Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2016, Article ID 6023273, 6 pages
http://dx.doi.org/10.1155/2016/6023273

Research Article

Completeness of Ordered Fields and a Trio of Classical Series Tests

Robert Kantrowitz[1] and Michael M. Neumann[2]

[1] *Department of Mathematics, Hamilton College, 198 College Hill Road, Clinton, NY 13323, USA*
[2] *Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, USA*

Correspondence should be addressed to Robert Kantrowitz; rkantrow@hamilton.edu

Received 12 April 2016; Revised 25 September 2016; Accepted 12 October 2016

Academic Editor: Bashir Ahmad

This article explores the fate of the infinite series tests of Dirichlet, Dedekind, and Abel in the context of an arbitrary ordered field. It is shown that each of these three tests characterizes the Dedekind completeness of an Archimedean ordered field; specifically, none of the three is valid in any proper subfield of $\mathbb{R}$. The argument hinges on a contractive-type property for sequences in Archimedean ordered fields that are bounded and strictly increasing. For an arbitrary ordered field, it turns out that each of the tests of Dirichlet and Dedekind is equivalent to the sequential completeness of the field.

1. Introduction

The main theorems of calculus derive their validity from the completeness property of the real numbers, but the extent of the interconnectedness of the theorems themselves is continuing to come into sharper focus. The "real analysis in reverse" program, as described in [1], is shedding light on the fact that so many of the theorems of real analysis are actually equivalent reformulations of the notion of completeness and hence of one another. In addition to [1], the flurry of recent articles [2–9] is dedicated to the study of abstract ordered fields, particularly the various ways in which the Archimedean and completeness properties are manifested.

An ordered field $\mathbb{F}$ has the *Archimedean property* if the canonically embedded copy of the natural numbers is not bounded above. Equivalent expressions of the Archimedean property are many, and several of the articles cited above treat them. Discussions of completeness center around *Dedekind completeness* and *Cauchy completeness*. The former means that $\mathbb{F}$ has the *supremum property*; namely, every nonempty subset of $\mathbb{F}$ that is bounded above has a least upper bound in $\mathbb{F}$; the latter, also called *sequential completeness*, entails the convergence of all Cauchy sequences in $\mathbb{F}$.

The ordered field $\mathbb{F}$ is topologized by the basic open intervals $(a, b) = \{x \in \mathbb{F} : a < x < b\}$, for all elements $a, b \in \mathbb{F}$, while the absolute value function for $\mathbb{F}$ is defined by $|a| = \max\{-a, a\}$ for all $a \in \mathbb{F}$. Absolute value works in conjunction with the order topology to lend a familiar meaning to analytic aspects of $\mathbb{F}$. For example, a sequence (x_n) of elements of $\mathbb{F}$ converges to $x \in \mathbb{F}$ if, for every positive element ε in $\mathbb{F}$, there is a natural number N for which $|x_n - x| < \varepsilon$ whenever $n \geq N$. The definition of a Cauchy sequence in $\mathbb{F}$ also parallels its counterpart from real analysis.

It is easy to prove that a field that is Dedekind complete has the Archimedean property, and it is well known that a field that is Dedekind complete is also Cauchy complete. Moreover, if an Archimedean ordered field is Cauchy complete, then it must be Dedekind complete (CA35 of [3] or Sections 1-2 of [8]); such a field is, up to isomorphism, the field $\mathbb{R}$ of real numbers. Behind all this, of course, is Cantor's classical construction of the real numbers as the Cauchy completion of the rationals ([10]). A remarkable fact about ordered fields with the Archimedean property is that they are precisely those that are isomorphic to subfields of $\mathbb{R}$ (Theorem 3.5 of [5] or Section 4 of [1]).

Series tests are among the many theorems from calculus that have been considered in the classification of ordered fields. An elementary example from the Classroom Capsule [6] exposes the equivalence of the geometric series test and the Archimedean property. Other familiar tests that enter the fray are given as follows.

Absolute Convergence Test. If (a_n) is a sequence of elements in the ordered field $\mathbb{F}$ for which the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

Comparison Test. If (a_n) and (b_n) are sequences of nonnegative elements in the ordered field $\mathbb{F}$ that satisfy $a_n \le b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges.

