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Research Article

Timelike Tangent Developable Surfaces and Bonnet Surfaces

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A criterion was given for a timelike surface to be a Bonnet surface in 3-dimensional Minkowski space by Chen and Li, 1999. In this study, we obtain a necessary and sufficient condition for a timelike tangent developable surface to be a timelike Bonnet surface by the aid of this criterion. This is examined under the condition of the curvature and torsion of the base curve of the timelike developable surface being nonconstant. Moreover, we investigate the nontrivial isometry preserving the mean curvature for a timelike flat helicoidal surface by considering the curvature and torsion of the base curve of the timelike developable surface as being constant.

1. Introduction

Surfaces which admit a one-parameter family of isometric deformations preserving the mean curvature are called Bonnet surfaces. In 1867, Bonnet proved that any surface with constant mean curvature in $\mathbb{R}^3$ (which is not totally umbilical) is a Bonnet surface [1]. Cartan obtained some detailed results for Bonnet surfaces in [2]. Lawson extended Bonnet's results to any surface with constant mean curvature in Riemannian 3-manifold of constant curvature. Also, it was proved that any Bonnet surface of nonconstant mean curvature depends on six arbitrary constants [3]. Characterization for isometric deformation preserving the principal curvatures of surfaces was obtained by the aid of differential forms by Chern in [4]. The geometric characterizations of helicoidal surfaces of constant mean curvature, helicoidal surfaces as Bonnet surfaces, and tangent developable surfaces as Bonnet surfaces were studied by Roussos in [5], [6] and [7], respectively. Roussos obtained a characterization for isometric deformation preserving the mean curvature by using the method of Chern. Soyuçok gave the necessary and sufficient condition of a surface to be Bonnet surface, which is to have a special system of isothermal parameters [8]. Moreover, Soyuçok proved that 3-dimensional hyperspace in 4-dimensional space is Bonnet surface if and only if hypersurface has orthogonal net [9]. Bağdatlı and Soyuçok studied hypersurfaces preserving the mean curvature and proved that a hypersurface in $\mathbb{R}^{n+1}$ is Bonnet surface if and only if hypersurface has orthogonal A-net [10]. On the other hand, Chen and Li studied 3-dimensional Minkowski space and classified timelike Bonnet surfaces [11].

2. Preliminaries

Let M be a timelike surface in 3-dimensional Minkowski space with nondegenerate metric tensor $g = -dx_1^2+dx_2^2+dx_3^2$, where $\{x_1, x_2, x_3\}$ is a system of the canonical coordinates in $\mathbb{R}^3$. Let $\Phi : M \to \mathbb{R}_1^3$ be a timelike immersion that admits a nontrivial isometry preserving the mean curvature. Nontriviality means that the immersion in the family is not in the form of $r \circ \Phi$, where $r : \mathbb{R}_1^3 \to \mathbb{R}_1^3$ is an immersion of $\mathbb{R}_1^3$. These kinds of surfaces are called timelike Bonnet surfaces by Chen and Li in [11]. Suppose that $\{e_1, e_2, e_3\}$ is a local orthonormal frame at the point $x \in M$, where e_1 is a timelike tangent vector, e_2 is a spacelike unit tangent vector, and e_3 is a spacelike unit normal vector field at $x \in M$. e_3 can be regarded as a map $e_3 : M \to S_1^2$, where $S_1^2 = \{x \in \mathbb{R}_1^3 : \langle x, x\rangle = 1\}$ is the de Sitter space. Let w^i, $1 \le i \le 3$, be dual 1-forms of e_i defined by $w_i(e_j) = \langle e_i, e_j\rangle = \varepsilon_i\delta_{ij}$, $1 \le i, j \le 3$, and let w_j^i, $1 \le i, j \le 3$, be connection forms; then

$$dx = w^1 e_1 + w^2 e_2,$$

$$de_1 = w_1^2 e_2 + w_1^3 e_3,$$

$$de_2 = w_2^1 e_1 + w_2^3 e_3,$$
$$de_3 = w_3^1 e_1 + w_3^2 e_2, \tag{1}$$

such that $w_1^3 = w_3^1$, $w_2^3 = -w_3^2$, and $w_1^2 = w_2^1$.

