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Research Article

A Simplified Proof of Uncertainty Principle for Quaternion Linear Canonical Transform

Mawardi Bahri[1] and Ryuichi Ashino[2]

[1] *Department of Mathematics, Hasanuddin University, Makassar 90245, Indonesia*
[2] *Division of Mathematical Sciences, Osaka Kyoiku University, Osaka 582-8582, Japan*

Correspondence should be addressed to Ryuichi Ashino; ashino@cc.osaka-kyoiku.ac.jp

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We provide a short and simple proof of an uncertainty principle associated with the quaternion linear canonical transform (QLCT) by considering the fundamental relationship between the QLCT and the quaternion Fourier transform (QFT). We show how this relation allows us to derive the inverse transform and Parseval and Plancherel formulas associated with the QLCT. Some other properties of the QLCT are also studied.

1. Introduction

It is well-known that the traditional linear canonical transform (LCT) plays an important role in many fields of optics and signal processing. It can be regarded as a generalization of many mathematical transforms such as the Fourier transform, Laplace transform, the fractional Fourier transform, and the Fresnel transform. Many fundamental properties of this extended transform are already known, including shift, modulation, convolution, and correlation and uncertainty principle, for example, in [1–6].

Recently, there are so many studies in the literature that are concerned with the generalization of the LCT within the context of quaternion algebra, which is the so-called quaternion linear canonical transform (QLCT) (see, e.g., [7–10]). They also established some important properties of the QLCT such as inversion formula and the uncertainty principle. An application of the QLCT to study of generalized swept-frequency filters was presented in [11]. In this paper, we will focus on the two-dimensional case and provide a new proof of uncertainty principle associated with the QLCT, the ones proposed in [8], the proof of which is much simpler using the component-wise and directional uncertainty principles for the QFT [12, 13]. Therefore, before proving this main result, we first derive the fundamental relationship between the QLCT and QFT. Using the relation, we obtain useful properties of the QLCT such as inverse transform and Parseval formula associated with the QLCT.

The quaternion algebra over $\mathbb{R}$, denoted by $\mathbb{H}$, is an associative noncommutative four-dimensional algebra:

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3;\ q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \quad (1)$$

which obeys the following multiplication rules:

$$\begin{aligned} \mathbf{ij} &= -\mathbf{ji} = \mathbf{k}, \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i}, \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j}, \\ \mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \end{aligned} \quad (2)$$

For a quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{H}$, q_0 is called the *scalar* part of q denoted by $\mathrm{Sc}(q)$ and $\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is called the *vector* (or *pure*) part of q. The vector part of q is conventionally denoted by $\mathbf{q}$. Let $p, q \in \mathbb{H}$ and $\mathbf{p}$, $\mathbf{q}$ be their vector parts, respectively. Equation (2) yields the quaternionic multiplication qp as

$$qp = q_0 p_0 - \mathbf{q} \cdot \mathbf{p} + q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{q} \times \mathbf{p}, \quad (3)$$

where $\mathbf{q} \cdot \mathbf{p} = (q_1 p_1 + q_2 p_2 + q_3 p_3)$ and $\mathbf{q} \times \mathbf{p} = \mathbf{i}(q_2 p_3 - q_3 p_2) + \mathbf{j}(q_3 p_1 - q_1 p_3) + \mathbf{k}(q_1 p_2 - q_2 p_1)$.

The quaternion conjugate of q, given by

$$\overline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}, \tag{4}$$

is an anti-involution; that is,

$$\overline{q}\,\overline{p} = \overline{p}\,\overline{q}. \tag{5}$$

From (4) we obtain the norm or modulus of $q \in \mathbb{H}$ defined as

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{6}$$

It is not difficult to see that

$$|qp| = |q|\,|p|, \quad \forall p, q \in \mathbb{H}. \tag{7}$$

Furthermore, it is easily seen that

$$|pqr| = |rqp|, \quad \forall p, q, r \in \mathbb{H}. \tag{8}$$

Using conjugate (4) and the modulus of q, we can define the inverse of $q \in \mathbb{H} \setminus \{0\}$ as

$$q^{-1} = \frac{\overline{q}}{|q|^2}, \tag{9}$$

which shows that $\mathbb{H}$ is a normed division algebra.

It is convenient to introduce an inner product for quaternion-valued (in the rest of the paper, we will always consider quaternion function) functions $f, g : \mathbb{R}^2 \to \mathbb{H}$ as

$$(f, g) = \int_{\mathbb{R}^2} f(\mathbf{x})\,\overline{g(\mathbf{x})}\,d\mathbf{x}, \quad d\mathbf{x} = dx_1 dx_2, \tag{10}$$

with symmetric real scalar part

$$\langle f, g\rangle = \frac{1}{2}\left[(f, g) + (g, f)\right] = \mathrm{Sc}\int_{\mathbb{R}^2} f(\mathbf{x})\,\overline{g}(\mathbf{x})\,d\mathbf{x}. \tag{11}$$

In particular, for $f = g$, we obtain the $L^2(\mathbb{R}^2; \mathbb{H})$-norm:

$$\|f\| = \sqrt{\langle f, f\rangle} = \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2\, d^2\mathbf{x}\right)^{1/2}. \tag{12}$$

2. Quaternion Linear Canonical Transform

In this section we begin by defining the two-sided QFT (for simplicity of notation we write the QFT instead of the two-sided QFT in the next section). We discus some properties, which will be used to prove the uncertainty principle.

Definition 1. The QFT of $f \in L^1(\mathbb{R}^2; \mathbb{H})$ is the transform $\mathcal{F}_q\{f\} : \mathbb{R}^2 \to \mathbb{H}$ given by the integral

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{(2\pi)^2}}\int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} f(\mathbf{x})\, e^{-\mathbf{j}\omega_2 x_2}\, d\mathbf{x}, \tag{13}$$

where $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, $\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2$, and the quaternion exponential product $e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2}$ is the quaternion Fourier kernel. Here $\mathcal{F}_q$ is called the quaternion Fourier transform operator.

Definition 2. If $f \in L^1(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})$, then the inverse transform of the QFT is given by

$$\begin{aligned} f(\mathbf{x}) &= \mathcal{F}_q^{-1}\left[\mathcal{F}_q\{f\}\right](\mathbf{x}) \\ &= \frac{1}{\sqrt{(2\pi)^2}}\int_{\mathbb{R}^2} e^{\mathbf{i}\omega_1 x_1}\mathcal{F}_q\{f\}(\boldsymbol{\omega})\, e^{\mathbf{j}\omega_2 x_2}\, d\boldsymbol{\omega}, \end{aligned} \tag{14}$$

where $\mathcal{F}_q^{-1}$ is called the inverse QFT operator.

An important property of the QFT is stated in the following lemma, which is needed to prove Parseval formula of the QLCT. For more details of the QFT, see [12–16].

Lemma 3 (QFT Parseval). *The quaternion product of $f, g \in L^1(\mathbb{R}^2; \mathbb{H}) \cap L^2(\mathbb{R}^2; \mathbb{H})$ and its QFT are related by*

$$\langle f, g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} = \left\langle \mathcal{F}_q\{f\}, \mathcal{F}_q\{g\}\right\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{15}$$

In particular, with $f = g$, we get the quaternion version of the Plancherel formula; that is,

$$\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} = \left\|\mathcal{F}_q\{f\}\right\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{16}$$

Based on the definition of the QFT mentioned above, we consider the two-sided QLCT which is defined as follows.

