v_n. Since the proof is similar to that of Lemma 6.1 in [9], here

we only give a sketch of it. From Lemma 5, for the solution concentrating sequence (u_n, v_n) of (18) with $\varepsilon = \varepsilon_n \to 0$, we have the local Pohozaev identity on

$$B_n := B_{t_n \sigma_n^{-1/2}}((x_n, 0)) \cap \mathscr{C}_\Omega \subset \mathbb{R}^{N+1} \qquad (48)$$

$$\frac{4\varepsilon_n N}{2_s^* (2_s^* - 2\varepsilon_n)} \int_{B_n \cap \{y=0\}} |u_n|^{\alpha_n} |v_n|^{\beta_n} dx$$

$$+ s \int_{B_n \cap \{y=0\}} (\lambda u_n^2 + \mu v_n^2) dx = \frac{2}{\alpha_n + \beta_n}$$

$$\cdot \int_{\partial(B_n \cap \{y=0\})} |u_n|^{\alpha_n} |v_n|^{\beta_n} ((X - z_0) \cdot \nu) d\sigma + \frac{1}{2}$$

$$\cdot \int_{\partial(B_n \cap \{y=0\})} (\lambda u_n^2 + \mu v_n^2) ((X - z_0) \cdot \nu) d\sigma$$

$$+ \frac{N - 2s}{2} \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left(u_n \frac{\partial u_n}{\partial \nu} + v_n \frac{\partial v_n}{\partial \nu} \right) d\sigma \qquad (49)$$

$$- \frac{1}{2} \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) ((X - z_0)$$

$$\cdot \nu) d\sigma$$

$$+ \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left((\nabla u_n \cdot X - z_0) \frac{\partial u_n}{\partial \nu} \right.$$

$$+ \left. (\nabla v_n \cdot X - z_0) \frac{\partial v_n}{\partial \nu} \right) d\sigma,$$

where ν is the outward unit normal to ∂B_n and $z_0 = (x_0, 0)$, $x_0 \in \mathbb{R}^N$. We decompose

$$\partial B_n \cap \{y > 0\} = \partial_i B_n \cup \partial_e B_n, \qquad (50)$$

where $\partial_i B_n := \partial B_n \cap \mathscr{C}_\Omega$ and $\partial_e B_n := B_n \cap \partial_L \mathscr{C}_\Omega$.

We consider two different cases:

(i) $B_{t_n \sigma_n^{-1/2}}((x_n, 0)) \cap \{0\} \cap (\mathbb{R}^{N+1} \setminus \mathscr{C}_\Omega) \neq \emptyset$.

(ii) $B_{t_n \sigma_n^{-1/2}}((x_n, 0)) \cap \{0\} \subset \mathscr{C}_\Omega$.

In *case (i)*, we take $x_0 \in \mathbb{R}^N \setminus \Omega$ with $|x_0 - x_n| \leq 2t_n \sigma_n^{-1/2}$ and $\nu \cdot (X - (x_0, 0)) \leq 0$ in $\partial_e B_n$, where ν is the outward unit normal to $\partial_L \mathscr{C}_\Omega$.

Since $(u_n, v_n) = (0, 0)$ on $\partial_L \mathscr{C}_\Omega$, we find

$$- \frac{1}{2} \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) ((X - z_0) \cdot \nu) d\sigma$$

$$+ \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left((\nabla u_n \cdot X - z_0) \frac{\partial u_n}{\partial \nu} \right.$$

$$+ \left. (\nabla v_n \cdot X - z_0) \frac{\partial v_n}{\partial \nu} \right) d\sigma = \frac{1}{2} \qquad (51)$$

$$\cdot \int_{\partial B_n \cap \{y>0\}} y^{1-2s} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) ((X - z_0)$$

$$\cdot \nu) d\sigma \leq 0.$$

In *case (ii)*, $\partial_e B_n = \emptyset$; we take a point $x_0 = x_n$.