The Weingarten map $A : T_xM \to T_xM$ is given by

$$Ae_1 = -ae_1 - be_2,$$
$$Ae_2 = be_1 - ce_2, \tag{2}$$

and A has real eigenvector if and only if $(a+c)^2/4-(ac+b^2) \geq 0$; that is, $H^2-K = (a-c)^2/4-b^2 \geq 0$ [11]. Here, $H = (a+c)/2$ and $K = ac + b^2$ are the mean and Gaussian curvatures of M, respectively.

Unless otherwise stated throughout this paper, we will assume that $H^2 > K$ and e_1, e_2 are the eigenvectors. Thus, $b = 0$ and

$$w_1^3 = -aw^1,$$
$$w_2^3 = cw^2, \tag{3}$$

where a and c are the principal curvatures throughout e_1 and e_2. Then, the mean and Gaussian curvatures of the surface M are

$$H = \frac{a+c}{2},$$
$$K = ac, \tag{4}$$

respectively, and $J = (a-c)/2 > 0$. Suppose that $w_1^2 = hw^1 + kw^2$; then h and k can be determined from the Cartan structure equations given by

$$dw^1 = w^2 \wedge w_2^1,$$
$$dw^2 = w^1 \wedge w_1^2. \tag{5}$$

The Gaussian and Codazzi equations are

$$dw_1^2 = w_1^3 \wedge w_3^2 = -Kw^1 \wedge w^2, \tag{6}$$

$$dw_1^3 = w_1^2 \wedge w_2^3,$$
$$dw_2^3 = w_2^1 \wedge w_1^3, \tag{7}$$

respectively [11]. If we substitute the equations of (3) into the Codazzi equations, respectively, and make them equal to the exterior differentiations of (3), then we give

$$\left(da + (a-c)\, hw^2\right) \wedge w^1 = 0,$$
$$\left(dc + (c-a)\, kw^1\right) \wedge w^2 = 0. \tag{8}$$

After reformulating the equations of (8), we can give

$$da = (c-a)\left(pw^1 + hw^2\right),$$
$$dc = (a-c)\left(kw^1 + qw^2\right). \tag{9}$$

Since $dH = (da + dc)/2$, these last two equations give us

$$2dH = (a-c)\left((k-p)\, w^1 + (q-h)\, w^2\right). \tag{10}$$

Here, if the functions u and v are defined as $u = k - p$ and $v = q - h$, then the last equation becomes

$$2dH = (a-c)\left(uw^1 + vw^2\right); \tag{11}$$

that is, $dH = J(uw^1 + vw^2)$. Moreover, from (9), we get

$$\frac{da}{a-c} = (u-k)\, w^1 - hw^2,$$
$$\frac{dc}{a-c} = kw^1 + (v+h)\, w^2. \tag{12}$$

Thus, by the aid of these last two equations, we see

$$d\ln(a-c) = (u-2k)\, w^1 - (v+2h)\, w^2. \tag{13}$$

If we take into consideration (11), then the gradient of the mean curvature function H is

$$\nabla H = \frac{a-c}{2}\left(-ue_1 + ve_2\right). \tag{14}$$

Thus, it is easily seen that

$$g(\nabla H, \nabla H) = \left(H^2 - K\right)\left(-u^2 + v^2\right). \tag{15}$$

Here, we assume that ∇H is nonnull vector field; that is, $\pm u \neq v$. By taking $\varepsilon = \operatorname{sgn} g(\nabla H, \nabla H) = \pm 1$, we find

$$\varepsilon\left(-u^2 + v^2\right) = \frac{\left|g(\nabla H, \nabla H)\right|}{H^2 - K}. \tag{16}$$

By the fact that Hodge $*$ operator is defined as

$$*w^1 = w^2,$$
$$*w^2 = w^1, \tag{17}$$
$$*^2 = 1,$$

the connection form w_1^2 given by (3) becomes

$$*w_1^2 = h * w^1 + k * w^2 = kw^1 + hw^2. \tag{18}$$

Let us define

$$\theta^1 = uw^1 + vw^2,$$
$$\theta^2 = vw^1 + uw^2, \tag{19}$$
$$\alpha^1 = uw^1 - vw^2,$$
$$\alpha^2 = -vw^1 + uw^2. \tag{20}$$