Definition 4 (QLCT). Let $A_1 = (a_1, b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$ be two matrix parameters satisfying $\det(A_s) = a_s d_s - b_s c_s = 1$, $s = 1, 2$. The QLCT of a quaternion signal $f \in L^1(\mathbb{R}^2; \mathbb{H})$ is defined by

$$L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = \begin{cases} \int_{\mathbb{R}^2} K_{A_1}(x_1, \omega_1)\, f(\mathbf{x})\, K_{A_2}(x_2, \omega_2)\, d\mathbf{x}, & b_1 b_2 \neq 0 \\ \sqrt{d_1}\, e^{\mathbf{i}(c_1 d_1/2)\omega_1^2} f(d_1\omega_1, x_2)\, K_{A_2}(x_2, \omega_2), & b_1 = 0,\ b_2 \neq 0 \\ \sqrt{d_2}\, e^{\mathbf{j}(c_2 d_2/2)\omega_1^2} f(x_1, d_2\omega_2)\, K_{A_1}(x_1, \omega_1), & b_1 \neq 0,\ b_2 = 0 \\ \sqrt{d_1 d_2}\, e^{\mathbf{i}(c_1 d_1/2)\omega_1^2} f(d_1\omega_1, d_2\omega_2)\, e^{\mathbf{j}(c_2 d_2/2)\omega_2^2}, & b_1 = b_2 = 0, \end{cases} \tag{17}$$

where the kernel functions of the QLCT are given by, respectively,

$$\begin{aligned} K_{A_1}(x_1, \omega_1) &= \frac{1}{\sqrt{2\pi b_1}} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2 - (2/b_1)x_1\omega_1 + (d_1/b_1)\omega_1^2 - (\pi/2))}, \\ &\qquad b_1 \neq 0, \end{aligned} \tag{18}$$

$$\begin{aligned} K_{A_2}(x_2, \omega_2) &= \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2 - (2/b_2)x_2\omega_2 + (d_2/b_2)\omega_2^2 - (\pi/2))}, \\ &\qquad b_2 \neq 0. \end{aligned} \tag{19}$$

From the definition of the QLCT, we can see easily that when $b_1 b_2 = 0$ and $b_1 = b_2 = 0$, the QLCT of a signal is essentially a quaternion chirp multiplication. Therefore, in this work we always assume that $b_1 b_2 \neq 0$. As a special case, when $A_1 = A_2 = (a_i, b_i, c_i, d_i) = (0, 1, -1, 0)$ for $i = 1, 2$, LCT definition (17) reduces to the QFT definition. That is,

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{2\pi}} f(\mathbf{x}) e^{-\mathbf{i}\omega_1 x_1} \frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{2\pi}} e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x} \\ &= e^{-\mathbf{i}(\pi/4)} \mathscr{F}_q\{f\}(\boldsymbol{\omega}) e^{-\mathbf{j}(\pi/4)}, \end{aligned} \tag{20}$$

where $\mathscr{F}_q\{f\}$ is the QFT of f given by (13).

We need the following important result (compare to [17, 18]), which will be useful in proving Theorem 15.

Theorem 5. *The QLCT of a quaternion function $f \in L^1(\mathbb{R}^2; \mathbb{H})$ with matrix parameters $A_1 = (a_1, b_1, c_1, d_1)$ and $A_2 = (a_2, b_2, c_2, d_2)$ can be reduced to the QFT*

$$\mathscr{F}_q\{g_f\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{-\mathbf{i}x_1\omega_1} g_f(\mathbf{x}) e^{-\mathbf{j}x_2\omega_2} d\mathbf{x}, \tag{21}$$

where

$$\mathscr{F}_q\{g_f\}(\boldsymbol{\omega}) = \widetilde{\mathscr{F}}(\mathbf{b}\boldsymbol{\omega}),$$

$$g_f(\mathbf{x}) = \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{b_1}} \widetilde{f}(\mathbf{x}) \frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{b_2}}, \tag{22}$$

$$\widetilde{f}(\mathbf{x}) = e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x}) e^{\mathbf{j}(a_2/2b_2)x_2^2},$$

with

$$\begin{aligned} \widetilde{\mathscr{F}}(\boldsymbol{\omega}) &= \frac{1}{\sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{-\mathbf{i}x_1(\omega_1/b_1)} g_f(\mathbf{x}) e^{-\mathbf{j}x_2(\omega_2/b_2)} d\mathbf{x}, \\ \widetilde{\mathscr{F}}(\boldsymbol{\omega}) &= e^{-\mathbf{i}(d_1/2b_1)\omega_1^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) e^{-\mathbf{j}(d_2/2b_2)\omega_2^2}. \end{aligned} \tag{23}$$

Proof. Simple computations using Definition 4 show that

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= \frac{1}{\sqrt{2\pi b_1}} \cdot \int_{\mathbb{R}^2} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2 - (2/b_1)x_1\omega_1 + (d_1/b_1)\omega_1^2 - (\pi/2))} f(\mathbf{x}) \\ &\cdot \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2 - (2/b_2)x_2\omega_2 + (d_2/b_2)\omega_2^2 - (\pi/2))} d\mathbf{x} \\ &= \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{2\pi b_1}} e^{\mathbf{i}(d_1/2b_1)\omega_1^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}x_1(\omega_1/b_1)} \left(e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x}) \right. \\ &\cdot \left. e^{\mathbf{j}(a_2/2b_2)x_2^2}\right) \frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{2\pi b_2 \mathbf{j}}} e^{-\mathbf{j}x_2(\omega_2/b_2)} e^{\mathbf{j}(d_2/2b_2)\omega_2^2} d\mathbf{x} \\ &= \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{2\pi b_1}} e^{\mathbf{i}(d_1/2b_1)\omega_1^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}x_1(\omega_1/b_1)} \widetilde{f}(\mathbf{x}) \\ &\cdot e^{-\mathbf{j}x_2(\omega_2/b_2)} d\mathbf{x} \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{2\pi b_2}} e^{e^{-\mathbf{j}(\pi/4)}(d_2/2b_2)\omega_2^2}. \end{aligned} \tag{24}$$

Then, multiplying both sides of (24) by $e^{-\mathbf{i}(d_1/2b_1)\omega_1^2} e^{-\mathbf{j}(d_2/2b_2)\omega_2^2}$ results in

$$\begin{aligned} e^{-\mathbf{i}(d_1/2b_1)\omega_1^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) e^{-\mathbf{j}(d_2/2b_2)\omega_2^2} &= \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{2\pi b_1}} \\ &\cdot \int_{\mathbb{R}^2} e^{-\mathbf{i}x_1(\omega_1/b_1)} \widetilde{f}(\mathbf{x}) e^{-\mathbf{j}x_2(\omega_2/b_2)} d\mathbf{x} \frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{2\pi b_2}} = \frac{1}{\sqrt{2\pi}} \\ &\cdot \int_{\mathbb{R}^2} e^{-\mathbf{i}x_1(\omega_1/b_1)} \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{b_1}} \widetilde{f}(\mathbf{x}) \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{b_2}} e^{-\mathbf{j}x_2(\omega_2/b_2)} d\mathbf{x} \\ &\cdot \frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{(2\pi)^2}} \\ &\cdot \int_{\mathbb{R}^2} e^{-\mathbf{i}x_1(\omega_1/b_1)} g_f(\mathbf{x}) e^{-\mathbf{j}x_2(\omega_2/b_2)} d\mathbf{x} = \mathscr{F}_q\{g_f\} \\ &\cdot \left(\frac{\boldsymbol{\omega}}{\mathbf{b}}\right). \end{aligned} \tag{25}$$

This is the desired result. □

Theorem 6. *If $f \in L^1(\mathbb{R}^2; \mathbb{H})$ and $L^{\mathbb{H}}_{A_1,A_2}\{f\} \in L^1(\mathbb{R}^2; \mathbb{H})$, then the inverse transform of the QLCT can be derived from that of the QFT.*