Since $(u_n, v_n) = (0, 0)$ on $\partial_L \mathscr{C}_\Omega$, and, for n large enough $4\varepsilon_n N / 2_s^* (2_s^* - 2\varepsilon_n) > 0$, we obtain

$$s \int_{B_n \cap \{y=0\}} (\lambda u_n^2 + \mu v_n^2) dx \leq \int_{\partial_i(B_n \cap \{0\})} \frac{2}{\alpha_n + \beta_n} |u_n|^{\alpha_n}$$

$$\cdot |v_n|^{\beta_n} ((X - z_0) \cdot \nu) d\sigma + \frac{1}{2}$$

$$\cdot \int_{(\partial_i B_n) \cap \{y=0\}} (\lambda u_n^2 + \mu v_n^2) ((X - z_0) \cdot \nu) d\sigma$$

$$+ \frac{N - 2s}{2} \int_{\partial_i B_n} y^{1-2s} \left(u_n \frac{\partial u_n}{\partial \nu} + v_n \frac{\partial v_n}{\partial \nu} \right) d\sigma - \frac{1}{2} \qquad (52)$$

$$\cdot \int_{\partial_i B_n} y^{1-2s} \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) ((X - z_0) \cdot \nu) d\sigma$$

$$+ \int_{\partial_i B_n} y^{1-2s} \left((\nabla u_n \cdot X - z_0) \frac{\partial u_n}{\partial \nu} \right.$$

$$+ \left. (\nabla v_n \cdot X - z_0) \frac{\partial v_n}{\partial \nu} \right) d\sigma.$$

By assumption, $\lambda, \mu \geq 0$ and $\lambda + \mu > 0$; we may assume that $\lambda > 0$. Set $B_n' = B_{\sigma_n^{-1}}(x_n, 0) \cap \mathscr{C}_\Omega$. Recalling that, by (36), we have the decomposition $u_n = u_n^0 + u_n^1 + u_n^2$, where $u_n^1 = u_0$, $u_n^2 = a_0^{-1} \sum_{j=1}^k (\sigma_n^i)^{(N-2s)/2} U(\sigma_n^i(x - x_n^i), 0)$, and $u_n^0 = u_n - u_n^1 - u_n^2$. $\|u_n^0\|_{E_0^s(\mathscr{C}_\Omega)} \to 0$ as $n \to \infty$. Then we deduce that, for n large enough, $B_n' \subset B_n$

$$\int_{B_n \cap \{y=0\}} u_n^2 dx \geq \int_{B_n' \cap \{y=0\}} u_n^2 dx$$

$$\geq \frac{1}{2} \int_{B_n' \cap \{y=0\}} |u_n^2|^2 dx \qquad (53)$$

$$- 2 \int_{B_n' \cap \{y=0\}} \left(|u_n^1|^2 + |u_n^0|^2 \right) dx.$$

After a direct calculation, we have

$$\int_{B_n' \cap \{y=0\}} |u_n^2|^2 dx \geq C\sigma_n^{-2s},$$

$$\int_{B_n' \cap \{y=0\}} |u_n^1|^2 dx \leq C\sigma_n^{-N}, \qquad (54)$$

$$\int_{B_n' \cap \{y=0\}} |u_n^0|^2 dx \leq C \|u_n^0\|_{L^{2_s^*}(\Omega)}^2 \sigma_n^{-2s}. \qquad (55)$$

Note that $\|u_n^0\|_{E_0^s(\mathscr{C}_\Omega)} \to 0$, as $n \to \infty$. Inserting (54) into (53), we get for n large enough

$$\int_{B_n \cap \{y=0\}} \lambda u_n^2 dx \geq C\sigma_n^{-2s}. \qquad (56)$$

By the choice of $z_0 = (x_0, 0)$, as in [10], we only need to consider the right-hand side of (52) on $\partial_i B_n$. By applying Hölder inequality and using Lemma 4, we get