Then, the following equations are obvious:

$$*\theta^1 = \theta^2,$$
$$*\theta^2 = \theta^1, \tag{21}$$
$$*\alpha^1 = \alpha^2,$$
$$*\alpha^2 = \alpha^1. \tag{22}$$

By considering the second equation of (21), if we rearrange (11), we get

$$2dH = (a - c)\,\theta^1. \tag{23}$$

In a similar way, from (18) and (20), (13) becomes

$$d\ln(a - c) = \left(uw^1 - vw^2\right) - 2\left(kw^1 + hw^2\right). \tag{24}$$

Thus, we find

$$d\ln(a - c) = \alpha^1 - 2 * w_1^2. \tag{25}$$

3. Timelike Bonnet Surfaces in Minkowski Space

Since we will make use of it in the following section, let us briefly recall construction of the criterion given by [11] for a timelike surface to be a Bonnet surface in 3-dimensional Minkowski space.

Let $\overline{M}$ be another timelike surface in $\mathbb{R}_1^3$ with the principal direction vectors such that it is an isometric deformation of M preserving the first fundamental form and principal curvatures. Suppose that $\{\overline{w}^1, \overline{w}^2\}$ is a principal coframe corresponding to orthonormal $\{\overline{e}_1, \overline{e}_2\}$ of $\overline{M}$; then the first fundamental form of $\overline{M}$ is

$$-\left(\overline{w}^1\right)^2 + \left(\overline{w}^2\right)^2 = -\left(w^1\right)^2 + \left(w^2\right)^2, \tag{26}$$

and the principal curvatures throughout $\overline{e}_1$ and $\overline{e}_2$ are

$$\begin{aligned} \overline{a} &= a, \\ \overline{c} &= c, \end{aligned} \tag{27}$$

respectively. It is seen from (26) that there is a function φ on $\overline{M}$ such that

$$\begin{aligned} \overline{w}^1 &= \cosh\varphi w^1 + \sinh\varphi w^2, \\ \overline{w}^2 &= \sinh\varphi w^1 + \cosh\varphi w^2. \end{aligned} \tag{28}$$

By direct calculations and the exterior derivative of (28), we get $d\overline{w}^1 = \overline{w}^2 \wedge (-d\varphi + w_1^2)$ and $d\overline{w}^2 = \overline{w}^1 \wedge (-d\varphi + w_1^2)$. From the first Cartan structure equations, we obtain

$$\overline{w}_1^2 = \overline{w}_2^1 = w_1^2 - d\varphi. \tag{29}$$

On the other hand, from (25), we write $d\ln(\overline{a} - \overline{c}) = \overline{\alpha}^1 - 2 * \overline{w}_1^2$. Equations (27) and (29) and the last equation give us $\alpha^1 - 2 * w_1^2 = \overline{\alpha}^1 - 2 * \overline{w}_1^2$. If we apply $*$ operator to this equality, we find $*\alpha^1 - 2w_1^2 = *\overline{\alpha}^1 - 2\overline{w}_1^2$. By the fact that $*\alpha^1 = \alpha^2$, $*\alpha^2 = \alpha^1$, $*^2 = 1$ we see $\alpha^2 - 2w_1^2 = \overline{\alpha}^2 - 2\overline{w}_1^2$. After the necessary arrangements, we find $(\overline{w}_1^2 - w_1^2) = (1/2)(\overline{\alpha}^2 - \alpha^2)$. If we consider (29), we get

$$d\varphi = \frac{1}{2}\left(\alpha^2 - \overline{\alpha}^2\right). \tag{30}$$

$2dH = (\overline{\alpha} - \overline{c})\overline{\theta}^1$ is satisfied for $\overline{M}$ with respect to (23). Considering (27) and comparing the last equation with (23), we see $\overline{\theta}^1 = \theta^1$. Thus, $\overline{u}\,\overline{w}^1 + \overline{v}\,\overline{w}^2 = uw^1 + vw^2$ and if we consider (28) we write

$$\begin{aligned} \overline{u} &= u\cosh\varphi - v\sinh\varphi, \\ \overline{v} &= -u\sinh\varphi + v\cosh\varphi. \end{aligned} \tag{31}$$

By taking (20), if we substitute (28) and (31) into $\overline{\alpha}^2 = -\overline{v}\,\overline{w}^1 + \overline{u}\,\overline{w}^2$, we obtain

$$\overline{\alpha}^2 = \sinh 2\varphi\alpha^1 + \cosh 2\varphi\alpha^2. \tag{32}$$

Let us define $T = \coth\varphi$. The differentiation of T is

$$dT = T\alpha^1 + \alpha^2. \tag{33}$$

The total differential equation (33) is satisfied by rotations of the principal directions with hyperbolic angles φ during isometric deformations. The deformation is nontrivial if and only if (33) is completely integrable [11].