Proof. Indeed, we have

$$\begin{aligned} g_f(\mathbf{x}) &= \frac{1}{\sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{\mathbf{i}x_1\omega_1} \mathscr{F}_q\{g_f\}(\boldsymbol{\omega}) e^{\mathbf{j}x_2\omega_2} d\boldsymbol{\omega} \\ &= \frac{1}{\sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{\mathbf{i}x_1\omega_1} \widetilde{\mathscr{F}}(\mathbf{b}\boldsymbol{\omega}) e^{\mathbf{j}x_2\omega_2} d\boldsymbol{\omega} \\ &= \frac{1}{b_1 b_2 \sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{\mathbf{i}x_1(\omega_1/b_1)} \widetilde{\mathscr{F}}(\boldsymbol{\omega}) e^{\mathbf{j}x_2(\omega_2/b_2)} d\boldsymbol{\omega} \\ &= \frac{1}{b_1 b_2 \sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{\mathbf{i}x_1(\omega_1/b_1)} e^{-\mathbf{i}(d_1/2b_1)\omega_1^2} L^{\mathbb{H}}_{A_1,A_2}\{f\} \\ &\cdot (\boldsymbol{\omega}) e^{-\mathbf{j}(d_2/2b_2)\omega_2^2} e^{\mathbf{j}x_2(\omega_2/b_2)} d\boldsymbol{\omega}. \end{aligned} \tag{26}$$

It means that

$$\begin{aligned} &\frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{b_1}} e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x}) e^{\mathbf{j}(a_2/2b_2)x_2^2} \frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{b_2}} \\ &= \frac{1}{b_1 b_2 \sqrt{(2\pi)^2}} \int_{\mathbb{R}^2} e^{\mathbf{i}x_1(\omega_1/b_1)} e^{-\mathbf{i}(d_1/2b_1)\omega_1^2} L^{\mathbb{H}}_{A_1,A_2}\{f\} \\ &\cdot (\boldsymbol{\omega}) e^{-\mathbf{j}(d_2/2b_2)\omega_2^2} e^{\mathbf{j}x_2(\omega_2/b_2)} d\boldsymbol{\omega}. \end{aligned} \tag{27}$$

Or, equivalently,

$$\begin{aligned} f(\mathbf{x}) &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{-\mathbf{i}(a_1/2b_1)x_1^2} e^{\mathbf{i}x_1(\omega_1/b_1)} e^{-\mathbf{i}(d_1/2b_1)\omega_1^2} e^{\mathbf{i}(\pi/4)} L^{\mathbb{H}}_{A_1,A_2}\{f\} \\ &\quad \cdot (\boldsymbol{\omega}) \frac{1}{\sqrt{2\pi b_2}} e^{-\mathbf{j}(a_2/2b_2)x_2^2} e^{\mathbf{j}x_2(\omega_2/b_2)} e^{-\mathbf{j}(d_2/2b_2)\omega_2^2} e^{-\mathbf{i}(\pi/4)} d\boldsymbol{\omega} \\ &= \int_{\mathbb{R}^2} \overline{K_{A_1}(x_1,\omega_1)} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \overline{K_{A_2}(x_2,\omega_2)}\, d\boldsymbol{\omega}, \end{aligned} \tag{28}$$

which is inverse transform of the QLCT. This proves the theorem. □

In following we give an alternative proof of Parseval formula for the QLCT (cf. [8]).

Theorem 7 (QLCT Parseval). *Two quaternion functions $f, h \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$ are related to their QLCT via the Parseval formula, given as*

$$\langle f, h\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} = \left\langle L^{\mathbb{H}}_{A_1,A_2}\{f\}, L^{\mathbb{H}}_{A_1,A_2}\{h\}\right\rangle_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{29}$$

For $f = h$, one has

$$\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} = \left\|L^{\mathbb{H}}_{A_1,A_2}\{f\}\right\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{30}$$

Proof. From Parseval formula (15), it follows that

$$\begin{aligned} \langle g_f, g_h\rangle &= \langle \mathscr{F}_q\{g_f\}, \mathscr{F}_q\{g_h\}\rangle \\ &= \mathrm{Sc}\int_{\mathbb{R}^2} \mathscr{F}_q\{g_f\}(\boldsymbol{\omega}) \overline{\mathscr{F}_q\{g_h\}(\boldsymbol{\omega})}\, d\boldsymbol{\omega} \\ &= \frac{1}{|b_1 b_2|}\mathrm{Sc}\int_{\mathbb{R}^2} \mathscr{F}_q\{g_f\}\left(\frac{\boldsymbol{\omega}}{\mathbf{b}}\right) \overline{\mathscr{F}_q\{g_h\}\left(\frac{\boldsymbol{\omega}}{\mathbf{b}}\right)}\, d\boldsymbol{\omega}. \end{aligned} \tag{31}$$

Applying the cyclic multiplication symmetry, we get

$$\begin{aligned} \langle g_f, g_h\rangle &= \frac{1}{|b_1 b_2|}\mathrm{Sc}\int_{\mathbb{R}^2} e^{-\mathbf{i}(d_1/2b_1)\omega_1^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \\ &\quad \cdot e^{-\mathbf{j}(d_2/2b_2)\omega_2^2} \overline{e^{-\mathbf{i}(d_1/2b_1)\omega_1^2} L^{\mathbb{H}}_{A_1,A_2}\{h\}(\boldsymbol{\omega}) e^{-\mathbf{j}(d_2/2b_2)\omega_2^2}}\, d\boldsymbol{\omega} \\ &= \frac{1}{|b_1 b_2|}\mathrm{Sc}\int_{\mathbb{R}^2} e^{-\mathbf{i}(d_1/2b_1)\omega_1^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \\ &\quad \cdot e^{-\mathbf{j}(d_2/2b_2)\omega_2^2} \overline{e^{-\mathbf{j}(d_2/2b_2)\omega_1^2}} \overline{L^{\mathbb{H}}_{A_1,A_2}\{h\}(\boldsymbol{\omega})} \overline{e^{-\mathbf{i}(d_1/2b_1)\omega_1^2}}\, d\boldsymbol{\omega} \\ &= \frac{1}{|b_1 b_2|}\mathrm{Sc}\int_{\mathbb{R}^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \overline{L^{\mathbb{H}}_{A_1,A_2}\{h\}(\boldsymbol{\omega})}\, d\boldsymbol{\omega}. \end{aligned} \tag{32}$$

On the other hand,

$$\begin{aligned} \langle g_f, g_h\rangle &= \mathrm{Sc}\int_{\mathbb{R}^2} g_f(\mathbf{x}) \overline{g_h(\mathbf{x})}\, d\mathbf{x} \\ &= \mathrm{Sc}\int_{\mathbb{R}^2} \frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{b_1}} \tilde{f}(\mathbf{x}) \\ &\quad \cdot \frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{b_2}} \overline{\frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{b_1}} \tilde{h}(\mathbf{x}) \frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{b_2}}}\, d\mathbf{x} \\ &= \frac{1}{|b_1 b_2|}\mathrm{Sc}\int_{\mathbb{R}^2} \tilde{f}(\mathbf{x}) \overline{\tilde{h}(\mathbf{x})}\, d\mathbf{x} \\ &= \frac{1}{|b_1 b_2|}\mathrm{Sc}\int_{\mathbb{R}^2} e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x}) \\ &\quad \cdot e^{\mathbf{j}(a_2/2b_2)x_2^2} \overline{e^{\mathbf{i}(a_1/2b_1)x_1^2} h(\mathbf{x}) e^{\mathbf{j}(a_2/2b_2)x_2^2}}\, d\mathbf{x} \\ &= \frac{1}{|b_1 b_2|}\mathrm{Sc}\int_{\mathbb{R}^2} f(\mathbf{x}) \overline{h(\mathbf{x})}\, d\mathbf{x}. \end{aligned} \tag{33}$$

The proof is complete. □

It is interesting to describe the relationship between the QLCT and QFT as shown in the following example.