We document the connection between completeness and these two series tests in the following theorem which may be viewed as a modest complement to the main result of [2]. Theorem 1 will be an important tool in this article.

Theorem 1. *For an ordered field $\mathbb{F}$, the following statements are equivalent:*

(a) $\mathbb{F}$ *is Cauchy complete.*

(b) *The absolute convergence test holds.*

(c) *The comparison test holds.*

Proof. The fact that the absolute convergence test characterizes Cauchy completeness for ordered fields is proven in the article [2].

The validity of the comparison test in fields that are Cauchy complete follows by noting that convergence of the series $\sum b_n$ implies that its sequence (B_n) of partial sums is convergent and hence Cauchy. The sequence (A_n) of partial sums of the series $\sum a_n$ is therefore also Cauchy, from which the convergence of the series $\sum a_n$ follows from the Cauchy completeness of the field.

Finally, to see that the absolute convergence test holds whenever the comparison test does, observe that, for all n, the inequalities

$$0 \le a_n + |a_n| \le 2|a_n| \tag{1}$$

are true in $\mathbb{F}$, setting the stage for the comparison test to ensure that the series $\sum(a_n+|a_n|)$ converges. Since also $\sum -|a_n|$ converges, it follows that $\sum a_n$ converges. □

Theorem 1 refines a result of [8] on the characterization of Dedekind complete ordered fields in terms of the Archimedean property together with either (b) or (c).

The equivalence of statements (a) and (b) of Theorem 1 appears *mutatis mutandis* in functional analysis where it is known that a normed linear space is a Banach space if and only if the absolute convergence test for series is valid (Theorem 2.8 of [11] or Statement (VIII) of [12]).

Our goal in the present article is to involve the series tests of Dirichlet, Dedekind, and Abel in the taxonomy of ordered fields. The classical versions of these may be found, for example, in [13] or [14], but each has meaning in the more general abstract context. We devote Section 2 to their statements and a short discussion of them. Like the absolute convergence test and the comparison test, Dirichlet's, Dedekind's, and Abel's tests are connected with completeness in the sense of both Dedekind and Cauchy. These relationships are detailed in Sections 4 and 5. Along the way, we explore the role of sequences of bounded variation in ordered fields.

2. The Series Tests of Dirichlet, Dedekind, and Abel

The hypothesis of Dedekind's test calls for the sequence (b_n) to have *bounded variation* which, in the setting of the ordered field $\mathbb{F}$, means that there is an element $M \in \mathbb{F}$ for which the inequality $\sum_{k=1}^{n} |b_{k+1} - b_k| \le M$ holds for all $n \in \mathbb{N}$. We will treat sequences of bounded variation in some detail in the next section. Without further ado, here is the trio of series tests under consideration for an arbitrary ordered field.

Dirichlet's Test. If $\sum a_n$ is a series whose partial sums form a bounded sequence and (b_n) is a decreasing sequence that converges to 0, then the series $\sum a_n b_n$ converges.

Dedekind's Test. If $\sum a_n$ is a convergent series and (b_n) is a sequence of bounded variation, then the series $\sum a_n b_n$ converges.

Abel's Test. If $\sum a_n$ is a convergent series and (b_n) is a monotone convergent sequence, then the series $\sum a_n b_n$ converges.

Dirichlet's test is recognizable as a generalization of Leibniz's alternating series test from calculus (Theorem 3.4.2 of [14]) and, as Apostol mentions in Section 8.15 of [13], these three tests are useful tools when one is confronted with the task of trying to determine convergence of real series that do not converge absolutely. Moreover, as Knopp points out in Section 5.5 of [14], a common feature of the tests in this trio is that they all allow conclusions to be drawn about the series $\sum a_n b_n$ from assumptions concerning the series $\sum a_n$ and the sequence (b_n). Knopp remarks that such tests may be construed as "comparison tests in the extended sense," and he also includes a few applications and examples of them.