Theorem 1. *Every timelike constant mean curvature surface M with $H^2 > K$ in $\mathbb{R}_1^3$ has one-parameter family of nontrivial isometric deformations preserving the mean curvature; that is, M is timelike Bonnet surface [11].*

In order to investigate the situations of H being constant or nonconstant, let

$$\begin{aligned} d\alpha^1 &= P\alpha^1 \wedge \alpha^2, \\ d\alpha^2 &= Q\alpha^1 \wedge \alpha^2, \end{aligned} \tag{34}$$

thus defining P and Q. By substituting (34) into the exterior derivative of (33), we get

$$(TP + Q - 1)\,\alpha^1 \wedge \alpha^2 = 0. \tag{35}$$

Thus, the following classification is satisfied:

$C_1 : H = \text{constant}.$

$C_2 : H \neq \text{constant}, P = 0, \text{and } Q = 1.$

$C_3 : H \neq \text{constant}, P \neq 0, \text{and } Q \neq 1.$

If these categories are investigated separately, then C_1, C_2, and C_3 can be interpreted as follows.

Since the mean curvature is constant in the case of C_1 from (34) it is obvious that $u = v = 0$. Thus, by (20), we see $\alpha^1 = \alpha^2 = 0$. Consequently, from (33), T is constant.

Since the mean curvature is nonconstant and $P = 0$ and $Q = 1$ in C_2, (35) is satisfied for all T.

The mean curvature is nonconstant and $P \neq 0$ and $Q \neq 1$ in C_3. In this case, by considering (35), we get

$$T = \frac{1 - Q}{P}. \tag{36}$$

T can be computed for any timelike surface with nonconstant mean curvature, nonnull ∇H and $H^2 > K$ but in order to find

the nontrivial isometry Φ preserving the mean curvature T given in (36) must satisfy (33). Thus,

$$d\left(\frac{1-Q}{P}\right)=\left(\frac{1-Q}{P}\right)\alpha^1+\alpha^2 \tag{37}$$

constitutes a criterion for being timelike Bonnet surface.

In the following section, we will check this criterion for timelike tangent developable surfaces and obtain the necessary and sufficient condition for timelike tangent developable surfaces to belong to case C_3.

4. Timelike Tangent Developable Surfaces

Timelike tangent developable surfaces can be investigated in two subcases when the curvature and torsion of base curve of these surfaces are nonconstant and constant.

4.1. Timelike Tangent Developable Surfaces with Base Curve with Nonconstant Curvature and Torsion. Let M be a timelike tangent developable surface given by

$$\begin{aligned} X: I\times\mathbb{R} &\longrightarrow \mathbb{R}_1^3\\ (s,t) &\longrightarrow X(s,t)=\eta(s)+te_1(s), \end{aligned} \tag{38}$$

where $\eta(s)$ is timelike curve parametrized by its arc-length s and $e_1(s)=\dot{\eta}(s)$ is timelike unit tangent vector field. Also assume that $t>0$. (In the same way, $t<0$ gives the second sheet of this surface.) Let $e_2(s)$ be principal normal vector field of $\eta(s)$ and of course it is spacelike. Then, $\dot{e}_1(s)=\kappa(s)e_2(s)$ such that $\kappa(s)\geq 0$ is the curvature of the base curve $\eta(s)$. From (38), it is seen that $X_s(s,t)=e_1(s)+t\kappa(s)e_2(s)$ and $X_t(s,t)=e_1(s)$. The first fundamental form of this surface is

$$I=\left(-1+t^2\kappa^2(s)\right)ds^2-2ds\,dt-dt^2. \tag{39}$$

Then, $dX(s,t)=(ds+dt)e_1(s)+(t\kappa(s)ds)e_2(s)$ can be given from $dX(s,t)=X_s ds+X_t dt$. Also, keeping in mind $dX(s,t)=w^1e_1(s)+w^2e_2(s)$, we see