Example 8. Let us now compute the QLCT of the two-dimensional Gaussian function $f(\mathbf{x}) = e^{-(k_1x_1^2 + k_2x_2^2)}$ with $k_1, k_2 > 0$.

From the definition of QLCT (17), we easily obtain

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= \frac{1}{\sqrt{2\pi b_1}} \\ &\quad \cdot \int_{\mathbb{R}^2} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2 - (2/b_1)x_1\omega + (d_1/b_1)\omega^2 - (\pi/2))} f(\mathbf{x}) \\ &\quad \cdot e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2 - (2/b_2)x_2\omega + (d_2/b_2)\omega^2 - (\pi/2))} \frac{1}{\sqrt{2\pi b_2}} d\mathbf{x} \\ &= \frac{1}{\sqrt{2\pi b_1}} \\ &\quad \cdot \int_{\mathbb{R}} e^{-k_1x_1^2} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2 - (2/b_1)x_1\omega + (d_1/b_1)\omega^2 - (\pi/2))} dx_1 \\ &\quad \cdot \frac{1}{\sqrt{2\pi b_2}} \\ &\quad \cdot \int_{\mathbb{R}} e^{-k_2x_2^2} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2 - (2/b_2)x_2\omega_2 + (d_2/b_2)\omega^2 - (\pi/2))} dx_2 \\ &= \frac{1}{\sqrt{2\pi b_1}} \\ &\quad \cdot e^{\mathbf{i}(1/2)(d_1/b_1)\omega^2} \int_{\mathbb{R}} e^{-(1/2b_1)(2k_1b_1 - \mathbf{i}a_1)x_1^2} e^{-\mathbf{i}(\omega/b_1)x_1} e^{-\mathbf{i}(\pi/2)} dx_1 \\ &\quad \cdot \frac{1}{\sqrt{2\pi b_2}} \\ &\quad \cdot e^{\mathbf{j}(1/2)(d_2/b_2)\omega^2} \int_{\mathbb{R}} e^{-(1/2b_2)(2k_2b_2 - \mathbf{j}a_2)x_2^2} e^{-\mathbf{j}(\omega/b_2)x_2} e^{-\mathbf{j}(\pi/2)} dx_2. \end{aligned} \tag{34}$$

Using the QFT of the Gaussian function,

$$\begin{aligned}\mathcal{F}_q\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} e^{-(k_1x_1^2+k_2x_2^2)} e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x} \\ &= \frac{\pi}{\sqrt{k_1k_2}} e^{-(\omega_1^2/4k_1+\omega_2^2/4k_2)}.\end{aligned} \tag{35}$$

We immediately obtain

$$\begin{aligned}L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} \\ &\cdot e^{\mathbf{i}(1/2)(d_1/b_1)\omega_1^2} \sqrt{\frac{2\pi b_1}{(2k_1b_1 - \mathbf{i}a_1)}} e^{-\omega_1^2/2b_1(2k_1b_1-\mathbf{i}a_1)} \\ &\cdot \frac{1}{\sqrt{2\pi b_1}} \\ &\cdot e^{\mathbf{j}(1/2)(d_2/b_2)\omega_2^2} \sqrt{\frac{2\pi b_2}{(2k_2b_2 - \mathbf{j}a_2)}} e^{-\omega_2^2/2b_2(2k_2b_2-\mathbf{j}a_2)} \\ &= \frac{1}{\sqrt{a_1 + 2k_1b_1}} e^{(\omega_1^2/2)((c_1+2k_1\mathbf{i}d_1)/(2k_1b_1-\mathbf{i}a_1))} \\ &\cdot \frac{1}{\sqrt{a_2 + 2k_2b_2}} e^{(\omega_2^2/2)((c_2+2k_2\mathbf{j}d_2)/(2k_2b_2-\mathbf{j}a_2))}.\end{aligned} \tag{36}$$

3. Properties of the QLCT

In this section we present useful properties of the QLCT in detail. We see that the results are generalizations of the properties of the LCT [5, 19]. In [9], the authors derived the asymptotic behavior of the QLCT. In the following, we shall provide a different proof of the results using the QLCT kernel properties.

3.1. Asymptotic Behavior of the QLCT. Like the classical Fourier transform, the Riemann-Lebesgue lemma is also valid for the QLCT, expressed as follows.

Theorem 9 (Riemann-Lebesgue lemma). *Suppose that $f \in L^1(\mathbb{R}^2;\mathbb{H})$. Then*

$$\begin{aligned}\lim_{|\omega_1|\to\infty} \left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right| &= 0, \\ \lim_{|\omega_2|\to\infty} \left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right| &= 0.\end{aligned} \tag{37}$$

Proof. It is not difficult to see that

$$\begin{aligned}e^{-\mathbf{i}(\omega_1x_1/b_1)} &= -e^{-\mathbf{i}(\omega_1/b_1)(x_1+b_1\pi/\omega_1)}, \\ e^{-\mathbf{j}(\omega_2x_2/b_2)} &= -e^{-\mathbf{j}(\omega_2/b_2)(x_2+b_2\pi/\omega_2)}.\end{aligned} \tag{38}$$

Now applying (38) gives

$$\begin{aligned}L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2-\pi/2)} f(\mathbf{x}) \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2-\pi/2)} d\mathbf{x} \\ &= -\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)(x_1+b_1\pi/\omega_1)\omega_1+(d_1/b_1)\omega_1^2-\pi/2)} f(\mathbf{x}) \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2-\pi/2)} d\mathbf{x}.\end{aligned} \tag{39}$$

Therefore, by making the change of variable $x_1 + b_1\pi/\omega_1 = t_1$ in the above identity, we immediately obtain

$$\begin{aligned}L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) &= -\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\mathbf{i}(1/2)((a_1/b_1)(t_1-b_1\pi/\omega_1)^2-(2/b_1)t_1\omega_1+(d_1/b_1)\omega_1^2-\pi/2)} f\left(t_1 - \frac{b_1\pi}{\omega_1}, t_2\right) \\ &\cdot \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)t_2^2-(2/b_1)t_2\omega_1+(d_1/b_2)\omega_2^2-\pi/2)} d\mathbf{t} = \frac{1}{2}\left[\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\mathbf{i}(1/2)((a_1/b_1)t_1^2-(2/b_1)t_1\omega_1+(d_1/b_1)\omega_1^2-\pi/2)} f(\mathbf{t})\right. \\ &\cdot \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)t_2^2-(2/b_2)t_2\omega_2+(d_2/b_2)\omega_2^2-\pi/2)} d\mathbf{x} \\ &\left. - \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\mathbf{i}(1/2)((a_1/b_1)(t_1-b_1\pi/\omega_1)^2-(2/b_1)t_1\omega_1+(d_1/b_1)\omega_1^2-\pi/2)} f\left(t_1 - \frac{b_1\pi}{\omega_1}, t_2\right) \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)t_2^2-(2/b_2)t_2\omega_2+(d_2/b_2)\omega_2^2-\pi/2)} d\mathbf{t}\right]\end{aligned}$$

$$= \frac{1}{2}\left[\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\mathbf{i}(1/2)((a_1/b_1)t_1^2-(2/b_1)t_1\omega_1+(d_1/b_1)\omega_1^2-\pi/2)} \left(f(\mathbf{t}) - e^{\mathbf{i}(a_1/2b_1)(-2t_1b_1\pi/\omega_1+(b_1\pi/\omega_1)^2)} f\left(t_1 - \frac{b_1\pi}{\omega_1}, t_2\right)\right)\right.$$
$$\left. \cdot \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)t_2^2-(2/b_2)t_2\omega_2+(d_{2s}/b_2)\omega_2^2-\pi/2)} d\mathbf{t}\right]. \tag{40}$$