It turns out that Abel's test can be inferred from Dedekind's test, since, as explained in the next section, every monotone convergent sequence has bounded variation. A moment's reflection reveals that Abel's test is also a corollary of Dirichlet's test. Indeed, the sequence of partial sums of a convergent series $\sum a_n$ is bounded, so attention needs only to be directed to the stipulation that (b_n) is a monotone convergent sequence. If (b_n) is decreasing and its limit is b, then the auxiliary sequence given by $c_n = b_n - b$ is also decreasing, and it converges to 0. Dirichlet's test thus ensures that the series $\sum a_n c_n$ converges. The convergence of the series $\sum a_n b_n$ follows easily from algebraic properties of convergent series. The case that (b_n) is increasing with limit b is handled analogously, since this time the sequence $(-c_n)$ decreases to 0.

None of the series tests of Dedekind, Dirichlet, or Abel is, on its own, strong enough to imply Dedekind completeness for an ordered field $\mathbb{F}$ in which it holds. If $\mathbb{F}$ is Archimedean, however, and any of these three tests is in force, then it will follow from Theorem 3 that $\mathbb{F}$ is Dedekind complete. The linchpin is a contractive-type property for certain sequences in Archimedean ordered fields which is detailed in Lemma 4.

As we shall see in Theorem 6 in the final section, the validity of either of the tests of Dedekind or Dirichlet in an ordered field is equivalent to the Cauchy completeness of the field. In a similar vein, it was recently confirmed in [15] that a normed linear space is a Banach space precisely when a suitable vector version of Dedekind's test holds and, moreover, that a unital normed algebra is a Banach algebra if and only if an algebra version of Dedekind's test is valid.

3. Sequences of Bounded Variation

The set of all sequences of elements of the ordered field $\mathbb{F}$ having bounded variation will be denoted by $bv_{\mathbb{F}}$. Examples abound. Indeed, sequences that are monotone and bounded automatically have bounded variation: if $(a_n)_{n\in\mathbb{N}}$ is increasing, for example, with $|a_n| \le K$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}\sum_{k=1}^{n} |a_{k+1} - a_k| &= \sum_{k=1}^{n} (a_{k+1} - a_k) = a_{n+1} - a_1 \\ &\le |a_{n+1}| - a_1 \le K - a_1,\end{aligned} \tag{2}$$

which shows that $(a_n)_{n\in\mathbb{N}}$ has bounded variation.

Sequences of bounded variation are always bounded since, for any $n \in \mathbb{N}$,

$$\begin{aligned}|a_n| &\le |a_1| + |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_n - a_{n-1}| \\ &\le |a_1| + M.\end{aligned} \tag{3}$$

But, in general, there is no relationship between $bv_{\mathbb{F}}$ and the set $c_{\mathbb{F}}$ of convergent sequences in $\mathbb{F}$. For example, in any Archimedean ordered field, the sequence

$$1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \ldots \tag{4}$$

converges to 0, but it does not have bounded variation, whereas, in any non-Archimedean ordered field, it has bounded variation but does not converge. In fact, in a non-Archimedean ordered field, any sequence of rational numbers has bounded variation.

Like their counterparts in the realm of functions on an interval (see Section 6.7 of [13]), sequences of bounded variation admit a *Jordan decomposition*. Specifically, a sequence $(a_n)_{n\in\mathbb{N}}$ in $bv_{\mathbb{F}}$ may be resolved into the difference of two bounded increasing sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ in a canonical way. Indeed, it is easily verified that this is accomplished by the choice $p_n = (v_n + a_n)/2$ and $q_n = (v_n - a_n)/2$ for all $n \in \mathbb{N}$, where $v_1 = 0$ and $v_n = \sum_{k=1}^{n-1} |a_{k+1} - a_k|$ for all $n \ge 2$.

Since, by Proposition 4 of [9], the Archimedean property of an ordered field $\mathbb{F}$ is characterized by the condition that all increasing sequences that are bounded above are Cauchy sequences, the Jordan decomposition now reveals that $\mathbb{F}$ has the Archimedean property if and only if every sequence of bounded variation in $\mathbb{F}$ is a Cauchy sequence.