$$\begin{aligned} w^1&=ds+dt,\\ w^2&=t\kappa(s)\,ds. \end{aligned} \tag{40}$$

By the fact that $w_1^2=g(de_1(s),e_2(s))$, we get $w_1^2=\kappa(s)ds$. On the other hand, since $w^2=t\kappa(s)ds$, that is, $\kappa(s)ds=w^2/t$, we find $w_1^2=(1/t)w^2$. By defining $e_3(s)=e_1(s)\wedge e_2(s)$, $e_3(s)$ is binormal vector field of base curve $\eta(s)$ and spacelike. Let us denote the torsion of $\eta(s)$ by $\tau(s)$. Considering the Serret-Frenet formulae, we obtain

$$\begin{aligned} w_1^3&=g\left(de_1(s),e_3(s)\right)=0=0w^1,\\ w_2^3&=g\left(de_2(s),e_3(s)\right)=-\tau(s)\,ds. \end{aligned} \tag{41}$$

Moreover, we get $w_2^3=-\tau(s)(w^2/t\kappa(s))$ since $ds=w^2/t\kappa(s)$. From the fact that $\{w^1,w^2\}$ is the dual coframe corresponding to principal frame field $\{e_1,e_2\}$, the principal curvatures throughout the principal direction vector fields a and c satisfy $a=0>c=-\tau(s)/t\kappa(s)$. That is, since $t>0$, it is assumed that $\tau(s)>0$. By taking the principal curvatures in terms of curvature and torsion of base curve, H and J can be given as

$$\begin{aligned} H&=\frac{1}{2}\left(\frac{-\tau(s)}{t\kappa(s)}\right),\\ J&=\frac{1}{2}\left(\frac{\tau(s)}{t\kappa(s)}\right), \end{aligned} \tag{42}$$

respectively. On the other hand, from (11), we get

$$d\left(\frac{-\tau(s)}{2t\kappa(s)}\right)=\frac{1}{2}\frac{\tau(s)}{t\kappa(s)}\left(uw^1+vw^2\right). \tag{43}$$

If we substitute (40) into this last equation, we find

$$d\left(\frac{-\tau(s)}{t\kappa(s)}\right)=\frac{\tau(s)}{t\kappa(s)}\left(u(ds+dt)+v(t\kappa(s))\,ds\right). \tag{44}$$

If we call

$$F(s)=\left(\ln\frac{\tau(s)}{\kappa(s)}\right)' \tag{45}$$

by considering (42) from (44) we obtain

$$\begin{aligned} u&=\frac{1}{t},\\ v&=\frac{-F(s)}{t\kappa(s)}-\frac{1}{t^2\kappa(s)}. \end{aligned} \tag{46}$$

If we substitute (40) and (46) into (20), we get

$$\begin{aligned} \alpha^1&=\left(F(s)+\frac{2}{t}\right)ds+\frac{1}{t}dt,\\ \alpha^2&=\left(\kappa(s)+\frac{F(s)}{t\kappa(s)}+\frac{1}{t^2\kappa(s)}\right)ds\\ &\quad+\left(\frac{F(s)}{t\kappa(s)}+\frac{1}{t^2\kappa(s)}\right)dt. \end{aligned} \tag{47}$$

The exterior differentiations of the equalities given in (47) are

$$\begin{aligned} d\alpha^1&=\frac{2}{t^2}ds\wedge dt,\\ d\alpha^2&=\left(\frac{t^2\left(F'(s)-F(s)(\ln\kappa(s))'\right)+t\left(F(s)-(\ln\kappa(s))'\right)+2}{t^3\kappa(s)}\right)ds\\ &\quad\wedge dt, \end{aligned} \tag{48}$$

respectively. The exterior product of the equalities given in (47) is

$$\alpha^1\wedge\alpha^2=\left(\frac{t^2\left(F^2(s)-\kappa^2(s)\right)+2tF(s)+1}{t^3\kappa(s)}\right)ds\wedge dt \tag{49}$$

such that it never vanishes. By putting (48) and (49) into the equalities in (34), we find

$$P = \frac{2t\kappa(s)}{t^2\left(F^2(s)-\kappa^2(s)\right)+2tF(s)+1},$$
$$Q = \frac{t^2\left(F'(s)-F(s)(\ln\kappa(s))'\right)+t\left(F(s)-(\ln\kappa(s))'\right)+2}{t^2\left(F^2(s)-\kappa^2(s)\right)+2tF(s)+1}. \tag{50}$$