$$
\begin{aligned}
&\frac{2}{\alpha_n + \beta_n} \int_{\partial_i(B_n \cap \{y=0\})} |u_n|^{\alpha_n} |v_n|^{\beta_n} \left((X - z_0) \cdot \nu\right) d\sigma + \frac{1}{2} \\
&\cdot \int_{(\partial_i B_n) \cap \{y=0\}} \left(\lambda u_n^2 + \mu v_n^2\right) \left((X - z_0) \cdot \nu\right) d\sigma \\
&+ \frac{N - 2s}{2} \int_{\partial_i B_n} y^{1-2s} \left(u_n \frac{\partial u_n}{\partial \nu} + v_n \frac{\partial v_n}{\partial \nu}\right) d\sigma - \frac{1}{2} \\
&\cdot \int_{\partial_i B_n} y^{1-2s} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \left((X - z_0) \cdot \nu\right) d\sigma \\
&+ \int_{\partial_i B_n} y^{1-2s} \left(\left(\nabla u_n \cdot X - z_0\right) \frac{\partial u_n}{\partial \nu}\right. \\
&\left. + \left(\nabla v_n \cdot X - z_0\right) \frac{\partial v_n}{\partial \nu}\right) d\sigma \\
&\leq C\sigma_n^{-1/2} \left(\int_{\partial_i(B_n \cap \{y=0\})} |u_n|^{\alpha_n + \beta_n} d\sigma\right)^{\alpha_n/(\alpha_n + \beta_n)} \\
&\cdot \left(\int_{\partial_i(B_n \cap \{y=0\})} |v_n|^{\alpha_n + \beta_n} d\sigma\right)^{\beta_n/(\alpha_n + \beta_n)} \\
&+ \int_{\partial_i(B_n \cap \{y=0\})} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) |x - x_0| d\sigma \\
&+ \left(\int_{\partial_i(B_n \cap \{y=0\})} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) d\sigma\right)^{1/2} \\
&\cdot \left(\int_{\partial_i(B_n \cap \{y=0\})} \left(|u_n|^2 + |v_n|^2\right) d\sigma\right)^{1/2} \\
&\leq C\sigma_n^{-(N-2s)/2}.
\end{aligned}
\tag{57}
$$

Inserting (56) and (57) into (52), we obtain

$$
\sigma_n^{-2s} \leq C\sigma_n^{-(N-2s)/2},
\tag{58}
$$

which is a contradiction for n large enough due to $N > 6s$. □

4. Proof of Theorem 1

For any positive integer k, define the Z_2-homotopy class $\mathcal{F}_k$ as follows:

$$
\begin{aligned}
\mathcal{F}_k := \{A : A \\
\subset E_0^s(\mathscr{C}_\Omega) \text{ is compact, } Z_2\text{-invariant, } \gamma(A) \geq k\},
\end{aligned}
\tag{59}
$$

where the genus $\gamma(A)$ is the smallest integer m, such that there exists an odd map $\phi \in \mathscr{C}(A, \mathbb{R}^m \setminus \{0\})$. For $k = 1, 2, \ldots$, we define the min–max value as (e.g., see p. 134 in [26])

$$
c_{k,\varepsilon} = \min_{A \in \mathcal{F}_k} \max_{(u,v) \in A} I_{\lambda,\mu}^\varepsilon(u, v).
\tag{60}
$$

It follows from Corollary 7.12 in [26] that, for each small $\varepsilon > 0, c_{k,\varepsilon}$ is critical value of $I_{\lambda,\mu}^\varepsilon$, since $I_{\lambda,\mu}^\varepsilon$ satisfies the Palais-Smale condition. Thus problem (18) has a solution $(u_{k,\varepsilon}, v_{k,\varepsilon})$ such that $I_{\lambda,\mu}^\varepsilon(u_{k,\varepsilon}, v_{k,\varepsilon}) = c_{k,\varepsilon}$. Note that $I_{\lambda,\mu}^\varepsilon(tu_{k,\varepsilon}, tv_{k,\varepsilon}) \to$

$-\infty$ uniformly with respect to $\varepsilon > 0$ as $t \to +\infty$; hence, $c_{k,\varepsilon}$ is uniformly bounded with respect to ε for each fixed k. By a direct calculation, we find $\|(u_{k,\varepsilon}, v_{k,\varepsilon})\|_{E_0^s(\mathscr{C}_\Omega)} \leq C$ uniformly with respect to ε for each fixed k. So now, we can apply Proposition 2 and obtain a subsequence of $(u_{k,\varepsilon_n}, v_{k,\varepsilon_n})_{n \geq 1}$, such that, as $n \to +\infty, (u_{k,\varepsilon_n}, v_{k,\varepsilon_n}) \to (u_k, v_k)$ strongly in $E_0^s(\mathscr{C}_\Omega)$ for some (u_k, v_k) and $c_{k,\varepsilon_n} \to c_k$. Then (u_k, v_k) is solution of (1) and $I_{\lambda,\mu}(u_k, v_k) = c_k$. We are now ready to show that $I_{\lambda,\mu}$ has infinitely many critical point solutions. Note that c_k is nondecreasing in k. By an argument similar to the one used in the proof of Theorem 1.1 in [9] we distinguish two cases.