This means that

$$\lim_{|\omega_1|\to\infty} \left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right| \le \frac{1}{|4\pi b_1 b_2|} \lim_{|\omega_1|\to\infty} \int_{\mathbb{R}^2} \left| f(\mathbf{t}) - e^{\mathbf{i}(a_1/2b_1)(-2t_1b_1\pi/\omega_1+(b_1\pi/\omega_1)^2)} f\left(t_1 - \frac{b_1\pi}{\omega_1}, t_2\right)\right| d\mathbf{t} = 0. \tag{41}$$

Similarly we can prove

$$\lim_{|\omega_2|\to\infty} \left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right| = 0. \tag{42}$$

□

Theorem 10 (continuity). *If $f \in L^1(\mathbb{R}^2;\mathbb{H})$, then $L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})$ is continuous on $\mathbb{R}^2$.*

Proof. Simple computations show that

$$\begin{aligned}
&\left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}+\mathbf{h}) - L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right| = \left|\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\mathbf{i}(1/2)((a_1/b_1)(x_1+h_1)^2-(2/b_1)(x_1+h_1)\omega_1+(d_1/b_1)\omega_1^2-\pi/2)} f(\mathbf{x})\right. \\
&\cdot \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)(x_2+h_2)^2-(2/b_2)(x_2+h_2)\omega_2+(d_2/b_2)\omega_2^2-\pi/2)} d\mathbf{x} - \frac{1}{\sqrt{2\pi b_1}} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2-\pi/2)} f(\mathbf{x}) \frac{1}{\sqrt{2\pi b_2}} \\
&\left.\cdot e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2-\pi/2)} d\mathbf{x}\right| \le \int_{\mathbb{R}^2} \left| \frac{1}{\sqrt{2\pi b_1}}\right. \\
&\cdot e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2-\pi/2)} \left(e^{\mathbf{i}(1/2)((a_1/b_1)(2x_1h_1+h_1^2)-(2/b_1)\omega_1h_1)} f(\mathbf{x})\, e^{\mathbf{j}(1/2)((a_2/b_2)(2x_2h_2+h_2^2)-(2/b_2)\omega_2h_2)} - f(\mathbf{x})\right) \\
&\left.\cdot \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2-\pi/2)}\right| d\mathbf{x} = \frac{1}{|2\pi b_1 b_2|} \\
&\cdot \int_{\mathbb{R}^2} \left|\left(e^{\mathbf{i}(1/2)((a_1/b_1)(2x_1h_1+h_1^2)-(2/b_1)\omega_1h_1)} f(\mathbf{x})\, e^{\mathbf{j}(1/2)((a_2/b_2)(2x_2h_2+h_2^2)-(2/b_2)\omega_2h_2)} - f(\mathbf{x})\right)\right| d\mathbf{x} \le \frac{1}{|\pi b_1 b_2|} \int_{\mathbb{R}^2} |f(\mathbf{x})|\, d\mathbf{x}.
\end{aligned} \tag{43}$$

By the Lebesgue dominated convergence theorem, we may conclude that

$$\left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}+\mathbf{h}) - L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right| \longrightarrow 0 \tag{44}$$

when $\mathbf{h} \to 0$. This proves that $L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})$ is continuous on $\mathbb{R}^2$. Again since (43) is independent of $\boldsymbol{\omega}$, $L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})$ is, in fact, uniformly continuous on $\mathbb{R}^2$. □

3.2. Useful Properties of the QLCT. Due to the noncommutativity of the kernel of the QLCT, we only have a left linearity property with specific constants

$$\alpha, \beta \in \{q \mid q = q_0 + \mathbf{i}q_1,\ q_0, q_1 \in \mathbb{R}\}, \tag{45}$$

which is

$$L^{\mathbb{H}}_{A_1,A_2}\{\alpha f + \beta g\}(\boldsymbol{\omega}) = \alpha L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) + \beta L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega}), \tag{46}$$

and a right linearity property with specific constants

$$\alpha', \beta' \in \{q \mid q = q_0 + \mathbf{j}q_2,\ q_0, q_2 \in \mathbb{R}\}. \tag{47}$$

Theorem 11 (shift property). *Given a quaternion function $f \in L^2(\mathbb{R}^2;\mathbb{H})$, let $\tau_{\mathbf{k}} f(\mathbf{x})$ denote the shifted (translated) function defined by $\tau_{\mathbf{k}} f(\mathbf{x}) = f(\mathbf{x}-\mathbf{k})$, where $\mathbf{k} \in \mathbb{R}^2$. Then one gets*

$$L^{\mathbb{H}}_{A_1,A_2}\{\tau_{\mathbf{k}} f\}(\boldsymbol{\omega}) = e^{-\mathbf{i}a_1c_1k_1^2/2+\mathbf{i}c_1k_1\omega_1} L^{\mathbb{H}}_{A_1,A_2}\{f\} \cdot (\omega_1 - a_1k_1, \omega_2 - a_2k_2) \cdot e^{-\mathbf{j}a_2c_2k_2^2/2+\mathbf{j}c_2k_2\omega_2}. \tag{48}$$

Proof. Taking into account the definition of QLCT (17), we get

$$L_{A_1,A_2}^{\mathbb{H}}\{\tau_{\mathbf{k}} f\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{2\pi b_1}} \cdot \int_{\mathbb{R}^2} e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2-\pi/2)} f(\mathbf{x}-\mathbf{k}) \cdot \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2-\pi/2)} d\mathbf{x}. \tag{49}$$

By making the change of a variable $\mathbf{x}-\mathbf{k}=\mathbf{m}$, we easily obtain

$$\begin{aligned} L_{A_1,A_2}^{\mathbb{H}}\{\tau_{\mathbf{k}} f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\mathbf{i}(1/2)((a_1/b_1)(m_1+k_1)^2-(2/b_1)(m_1+k_1)\omega_1+(d_1/b_1)\omega_1^2-\pi/2)} f(\mathbf{m}) \\ &\cdot \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)(m_2+k_2)^2-(2/b_2)(m_2+k_2)\omega_2+(d_2/b_2)\omega_2^2-\pi/2)} d\mathbf{m} = \frac{1}{\sqrt{2\pi b_1}} \\ &\cdot \int_{\mathbb{R}^2} e^{\mathbf{i}(1/2)((a_1/b_1)m_1^2-(2/b_1)m_1(\omega_1-k_1a_1)+(d_1/b_1)\omega_1^2-\pi/2)} e^{\mathbf{i}((1/2)(a_1/b_1)k_1^2)} e^{\mathbf{i}(-(1/2)(2k_1\omega_1/b_1))} f(\mathbf{m}) \\ &\cdot \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)m_2^2-(2/b_2)m_2(\omega_2-k_2a_2)+(d_2/b_2)\omega_2^2-\pi/2)} e^{\mathbf{j}((1/2)(a_2/b_2)k_2^2)} e^{\mathbf{j}(-(1/2)(2k_2\omega_2/b_2))} d\mathbf{m}. \end{aligned} \tag{50}$$