In order to check whether the criterion of being a timelike Bonnet surface given in the case of C_3 is satisfied or not, let us substitute the equations given in (50) into (36) and find

$$T = \frac{t^2\left(F^2(s)-\kappa^2(s)-F'(s)+F(s)(\ln\kappa(s))'\right)+t\left(F(s)+(\ln\kappa(s))'\right)-1}{2t\kappa(s)}. \tag{51}$$

By arranging (45), we get

$$F(s) = (\ln\tau(s))' - (\ln\kappa(s))'. \tag{52}$$

Thus, (51) becomes

$$T = \frac{t^2\left(-\kappa^2(s)-F'(s)+F(s)(\ln\tau(s))'\right)+t(\ln\tau(s))'-1}{2t\kappa(s)}. \tag{53}$$

From exterior derivative of (53), we obtain

$$\begin{aligned} dT = {} & \left(\frac{t}{2\kappa(s)}\left(-F'(s)+F(s)(\ln\tau(s))'-\kappa^2(s)\right)'\right. \\ & -\frac{t}{2\kappa^2(s)}\left(-F'(s)+F(s)(\ln\tau(s))'-\kappa^2(s)\right) \\ & +\frac{1}{2\kappa(s)}(\ln\tau(s))''-\frac{t}{2t\kappa^2(s)}(\ln\tau(s))' \\ & \left.+\frac{1}{2t\kappa^2(s)}\right)ds \\ & +\left(\frac{t}{2\kappa(s)}\left(-F'(s)+F(s)(\ln\tau(s))'-\kappa^2(s)\right)\right. \\ & \left.+\frac{1}{2t^2\kappa(s)}\right)dt. \end{aligned} \tag{54}$$

In order to find the right side of (33), we consider (47) and (53) and find

$$\begin{aligned} T\alpha^1+\alpha^2 = {} & \left(\frac{t^2F(s)\left(-F'(s)+F(s)(\ln\tau(s))'-\kappa^2(s)\right)+t\left(3F(s)(\ln\tau(s))'-2F'(s)\right)+F(s)+2(\ln\tau(s))'}{2t\kappa(s)}\right)ds \\ & +\left(\frac{t^2\left(-F'(s)+F(s)(\ln\tau(s))'-\kappa^2(s)\right)+t\left((\ln\tau(s))'+2F(s)\right)+1}{2t^2\kappa(s)}\right)dt. \end{aligned} \tag{55}$$

If (54) and (55) are compared,

$$(\ln\tau(s))' = -2F(s) \tag{56}$$

is obtained since $dT = T\alpha^1+\alpha^2$. If we write this equation into (52), we get

$$(\ln\tau(s))' = -3F(s). \tag{57}$$

If we substitute (56) into (52) and solve it, we find

$$A\tau^3(s) = \kappa^2(s), \tag{58}$$

where A is a constant value such that $A\tau > 0$. If (56) and (57) are written in (54),

$$\begin{aligned} dT = {} & \frac{t^2\left(\left(-F'(s)-2F^2(s)-\kappa^2(s)\right)'+3F(s)\left(-F'(s)-2F^2(s)-\kappa^2(s)\right)\right)-t\left(2F'(s)+6F^2(s)\right)-3F(s)}{2t\kappa(s)}ds \\ & +\frac{t^2\left(F(s)(\ln\tau(s))'-\kappa^2(s)-F'(s)\right)+1}{2t^2\kappa(s)}dt \end{aligned} \tag{59}$$

is obtained. By comparing (55) with (59) and substituting into (56), we get $(-F'(s)-2F^2(s)-\kappa^2(s))'+3F(s)(-F'(s)-2F^2(s)-\kappa^2(s)) = F(s)(-F'(s)-2F^2(s)-\kappa^2(s))$. It is easily seen that

$$\left(\frac{-F'(s)-2F^2(s)-\kappa^2(s)}{\tau(s)}\right)' = 0; \tag{60}$$

that is,

$$\left(\frac{-F'(s)-2F^2(s)-\kappa^2(s)}{\tau(s)}\right) = B, \tag{61}$$

where B is constant. From this last equation, we get

$$-F'(s)-2F^2(s) = \kappa^2(s)+B\tau(s). \tag{62}$$

If we put (56) and (58) into (62), then we find

$$\tau''(s)\tau(s)-2\left(\tau'(s)\right)^2 = 2A\tau^5(s)+2B\tau^3(s). \tag{63}$$