Case 1. Suppose that there are $1 < k_1 < \cdots k_i < \cdots$, satisfying

$$
c_{k_1} < \cdots < c_{k_i} < \cdots.
\tag{61}
$$

In this case, we have infinitely many distinct critical points and, therefore, infinitely many solutions.

Case 2. We assume, in this case, that for some positive integer $m, c_k = c$ for all $k \geq m$. Suppose that, for any $\delta > 0, I_{\lambda,\mu}$ has a critical point (u, v) with $I_{\lambda,\mu}(u, v) \in (c - \delta, c + \delta)$ and $I_{\lambda,\mu}(u, v) \neq c$. In this case, we are done. So from now on we assume that there exists $\delta > 0$ such that $I_{\lambda,\mu}$ has no critical point (u, v) with $I_{\lambda,\mu}(u, v) \in (c - \delta, c) \cup (c, c + \delta)$. In this case, using the deformation argument, we can prove that

$$
\gamma(K_c) \geq 2,
\tag{62}
$$

where $K_c = \{(u, v) \in X : I_{\lambda,\mu}(u, v) = c, I_{\lambda,\mu}'(u, v) = 0\}$. As a consequence, $I_{\lambda,\mu}$ has infinitely many critical points. Thus we can obtain infinitely many solutions for problem (1).

Appendix

In this section, we give a global compactness result in the following proposition.

Proposition 6. *Suppose that $(u_n, v_n)_{n \geq 1}$ is a solution of (18) with $\varepsilon = \varepsilon_n \to 0$, satisfying $\|(u_n, v_n)\|_{E_0^s(\mathscr{C}_\Omega)} \leq C$ for that C is the constant independent of n. Then*

(i) *(u_n, v_n) can be decomposed as*

$$
\begin{aligned}
u_n &= u_0 + a_0^{-1} \sum_{i=1}^k \left(\sigma_n^i\right)^{(N-2s)/2} U\left(\sigma_n^i \left(x - x_n^i\right), 0\right) \\
&+ \omega_n^1, \\
v_n &= v_0 + b_0^{-1} \sum_{i=1}^k \left(\sigma_n^i\right)^{(N-2s)/2} U\left(\sigma_n^i \left(x - x_n^i\right), 0\right) \\
&+ \omega_n^2,
\end{aligned}
\tag{A.1}
$$

where $(u_0, v_0) \in E_0^s(\mathscr{C}_\Omega)$ is a weak solution of problem (12),
$\|(\omega_n^1, \omega_n^2)\|_{E_0^s(\mathscr{C}_\Omega)} \to 0$, as $n \to +\infty, U$ achieves the constant $\mathcal{S}(s, N)$, which is given in (28); and

$$
\begin{aligned}
a_0 &= \left(\left(\frac{2\beta}{\alpha + \beta}\right)^2 \left(\frac{\beta}{\alpha}\right)^{\beta - 2}\right)^{1/2(\alpha + \beta - 2)}, \\
b_0 &= \left(\left(\frac{2\alpha}{\alpha + \beta}\right)^2 \left(\frac{\alpha}{\beta}\right)^{\alpha - 2}\right)^{1/2(\alpha + \beta - 2)},
\end{aligned}
\tag{A.2}
$$

for $i = 1, \ldots, k$, $x_{n,i} \in \Omega$ with $\sigma_n^i d(x_{n,i}, \partial\Omega) \to +\infty$, $\sigma_n^i|x_{n,i}| \to +\infty$;

(ii) *for* $i, j = 1, \ldots, k$, *if* $i \neq j$, *then, as* $n \to +\infty$,

$$\frac{\sigma_n^j}{\sigma_n^i} + \frac{\sigma_n^i}{\sigma_n^j} + \sigma_n^i\sigma_n^j\left|x_{n,i} - x_{n,j}\right|^2 \longrightarrow +\infty. \qquad \text{(A.3)}$$

Proof. The proof follows without difficulty by modifying the proof of the concentration compactness result for (2) (see [10, 15]) and using Lemma 3. We omit the details for the sake of simplicity. □

Competing Interests

The authors declare that they have no competing interests.

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