Moreover, if every sequence of bounded variation in $\mathbb{F}$ happens to converge, then, because we have seen that every monotone bounded sequence lies in $bv_{\mathbb{F}}$, the *monotone convergence theorem* holds for $\mathbb{F}$. The monotone convergence theorem is a stalwart among the statements that are equivalent to Dedekind completeness (CA13 of [3] or Section 1 of [8]), so we conclude that a field for which the inclusion $bv_{\mathbb{F}} \subseteq c_{\mathbb{F}}$ holds must be Dedekind complete.

The hypotheses for Abel's test seem to vary slightly in the literature. The version we work with follows Theorem 8.29 of [13] in requiring a convergent series $\sum a_n$ and a monotone convergent sequence (b_n). However, in the statement of Abel's test that is Theorem 5.5.1 of [14], the series $\sum a_n$ is still required to be convergent, whereas the sequence (b_n) is only assumed to be monotone and bounded. The two versions are equivalent precisely when the underlying field is $\mathbb{R}$. Moreover, on account of the Jordan decomposition, it turns out that, in any field, the latter version is equivalent to Dedekind's test.

As we see, facts about sequences of bounded variation may often be deduced from properties of monotone sequences. Though they find themselves somewhat overshadowed by monotone sequences, sequences of bounded variation still play a role in analysis. It turns out that Dedekind's test is the easier half of the result that says that the topological dual space of the Banach space $cs_{\mathbb{R}}$ of all real sequences (a_n) for which the associated series $\sum a_n$ converges may be identified with $bv_{\mathbb{R}}$ (Exercise IV.13.12 of [16]). Sequences of bounded variation rear their heads again in the related context of summability theory: in order that an infinite matrix defines a transformation that maps $cs_{\mathbb{R}}$ into $cs_{\mathbb{R}}$, one of the necessary and sufficient conditions is that each of its rows is in $bv_{\mathbb{R}}$ (Theorem 5.6.1 of [14] or Exercise II.4.45 of [16]).

Lemma 2. *If the ordered field $\mathbb{F}$ is Cauchy complete, then the series tests of Dirichlet, Dedekind, and Abel all hold.*

Proof. The identity for *summation by parts* due to Abel (Theorem 8.27 of [13]), a well-known analogue for integration by parts in a Riemann-Stieltjes integral, carries over to the setting of an abstract field (or, in fact, of any ring) verbatim: for elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of $\mathbb{F}$ and $A_k = \sum_{j=1}^{k} a_j$ for $k = 1, \ldots, n$, the following equality holds:

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k). \tag{5}$$

As in the classical approach, it seems natural to deduce the convergence of the series $\sum a_n b_n$ by establishing the convergence of the two sequences on the right hand side of the preceding formula.

In the case of Dirichlet's test, this method turns out to be successful in an arbitrary Cauchy complete field, thanks to Theorem 1. Indeed, the hypotheses of Dirichlet's test easily ensure that $(A_n b_{n+1})$ and $\sum |b_{k+1} - b_k|$ both converge. Hence, if M denotes a positive upper bound on the sequence of partial sums (A_n), then the series $\sum_{k=1}^{\infty} M|b_{k+1} - b_k|$ converges. By Theorem 1, we obtain the convergence of the series $\sum_{k=1}^{\infty} |A_k(b_{k+1} - b_k)|$ from the comparison test and then the desired convergence of the series $\sum_{k=1}^{\infty} A_k(b_{k+1} - b_k)$ from the absolute convergence test.

However, more work is needed for Dedekind's test, since the convergence of sequences of bounded variation is guaranteed only in the setting of a Dedekind complete ordered field. We therefore separate the following two cases.

If the Cauchy complete field $\mathbb{F}$ happens to be Archimedean, then $\mathbb{F}$ is Dedekind complete, and we are in the classical setting $\mathbb{F} = \mathbb{R}$ for which Dedekind's and Abel's tests are, of course, immediate from the preceding partial summation formula.