By multiplying each side of the last equation by $2/\tau(s)$, we get

$$2\tau''(s)-\frac{4}{\tau(s)}\left(\tau'(s)\right)^2 = 4A\tau^4(s)+4B\tau^2(s). \tag{64}$$

Here, assume that $d\tau/ds = \tau'(s) = \sigma(\tau)$. Thus, (68) becomes

$$\left(\sigma^2\right)'-\frac{4}{\tau(s)}\sigma^2 = 4A\tau^4(s)+4B\tau^2(s). \tag{65}$$

The solution of this differential equation is

$$\sigma^2 = \left(\tau'\right)^2 = 4A\tau^5+C\tau^4-4B\tau^3, \quad C = \text{constant}. \tag{66}$$

Here, $(\tau')^2 \geq 0$ and $A\tau^3(s) = \kappa^2(s) > 0$. Thus, there is the inequality

$$4\tau^2+\frac{C}{A}\tau-4\frac{B}{A} \geq 0. \tag{67}$$

If $\Delta = C^2-4(4A)(-4B) > 0$, that is, $C^2 > -64AB$, then $\tau < r_1 < r_2$ or $r_1 < r_2 < \tau$ and $\tau \neq 0$, where r_1 and r_2 are the roots of the quadratic trinomial $4\tau^2+(C/A)\tau-4(B/A) = 0$.

Thus, the following theorem can be given.

Theorem 2. *Timelike tangent developable surface $X(s,t)$ is a timelike Bonnet surface if and only if*

$$\begin{aligned}
&\text{(i)} \quad A\tau^3(s) = k^2(s), \quad A\tau(s) > 0, \ A = \textit{constant},\\
&\text{(ii)} \quad \tau''(s)\tau(s)-2\left(\tau'(s)\right)^2 = 2A\tau^5(s)+2B\tau^3(s),\\
&B = \textit{constant} \quad \textit{or} \quad \left(\tau'\right)^2 = A\tau^3\left(4\tau^2+\frac{C}{A}\tau-4\frac{B}{A}\right),\\
&\text{(iii)} \quad C^2 > -64AB, \quad C = \textit{constant}.
\end{aligned} \tag{68}$$

4.2. Timelike Tangent Developable Surfaces with Base Curve with Constant Curvature and Torsion. If the curvature and torsion of timelike base curve of timelike tangent developable surface $X(s,t) = \eta(s)+te_1(s)$ are constants, then the base curve $\eta(s)$ is timelike circular helix. All t-constant curves are helices and $X(s,t)$ is a timelike flat helicoidal surface. Since $w^1, w^2, w_1^2, w_1^3, w_2^3, a$, and c are determined as in Section 4.1, we have

$$\begin{aligned}
&F(s) = 0,\\
&T = \coth\varphi = \frac{-t^2\kappa^2-1}{2t\kappa},\\
&\sinh\varphi = \frac{2t\kappa}{t^2\kappa^2-1},\\
&\cosh\varphi = -\frac{t^2\kappa^2+1}{t^2\kappa^2-1},\\
&\varphi \in (0,\infty).
\end{aligned} \tag{69}$$