Therefore, we further get

$$\begin{aligned} L_{A_1,A_2}^{\mathbb{H}}\{\tau_{\mathbf{k}} f\}(\boldsymbol{\omega}) &= \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} e^{\mathbf{i}(1/2)((a_1/b_1)m_1^2-(2/b_1)m_1(\omega_1-k_1a_1)+(d_1/b_1)(\omega_1-k_1a_1+k_1a_1)^2-\pi/2)} e^{\mathbf{i}((1/2)(a_1/b_1)k_1^2)} e^{\mathbf{i}(-(1/2)(2k_1\omega_1/b_1))} f(\mathbf{m}) \\ &\cdot \sqrt{2\pi b_2} e^{\mathbf{j}(1/2)((a_2/b_2)m_2^2-(2/b_2)m_2(\omega_2-k_2a_2)+(d_2/b_2)(\omega_2-k_2a_2+k_2a_2)^2-\pi/2)} e^{\mathbf{j}((1/2)(a_1/b_1)k_1^2)} e^{\mathbf{j}(-(1/2)(2k_2\omega_2/b_2))} d\mathbf{m} \\ &= e^{\mathbf{i}(1/2)(d_1/b_1)(2(\omega_1-k_1a_1)k_1a_1+(k_1a_1)^2)} e^{\mathbf{i}((1/2)(a_1/b_1)k_1^2)} e^{\mathbf{i}(-(1/2)(2k_1\omega_1/b_1))} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\mathbf{i}(1/2)((a_1/b_1)m_1^2-(2/b_1)m_1(\omega_1-k_1a_1)+(d_1/b_1)(\omega_1-k_1a_1)^2)} f(\mathbf{m}) \\ &\cdot \frac{1}{\sqrt{2\pi b_2}} e^{\mathbf{j}(1/2)((a_2/b_2)m_2^2-(2/b_2)m_2(\omega_2-k_2a_2)+(d_2/b_2)(\omega_2-k_2a_2)^2)} d\mathbf{m} e^{\mathbf{j}(1/2)(d_2/b_2)(2(\omega_2-k_2a_2)k_2a_2+(k_2a_2)^2)} e^{\mathbf{j}((1/2)(a_2/b_2)k_2^2)} e^{\mathbf{j}(-(1/2)(2k_2\omega_2/b_2))}. \end{aligned} \tag{51}$$

Applying the definition of the QLCT (17), the above expression can be rewritten in the form

$$\begin{aligned} L_{A_1,A_2}^{\mathbb{H}}\{\tau_{\mathbf{k}} f\}(\boldsymbol{\omega}) &= e^{\mathbf{i}(1/2)(d_1/b_1)(2(\omega_1-k_1a_1)k_1a_1+(k_1a_1)^2)} e^{\mathbf{i}((1/2)(a_1/b_1)k_1^2)} e^{\mathbf{i}(-(1/2)(2k_1\omega_1/b_1))} L_{A_1,A_2}^{\mathbb{H}}\{f\}(\omega_1-a_1k_1,\omega_2-a_2k_2) \\ &\cdot e^{\mathbf{j}(1/2)(d_2/b_2)(2(\omega_1-k_2a_2)k_2a_2+(k_2a_2)^2)} e^{\mathbf{j}((1/2)(a_2/b_2)k_2^2)} e^{\mathbf{j}(-(1/2)(2k_2\omega_2/b_2))}. \end{aligned} \tag{52}$$

We notice that

$$\begin{aligned} & e^{\mathbf{i}(1/2)(d_1/b_1)(2(\omega_1-k_1a_1)k_1a_1+(k_1a_1)^2)} e^{\mathbf{i}((1/2)(a_1/b_1)k_1^2)} e^{\mathbf{i}(-(1/2)(2k_1\omega_1/b_1))} \\ &\quad = e^{\mathbf{i}k_1\omega_1(d_1a_1/b_1-1/b_1)} e^{-\mathbf{i}(1/2)k_1^2a_1(d_1a_1/b_1-1/b_1)}, \\ & e^{\mathbf{j}(1/2)(d_2/b_2)(2(\omega_2-k_2a_2)k_2a_2+(k_2a_2)^2)} e^{\mathbf{j}((1/2)(a_2/b_2)k_2^2)} e^{\mathbf{j}(-(1/2)(2k_2\omega_2/b_2))} \\ &\quad = e^{\mathbf{j}k_2\omega_2(d_2a_2/b_2-1/b_2)} e^{-\mathbf{j}(1/2)k_2^2a_2(d_2a_2/b_2-1/b_2)}. \end{aligned} \tag{53}$$

Because $a_id_i - b_ic_i = 1$, then $d_ia_i/b_i - 1/b_i = c_i$ for $i = 1, 2$. It means that we get

$$\begin{aligned} & e^{\mathbf{i}(1/2)(d_1/b_1)(2(\omega_1-k_1a_1)k_1a_1+(k_1a_1)^2)} e^{\mathbf{i}((1/2)(a_1/b_1)k_1^2)} e^{\mathbf{i}(-(1/2)(2k_1\omega_1/b_1))} \\ &\quad = e^{\mathbf{i}k_1\omega_1c_1} e^{-\mathbf{i}(a_1k_1^2/2)c_1}. \end{aligned} \tag{54}$$

By the above equalities, we finally arrive at

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{\tau_{\mathbf{k}}f\}(\boldsymbol{\omega}) &= e^{\mathbf{i}k_1\omega_1c_1}e^{-\mathbf{i}(a_1k_1^2/2)c_1}L^{\mathbb{H}}_{A_1,A_2}\{f\} \\ &\cdot(\omega_1-a_1k_1,\omega_2-a_2k_2) \\ &\cdot e^{\mathbf{j}k_2\omega_2c_2}e^{-\mathbf{j}(a_2k_2^2/2)c_2}. \end{aligned} \tag{55}$$

This completes the proof of theorem. □

Next, we are concerned with the behavior of the QLCT under modulation.

Theorem 12 (modulation property). *Let $\mathbb{M}_{\boldsymbol{\omega}_0}f$ be modulation operator defined by $\mathbb{M}_{\boldsymbol{\omega}_0}f(\mathbf{x}) = e^{\mathbf{i}x_1u_0}f(\mathbf{x})e^{\mathbf{j}x_2v_0}$ with $\boldsymbol{\omega}_0 = u_0\mathbf{e}_1+v_0\mathbf{e}_2$. Then*

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{\mathbb{M}_{\boldsymbol{\omega}_0}f\}(\boldsymbol{\omega}) &= L^{\mathbb{H}}_{A_1,A_2}\{e^{\mathbf{i}x_1u_0}f(\mathbf{x})e^{\mathbf{j}x_2v_0}\}(\boldsymbol{\omega}) \\ &= e^{-\mathbf{i}b_1d_1u_0^2/2+\mathbf{i}d_1u_0\omega_1}L^{\mathbb{H}}_{A_1,A_2}\{f\} \\ &\cdot(\omega_1-u_0b_1,\omega_2-v_0b_2)\,e^{-\mathbf{j}b_2d_2v_0^2/2+\mathbf{j}d_2v_0\omega_2}. \end{aligned} \tag{56}$$