It thus remains to address the case that $\mathbb{F}$ is non-Archimedean. Since the hypotheses for Dedekind's and Abel's tests ensure that $a_n \to 0$ as $n \to \infty$ and that the sequence (b_n) is bounded, we have that $a_n b_n \to 0$ as $n \to \infty$. Convergence of the series $\sum a_n b_n$ thus follows from part (b) of the main theorem of [2]. □

4. Dedekind Completeness and Series Tests

We now involve the trio of series tests in a list of statements about an ordered field that are equivalent to Dedekind completeness.

Theorem 3. *For an ordered field $\mathbb{F}$, the following statements are equivalent:*

(a) $\mathbb{F}$ *is Dedekind complete.*

(b) *Sequences in $\mathbb{F}$ that are monotone and bounded are convergent.*

(c) $bv_{\mathbb{F}} \subseteq c_{\mathbb{F}}$.

(d) $\mathbb{F}$ *is Archimedean and Dirichlet's test holds.*

(e) $\mathbb{F}$ *is Archimedean and Dedekind's test holds.*

(f) $\mathbb{F}$ *is Archimedean and Abel's test holds.*

As indicated in the previous section, the equivalence of statements (a) and (b) is well known, and the equivalence of (b) and (c) is clear from the Jordan decomposition for sequences of bounded variation.

Because a field that is Dedekind complete is Archimedean and Cauchy complete, the implications (a) ⇒ (d) and (a) ⇒ (e) follow from Lemma 2. The implications (d) ⇒ (f) and (e) ⇒ (f) are consequences of the fact that Dirichlet's test and Dedekind's test both imply Abel's test, as detailed toward the end of Section 2.

It thus remains to establish that (f) ⇒ (b). Our proof will be based on the following result.

Lemma 4. *Let r be a positive element of the Archimedean ordered field $\mathbb{F}$, and let $(c_n)_{n\in\mathbb{N}}$ be a strictly increasing bounded sequence in $\mathbb{F}$. Then there exists a subsequence $(c_{n_k})_{k\in\mathbb{N}}$ such that*

$$c_{n_{k+2}} - c_{n_{k+1}} < r\left(c_{n_{k+1}} - c_{n_k}\right) \quad \forall k \in \mathbb{N}. \tag{6}$$

Proof. By the characterization of Archimedean fields mentioned earlier, we may and do assume that $\mathbb{F}$ is a subfield of $\mathbb{R}$. Then there exists an element $L \in \mathbb{R}$ for which $c_n \to L$ as $n \to \infty$. The inductive choice of the desired subsequence will be straightforward once we know that for every $p \in \mathbb{N}$ there exists some $q \in \mathbb{N}$ with $q > p$ such that the estimate

$$c_j - c_q < r\left(c_q - c_p\right) \tag{7}$$

holds for all $j \in \mathbb{N}$ with $j > q$. For this, fix $p \in \mathbb{N}$, and note that whenever $q \in \mathbb{N}$ satisfies $q > p$ the inequality $r(c_{p+1} - c_p) \leq r(c_q - c_p)$ automatically holds. Now, take $q \in \mathbb{N}$ so large that $q > p$ and $L - c_q < r(c_{p+1} - c_p)$. Then, for all $j \in \mathbb{N}$ with $j > q$, we obtain

$$c_j - c_q < L - c_q < r\left(c_{p+1} - c_p\right) \leq r\left(c_q - c_p\right). \tag{8}$$

To complete the proof of the lemma, we choose $n_1 = 1$ and then, by induction, a sequence of integers $n_k \in \mathbb{N}$ with $n_k < n_{k+1}$ such that the *a priori* estimate

$$c_j - c_{n_{k+1}} < r\left(c_{n_{k+1}} - c_{n_k}\right) \tag{9}$$

holds for all $k \in \mathbb{N}$ and $j \in \mathbb{N}$ with $j > n_{k+1}$. Once a subsequence $(c_{n_k})_{k\in\mathbb{N}}$ with the preceding property has been chosen, for each $k \in \mathbb{N}$ we may take $j = n_{k+2}$ to conclude that the sequence satisfies the conclusion of the lemma. □

We mention in passing that the preceding result remains valid for increasing, rather than strictly increasing, sequences provided that the strict inequality $<$ in the assertion is replaced by $\leq$. Indeed, this is obvious in the case of an increasing sequence that is eventually constant, while every increasing bounded sequence that fails to be eventually constant admits a strictly increasing subsequence to which Lemma 4 may be applied.