By considering the possibility of reflection, the forms of the image surface are

$$\begin{aligned}
\overline{I} &= \left(-1+t^2\kappa^2(s)\right)ds^2-2ds\,dt-dt^2,\\
\overline{w}^1 &= \pm\left(\cosh\varphi w^1+\sinh\varphi w^2\right)\\
&= \pm\left(ds+\frac{t^2\kappa^2+1}{t^2\kappa^2-1}dt\right),\\
\overline{w}^2 &= \pm\left(\sinh\varphi w^1+\cosh\varphi w^2\right)\\
&= \pm\left(-t\kappa ds+\frac{2t\kappa}{t^2\kappa^2-1}dt\right),\\
\overline{w}_1^2 &= \pm\left(-d\varphi+w_1^2\right) = \pm\left(-\frac{2\kappa}{t^2\kappa^2-1}dt+\kappa ds\right)\\
&= \pm\frac{1}{t}\overline{w}^2,\\
\overline{w}_1^3 &= 0\overline{w}^1 = 0,\\
\overline{w}_2^3 &= -\frac{\tau(s)}{t\kappa(s)}\overline{w}^2 = \pm\left(\tau ds-\frac{2t\kappa}{t^2\kappa^2-1}dt\right).
\end{aligned} \tag{70}$$

Moreover, the image surface $\Phi(X(s,t))$ is flat but it is not a cylinder or a cone because of $d\alpha^1 = 0$; that is, $P = 0$ is satisfied for cylinders and cones. Thus, a timelike developable surface cannot be a cylinder or a cone. Therefore, image surface is timelike developable surface given by $Y(\tilde{s},\tilde{t}) = \tilde{\eta}(\tilde{s})+\tilde{t}\tilde{e}_1(\tilde{s})$, $\tilde{t} > 0$. Here, $\tilde{s} = -s-(2/\kappa)\operatorname{arctanh}(\kappa t)$ is the arc-length of the timelike base curve $\tilde{\eta}(\tilde{s})$ and $\dot{\tilde{\eta}}(\tilde{s}) = \tilde{e}_1(\tilde{s})$ is timelike unit tangent vector. The forms of timelike surfaces $Y(\tilde{s},\tilde{t})$ are

$$\begin{aligned}
\widetilde{I} &= \left(-1+\tilde{t}^2\tilde{\kappa}^2(\tilde{s})\right)d\tilde{s}^2-2d\tilde{s}\,d\tilde{t}-d\tilde{t},\\
\widetilde{w}^1 &= d\tilde{s}+d\tilde{t},\\
\widetilde{w}^2 &= \tilde{t}\tilde{\kappa}(\tilde{s})\,d\tilde{s},
\end{aligned}$$

$$\widetilde{w}_1^2 = \widetilde{\kappa}(\widetilde{s})\, d\widetilde{s} = \frac{\widetilde{w}^2}{\widetilde{t}},$$
$$\widetilde{w}_1^3 = 0\widetilde{w}^1,$$
$$\widetilde{w}_2^3 = -\frac{\widetilde{\tau}(\widetilde{s})}{\widetilde{t}\widetilde{\kappa}(\widetilde{s})}\widetilde{w}^2. \tag{71}$$

By comparing the forms given in (70) with (71), one can see that $\overline{w}^1$ should be taken with the sign + whereas $\overline{w}^2$ should be taken with the sign −; that is, the reflection is necessary and $\widetilde{t} = t$, $\widetilde{\kappa} = \kappa$, $\widetilde{\tau} = \tau$, $\widetilde{s} = -s - (2/\kappa)\,\mathrm{arctanh}\,(\kappa t)$. Thus, the image surface is the original surface and the nontrivial isometry preserving the mean curvature is

$$\Phi : X(s,t) \longrightarrow X(s,t)$$
$$(s,t) \longrightarrow \Phi(s,t) = \left(-s - \frac{2}{\kappa}\,\mathrm{arctanh}\,(\kappa t), t\right) \tag{72}$$

or

$$\Phi\left(\eta(s) + te_1(s)\right) = \eta\left(-s - \frac{2}{\kappa}\,\mathrm{arctanh}\,(\kappa t)\right) + te_1\left(-s - \frac{2}{\kappa}\,\mathrm{arctanh}\,(\kappa t)\right). \tag{73}$$

Thus, the following corollaries can be given.

Corollary 3. *The mapping $\Phi : X(s,t) \to X(s,t)$ is a timelike isometry preserving the mean curvature. The second fundamental forms of corresponding surfaces are different; that is, Φ is nontrivial.*

Corollary 4. *$\widetilde{s} = -s - (2/\kappa)\,\mathrm{arctanh}\,(\kappa t)$, $\widetilde{t} = -t < 0$, is also a timelike nontrivial isometry from X to its other sheet preserving mean curvature.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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