Proof. From Definition 4, it follows that

$$\begin{aligned} &L^{\mathbb{H}}_{A_1,A_2}\{\mathbb{M}_{\boldsymbol{\omega}_0}f\}(\boldsymbol{\omega}) \\ &= \int_{\mathbb{R}^2}\frac{1}{\sqrt{2\pi b_1}}e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2-\pi/2)}e^{\mathbf{i}x_1u_0}f(\mathbf{x}) \\ &\cdot e^{\mathbf{j}x_2v_0}\frac{1}{\sqrt{2\pi b_2}}e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2-\pi/2)}d\mathbf{x} \\ &= \frac{1}{\sqrt{2\pi b_1}} \\ &\cdot\int_{\mathbb{R}^2}e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1\omega_1+(d_1/b_1)\omega_1^2+2u_0x_1-\pi/2)}f(\mathbf{x}) \\ &\cdot\frac{1}{\sqrt{2\pi b_2}}e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2\omega_2+(d_2/b_2)\omega_2^2+2v_0x_2-\pi/2)}d\mathbf{x} \\ &= \frac{1}{\sqrt{2\pi b_1}} \\ &\cdot\int_{\mathbb{R}^2}e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1(\omega_1-u_0b_1)+(d_1/b_1)\omega_1^2-\pi/2)}f(\mathbf{x}) \\ &\cdot\frac{1}{\sqrt{2\pi b_2}}e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2(\omega_2-v_0b_2)+(d_2/b_2)\omega_2^2-\pi/2)}d\mathbf{x}. \end{aligned} \tag{57}$$

Subsequent calculations reveal that

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{\mathbb{M}_{\boldsymbol{\omega}_0}f\}(\boldsymbol{\omega}) &= \frac{1}{\sqrt{2\pi b_1}}\int_{\mathbb{R}^2}e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1(\omega_1-u_0b_1)+(d_1/b_1)((\omega_1-u_0b_1)+u_0b_1)^2-\pi/2)}f(\mathbf{x}) \\ &\cdot\frac{1}{\sqrt{2\pi b_2}}e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2(\omega_2-v_0b_2)+(d_2/b_2)((\omega_2-v_0b_2)+v_0b_2)^2-\pi/2)}d\mathbf{x} = \frac{1}{\sqrt{2\pi b_1}} \\ &\cdot\int_{\mathbb{R}^2}e^{\mathbf{i}(1/2)((a_1/b_1)x_1^2-(2/b_1)x_1(\omega_1-u_0b_1)+(d_1/b_1)((\omega_1-u_0b_1)^2+2(\omega_1-u_0b_1)u_0b_1+u_0^2b_1^2)-\pi/2)}f(\mathbf{x})\frac{1}{\sqrt{2\pi b_2}} \\ &\cdot e^{\mathbf{j}(1/2)((a_2/b_2)x_2^2-(2/b_2)x_2(\omega_2-v_0b_2)+(d_2/b_2)((\omega_2-v_0b_2)^2+2(\omega_2-v_0b_2)v_0b_2+v_0^2b_2^2)-\pi/2)}d\mathbf{x}. \end{aligned} \tag{58}$$

Hence,

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{\mathbb{M}_{\boldsymbol{\omega}_0}f\}(\boldsymbol{\omega}) &= e^{\mathbf{i}(\omega_1-u_0b_1)u_0d_1+\mathbf{i}d_1u_0^2b_1/2}L^{\mathbb{H}}_{A_1,A_2}(\omega_1 \\ &-u_0b_1,\omega_2-v_0b_2)\,e^{\mathbf{j}(\omega_2-v_0b_2)v_0d_2+\mathbf{j}d_2v_0^2b_2/2} \\ &= e^{\mathbf{i}\omega_1u_0d_1}e^{-\mathbf{i}((2b_1d_1u_0^2-b_1d_1u_0^2)/2)}L^{\mathbb{H}}_{A_1,A_2}(\omega_1-u_0b_1,\omega_2 \\ &-v_0b_2)\,e^{\mathbf{j}\omega_2v_0d_2}e^{-\mathbf{j}((2b_2d_2v_0^2-b_2d_2v_0^2)/2)} \\ &= e^{\mathbf{i}\omega_1u_0d_1}e^{-\mathbf{i}(b_1d_1u_0^2/2)}L^{\mathbb{H}}_{A_1,A_2}(\omega_1-u_0b_1,\omega_2-v_0b_2) \\ &\cdot e^{\mathbf{j}\omega_2v_0d_2}e^{-\mathbf{j}(b_2d_2v_0^2/2)}. \end{aligned} \tag{59}$$

This is desired result. □

Theorem 13 (time-frequency shift). *If quaternion function $f\in L^2(\mathbb{R}^2;\mathbb{H})$, then one gets*

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{\mathbb{M}_{\boldsymbol{\omega}_0}\tau_{\mathbf{k}}f\}(\boldsymbol{\omega}) &= L^{\mathbb{H}}_{A_1,A_2}\{e^{\mathbf{i}x_1u_0}f(\mathbf{x}-\mathbf{k})e^{\mathbf{j}x_2v_0}\} \\ &\cdot(\boldsymbol{\omega}) \\ &= e^{-\mathbf{i}(a_1c_1k_1^2+b_1d_1u_0^2)/2+\mathbf{i}(c_1k_1+d_1u_0)\omega_1-\mathbf{i}b_1c_1k_1u_0}L^{\mathbb{H}}_{A_1,A_2}\{f\} \\ &\cdot(\omega_1-a_1k_1-u_0b_1,\omega_2-a_2k_2-v_0b_2) \\ &\cdot e^{-\mathbf{j}(a_2c_2k_2^2+b_2d_2v_0^2)/2+\mathbf{j}(c_2k_2+d_2v_0)\omega_2-\mathbf{j}b_2c_2k_2v_0}. \end{aligned} \tag{60}$$

Proof. The proof directly follows from two previous theorems. □

The above properties of the QLCT are summarized in Table 1.

TABLE 1: Properties of the QLCT of $f, g \in L^2(\mathbb{R}^2;\mathbb{H})$, where $\alpha, \beta, \alpha', \beta' \in \mathbb{H}$ are constants and $\boldsymbol{\omega}_0 = u_0\mathbf{e}_1 + v_0\mathbf{e}_2 \in \mathbb{R}^2$.

Property	Quaternion func.	QLCT
Left linearity	$\alpha f + \beta g$	$\alpha L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) + \beta L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})$
Right linearity	$f\alpha' + g\beta'$	$L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\,\alpha' + L^{\mathbb{H}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\,\beta'$
Shift	$f(\mathbf{x}-\mathbf{k})$	$e^{-\mathbf{i}a_1c_1k_1^2/2+\mathbf{i}c_1k_1\omega_1} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\omega_1 - a_1k_1, \omega_2 - a_2k_2)\, e^{-\mathbf{j}a_2c_2k_2^2/2+\mathbf{j}c_2k_2\omega_2}$
Modulation	$e^{\mathbf{i}x_1u_0} f(\mathbf{x})\, e^{\mathbf{j}x_2v_0}$	$e^{-\mathbf{i}b_1d_1u_0^2/2+\mathbf{i}d_1u_0\omega_1} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\omega_1 - u_0b_1, \omega_2 - v_0b_2)\, e^{-\mathbf{j}b_2d_2v_0^2/2+\mathbf{j}d_2v_0\omega_2}$
Time-frequency	$e^{\mathbf{i}x_1u_0} f(\mathbf{x}-\mathbf{k})\, e^{\mathbf{j}x_2v_0}$	$e^{-\mathbf{i}(a_1c_1k_1^2+b_1d_1u_0^2)/2+\mathbf{i}(c_1k_1+d_1u_0)\omega_1-\mathbf{i}b_1c_1k_1u_0} L^{\mathbb{H}}_{A_1,A_2}\{f\}(\omega_1 - a_1k_1 - u_0b_1, \omega_2 - a_2k_2 - v_0b_2)$ $\cdot e^{-\mathbf{j}(a_2c_2k_2^2+b_2d_2v_0^2)/2+\mathbf{j}(c_2k_2+d_2v_0)\omega_2-\mathbf{j}b_2c_2k_2v_0}$
Gaussian function	$e^{-(k_1x_1^2+k_2x_2^2)}$	$\dfrac{1}{\sqrt{a_1+2k_1b_1\mathbf{i}}} e^{(\omega_1^2/2)((c_1+2k_1\mathbf{i}d_1)/(2k_1b_1-\mathbf{i}a_1))} \dfrac{1}{\sqrt{a_2+2k_2b_2\mathbf{j}}} e^{(\omega_2^2/2)((c_2+2k_2\mathbf{j}d_2)/(2k_2b_2-\mathbf{j}a_2))}$