We are now ready to finish the proof of Theorem 3.

Proof of $(f) \Rightarrow (b)$. Suppose that $\mathbb{F}$ is an Archimedean ordered field for which Abel's test holds, and consider an arbitrary increasing bounded sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$. To see that this sequence converges in $\mathbb{F}$, we may suppose that it is not eventually constant and thus admits a strictly increasing subsequence. Since an increasing sequence converges in $\mathbb{F}$ precisely when it contains a convergent subsequence, we may, without loss of generality, assume that $(c_n)_{n\in\mathbb{N}}$ is actually strictly increasing. We then apply Lemma 4 with the choice $r = 1/4$ to obtain a subsequence $(c_{n_k})_{k\in\mathbb{N}}$ for which

$$c_{n_{k+2}} - c_{n_{k+1}} < \frac{1}{4}\left(c_{n_{k+1}} - c_{n_k}\right) \tag{10}$$

for all $k \in \mathbb{N}$. As before, it suffices to show that this subsequence converges in $\mathbb{F}$. This will now be established by an application of Abel's test and the Archimedean property for the field $\mathbb{F}$. For this, we introduce

$$\begin{aligned} a_k &= \frac{1}{2^k}, \\ b_k &= 2^k\left(c_{n_{k+1}} - c_{n_k}\right) \end{aligned} \tag{11}$$

for all $k \in \mathbb{N}$. Because $\mathbb{F}$ is Archimedean, we know from [6] that the geometric series

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \tag{12}$$

converges in $\mathbb{F}$. Moreover, the choice of the subsequence $(c_{n_k})_{k\in\mathbb{N}}$ ensures that

$$b_{k+1} = 2^{k+1}\left(c_{n_{k+2}} - c_{n_{k+1}}\right) < 2^{k-1}\left(c_{n_{k+1}} - c_{n_k}\right) = \frac{1}{2}b_k, \quad (13)$$

and therefore $0 < b_{k+1} < (1/2^k)b_1$ for all $k \in \mathbb{N}$. Thus (b_k) decreases to 0. By Abel's test, we now conclude that the sequence of partial sums $\sum_{k=1}^{m} a_k b_k$ converges in $\mathbb{F}$. Since telescoping shows that

$$\sum_{k=1}^{m} a_k b_k = \sum_{k=1}^{m}\left(c_{n_{k+1}} - c_{n_k}\right) = c_{n_{m+1}} - c_{n_1} \quad (14)$$

for all $m \in \mathbb{N}$, this confirms the desired convergence of $(c_{n_k})_{k\in\mathbb{N}}$. □

While the geometric series test is certainly not, on its own, strong enough to guarantee completeness, the Classroom Capsule [6], together with Theorem 3, shows that if $\mathbb{F}$ is an ordered field in which the geometric series test, along with any of the tests of Dirichlet, Dedekind, or Abel holds, then $\mathbb{F}$ is Dedekind complete. Theorem 3 also reveals that the only subfield of $\mathbb{R}$ in which Dirichlet's, Dedekind's, or Abel's test holds is the field $\mathbb{R}$ of real numbers itself. There are, however, many non-Archimedean ordered fields in which each of these three tests holds. We will take up this issue after an example.

Example 5. An example that simultaneously illustrates the failure of each of the three tests in the field $\mathbb{Q}$ of rational numbers is provided by the geometric series $\sum_{n=1}^{\infty} 1/2^n$ and the harmonic sequence $(1/n)_{n\in\mathbb{N}}$. The series converges in $\mathbb{Q}$, so its partial sums are bounded, and the sequence is a decreasing sequence that converges to 0. The hypotheses of Dirichlet's test are thus satisfied. We invoke some calculus to show that the series $\sum_{n=1}^{\infty} 1/(n2^n)$ does not, however, converge in $\mathbb{Q}$. The Taylor series for $(1-x)^{-1}$, valid whenever $|x| < 1$, is the familiar geometric series $1+x+x^2+x^3+\cdots$. Term-by-term integration yields the series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = -\log(1-x), \quad (15)$$

where the equality is valid for the same radius of convergence. The choice $x = 1/2$ reveals $\sum_{n=1}^{\infty} 1/(n2^n) = \log 2$. But $\log 2$ is irrational, since otherwise the equation $\log 2 = p/q$ with $p, q \in \mathbb{N}$ would entail that the number e is a root of the polynomial $x^p - 2^q$, contradicting the fact that e is transcendental, a celebrated result of Hermite dating back to 1873 (Chapter 15 of [17]). The conclusion of Dirichlet's test thus does not hold for $\mathbb{Q}$. These data also show that Dedekind's and Abel's tests fail in $\mathbb{Q}$.

5. Cauchy Completeness and Series Tests

We conclude with a return to the case of Cauchy completeness and the following extension of Theorem 1.

Theorem 6. *For an ordered field $\mathbb{F}$, the following statements are equivalent:*

(a) *$\mathbb{F}$ is Cauchy complete.*

(b) *The absolute convergence test holds.*

(c) *The comparison test holds.*

(d) *Dedekind's test holds.*

(e) *Dirichlet's test holds.*

Proof. The equivalence of statements (a), (b), and (c) was handled in Theorem 1, and the implications (a) ⇒ (d) and (a) ⇒ (e) were addressed in Lemma 2. By Theorem 3, it thus suffices to establish (d) ⇒ (b) and (e) ⇒ (b) in the non-Archimedean case. To this end, we consider an arbitrary sequence (x_n) in a non-Archimedean field $\mathbb{F}$ for which the series $\sum |x_n|$ converges.

If (d) holds, then we choose $b_n \in \{-1, 1\}$ such that $x_n = b_n|x_n|$ for all $n \in \mathbb{N}$ and observe that the sequence of partial sums $\sum_{k=1}^{n} |b_{k+1} - b_k|$ is bounded above in $\mathbb{F}$, since $\mathbb{F}$ fails to be Archimedean. Thus Dedekind's test ensures that $\sum x_n = \sum b_n|x_n|$ converges in $\mathbb{F}$, which confirms that (d) implies (b).

To prove the convergence of $\sum x_n$ under condition (e), we first note that the convergence of the series $\sum |x_n|$ entails that the sequence of partial sums of the series $\sum x_n$ is a Cauchy sequence. So, to meet our goal, it suffices to show that the sequence of partial sums of the series $\sum x_n$ has a convergent subsequence. This will be accomplished by the following construction.

The supposition that $\sum |x_n|$ converges implies that $|x_n| \to 0$ as $n \to \infty$, and, without loss of generality, we assume that the sequence $(|x_n|)$ is not eventually 0. Invoking the procedure detailed in the proof of the classical monotone convergence theorem (Theorem 3.4.7 of [18]), we inductively extract a decreasing "peak" subsequence $(|x_{n_k}|)$ as follows. We start by selecting n_1 as the least positive integer κ for which $|x_\kappa| \geq |x_j|$ for all $j \in \mathbb{N}$. Thereafter, once n_k is designated, n_{k+1} is chosen to be the least positive integer κ beyond n_k for which $|x_\kappa| \geq |x_j|$ for all $j > n_k$. Evidently, $|x_{n_k}| > 0$ for all $k \in \mathbb{N}$, and $(|x_{n_k}|)$ decreases to 0.

Now, for each $k \in \mathbb{N}$, let

$$a_k = \frac{x_{n_k} + \cdots + x_{n_{k+1}-1}}{\left|x_{n_k}\right|} \quad (16)$$

and observe that

$$\left|a_k\right| \leq \frac{\left|x_{n_k}\right| + \cdots + \left|x_{n_{k+1}-1}\right|}{\left|x_{n_k}\right|} \leq n_{k+1} - n_k. \quad (17)$$

The sequence of partial sums of the series $\sum a_k$ is thus bounded in $\mathbb{F}$ on account of the field $\mathbb{F}$ being non-Archimedean. Hence, by Dirichlet's test, the series $\sum a_k|x_{n_k}|$ converges. Since

$$\sum_{k=1}^{n_{m+1}-1} x_k = x_1 + \cdots + x_{n_1-1} + \sum_{k=1}^{m} a_k \left|x_{n_k}\right| \quad (18)$$

for all $m \in \mathbb{N}$, we conclude that the sequence of partial sums of the series $\sum x_n$ has indeed a convergent subsequence, so we have reached our stated goal. □

An anonymous referee graciously offered the equivalence of statements (a), (d), and (e) of Theorem 6 together with an outline of a proof. The suggested argument to establish the fact that Cauchy completeness is a consequence of Dirichlet's test was based on the main theorem of [2] along with results about rearrangements of series with nonnegative terms. In the end, we chose our own version of the proof of (e) $\Rightarrow$ (b) on account of its direct focus on the absolute convergence test. The question of whether the validity of Abel's test in an arbitrary ordered field implies that the field is Cauchy complete remains open.

Competing Interests

The authors declare that they have no competing interests.

References

[1] J. Propp, "Real analysis in reverse," *American Mathematical Monthly*, vol. 120, no. 5, pp. 392–408, 2013.

[2] P. L. Clark and N. J. Diepeveen, "Absolute convergence in ordered fields," *American Mathematical Monthly*, vol. 121, no. 10, pp. 909–916, 2014.

[3] M. Deveau and H. Teismann, "72 + 42: characterizations of the completeness and Archimedean properties of ordered fields," *Real Analysis Exchange*, vol. 39, no. 2, pp. 261–304, 2013.

[4] M. Deveau and H. Teismann, "Would real analysis be complete without the fundamental theorem of calculus?" *Elemente der Mathematik*, vol. 70, no. 4, pp. 161–172, 2015.

[5] J. F. Hall, "Completeness of ordered fields," https://arxiv.org/abs/1101.5652v1.

[6] R. Kantrowitz and M. Neumann, "Another face of the Archimedean property," *The College Mathematics Journal*, vol. 46, no. 2, pp. 139–141, 2015.

[7] R. Kantrowitz and M. Schramm, "Series that converge absolutely but don't converge," *The College Mathematics Journal*, vol. 43, no. 4, pp. 331–333, 2012.

[8] O. Riemenschneider, "37 elementare axiomatische Charakterisierungen des reellen Zahlkörpers," *Mitteilungen der Mathematischen Gesellschaft in Hamburg*, vol. 20, pp. 71–95, 2001.

[9] H. Teismann, "Toward a more complete list of completeness axioms," *American Mathematical Monthly*, vol. 120, no. 2, pp. 99–114, 2013.

[10] G. Cantor, "Über unendliche, lineare Punktmannichfaltigkeiten," *Mathematische Annalen*, vol. 21, no. 4, pp. 545–591, 1883.

[11] B. Bollobás, *Linear Analysis: An Introductory Course*, Cambridge University Press, Cambridge, UK, 1990.

[12] G. L. Cohen, "Is every absolutely convergent series convergent?" *The Mathematical Gazette*, vol. 61, no. 417, pp. 204–213, 1977.

[13] T. M. Apostol, *Mathematical Analysis*, Addison-Wesley, Reading, Mass, USA, 2nd edition, 1974.

[14] K. Knopp, *Infinite Sequences and Series*, Dover Publications, New York, NY, USA, 1956.

[15] R. Kantrowitz and M. M. Neumann, "More of Dedekind: his series test in normed spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2016, Article ID 2508172, 3 pages, 2016.

[16] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, John Wiley & Sons, New York, NY, USA, 1958.

[17] E. Maor, *e: The Story of a Number*, Princeton University Press, Princeton, NJ, USA, 1994.

[18] R. G. Bartle and D. S. Sherbert, *Introduction to Real Analysis*, John Wiley & Sons, New York, NY, USA, 3rd edition, 2000.