4. Heisenberg Uncertainty Principle for QLCT

The classical uncertainty principle of harmonic analysis states that a nontrivial function and its Fourier transform cannot be sharply localized simultaneously. In quantum mechanics, the uncertainty principle asserts that one cannot at the same time be certain of the position and of the velocity of an electron (or any particle) [20]. Let us now give an alternative proof of the Heisenberg type uncertainty principle for the QLCT, which is recently studied in [8] (the uncertainty principle of the QCT was proved using the exponential form of a 2D quaternion function and proposed proof of this paper uses the relationship between the QFT and QLCT). However, before proceeding with the statement of this main result, we need to introduce the component-wise uncertainty principle for the QFT as follows (see [12] for more details).

Theorem 14 (the QFT component-wise uncertainty principle). *Suppose that* $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$. *If* $\partial f/\partial x_k$ *and* $\omega_k(\partial f/\partial x_k) \in L^2(\mathbb{R}^2;\mathbb{H})$, *then one has*

$$\int_{\mathbb{R}^2} x_k^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_k^2 \left|\mathcal{F}_q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \geq \frac{1}{4}\left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\right)^2, \quad k = 1, 2. \tag{61}$$

The generalization of the above uncertainty principle to the the QLCT domain is given by the following theorem (for more detailed information, see [8]).

Theorem 15 (the QLFT component-wise uncertainty principle). *Assume that* $f \in L^1(\mathbb{R}^2;\mathbb{H}) \cap L^2(\mathbb{R}^2;\mathbb{H})$, $\partial f/\partial x_k \in L^2(\mathbb{R}^2;\mathbb{H})$ *and that* $L^{\mathbb{H}}_{A_1,A_2}\{f\}$, $\omega_k L^{\mathbb{H}}_{A_1,A_2}\{f\} \in L^2(\mathbb{R}^2;\mathbb{H})$, $k = 1, 2$. *Then, the following inequality holds:*

$$\int_{\mathbb{R}^2} x_k^2 |f(\mathbf{x})|^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_l^2 \left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \geq \frac{b_k^2}{4}\left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x}\right)^2, \quad k = 1, 2. \tag{62}$$

Proof. Substituting the quaternion function f by g_f defined by (21) on both sides of (62), we easily obtain

$$\int_{\mathbb{R}^2} x_k^2 \left|g_f(\mathbf{x})\right|^2 d\mathbf{x} \int_{\mathbb{R}^2} \omega_k^2 \left|\mathcal{F}_q\{g_f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega} \geq \frac{1}{4}\left(\int_{\mathbb{R}^2} \left|g_f(\mathbf{x})\right|^2 d\mathbf{x}\right)^2. \tag{63}$$

Now setting $\boldsymbol{\omega} = \boldsymbol{\omega}/\mathbf{b}$, we further have

$$\begin{aligned}\int_{\mathbb{R}^2} x_k^2 \left|\frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{b_1}}\tilde{f}(\mathbf{x})\frac{e^{-\mathbf{j}(\pi/4)}}{\sqrt{b_2}}\right|^2 d\mathbf{x} &\cdot \int_{\mathbb{R}^2} \frac{\omega_k^2}{b_k^2}\left|\mathcal{F}_q\{g_f\}\left(\frac{\boldsymbol{\omega}}{\mathbf{b}}\right)\right|^2 d\frac{\boldsymbol{\omega}}{\mathbf{b}} \\ &\geq \frac{1}{4}\left(\int_{\mathbb{R}^2}\left|\frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{b_1}}\tilde{f}(\mathbf{x})\frac{e^{-\mathbf{i}(\pi/4)}}{\sqrt{b_2}}\right|^2 d\mathbf{x}\right)^2,\end{aligned} \tag{64}$$

and thus

$$\int_{\mathbb{R}^2} \frac{x_k^2}{|b_1b_2|^2}\left|\tilde{f}(\mathbf{x})\right|^2 d\mathbf{x} \int_{\mathbb{R}^2} \frac{\omega_k^2}{b_k^2}\left|\mathcal{F}_q\{g_f\}\left(\frac{\boldsymbol{\omega}}{\mathbf{b}}\right)\right|^2 d\boldsymbol{\omega} \geq \frac{1}{4|b_1b_2|^2}\left(\int_{\mathbb{R}^2}\left|\tilde{f}(\mathbf{x})\right|^2 d\mathbf{x}\right)^2. \tag{65}$$

Hence,

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{x_k^2}{|b_1b_2|^2}\left|e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x})\, e^{\mathbf{j}(a_2/2b_2)x_2^2}\right|^2 d\mathbf{x} &\cdot \int_{\mathbb{R}^2} \frac{\omega_k^2}{b_k^2}\left|\mathcal{F}_q\{g_f\}\left(\frac{\boldsymbol{\omega}}{\mathbf{b}}\right)\right|^2 d\boldsymbol{\omega} \\ &\geq \frac{1}{4|b_1b_2|^2}\left(\int_{\mathbb{R}^2}\left|e^{\mathbf{i}(a_1/2b_1)x_1^2} f(\mathbf{x})\, e^{\mathbf{j}(a_2/2b_2)x_2^2}\right|^2 d\mathbf{x}\right)^2.\end{aligned} \tag{66}$$

By inserting (23) into (66), we immediately obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}\frac{x_k^2}{|b_1b_2|^2}\left|e^{\mathbf{i}(a_1/2b_1)x_1^2}f(\mathbf{x})\,e^{\mathbf{j}(a_2/2b_2)x_2^2}\right|^2 d\mathbf{x}\\&\cdot\int_{\mathbb{R}^2}\frac{\omega_k^2}{b_k^2}\left|e^{-\mathbf{i}(d_1/2b_1)\omega_1^2}L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right.\\&\left.\cdot\, e^{-\mathbf{j}(d_2/2b_2)\omega_2^2}\right|^2 d\boldsymbol{\omega}\\&\geq\frac{1}{4|b_1b_2|^2}\left(\int_{\mathbb{R}^2}\left|e^{\mathbf{i}(a_1/2b_1)x_1^2}f(\mathbf{x})\,e^{\mathbf{j}(a_2/2b_2)x_2^2}\right|^2 d\mathbf{x}\right)^2.\end{aligned}\tag{67}$$

Simplifying it gives

$$\begin{aligned}&\int_{\mathbb{R}^2}x_k^2|f(\mathbf{x})|^2 d\mathbf{x}\int_{\mathbb{R}^2}\omega_k^2\left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\\&\quad\geq\frac{b_k^2}{4}\left(\int_{\mathbb{R}^2}|f(\mathbf{x})|^2 d\mathbf{x}\right)^2.\end{aligned}\tag{68}$$

This finishes the proof of theorem. □

It is not difficult to check that directional uncertainty principle for the QFT takes the following form (cf. [21, 22]).

Theorem 16. *Suppose that $f \in L^1(\mathbb{R}^2;\mathbb{H})\cap L^2(\mathbb{R}^2;\mathbb{H})$. If $\partial f/\partial x_k$ and $\omega_k(\partial f/\partial x_k)\in L^2(\mathbb{R}^2;\mathbb{H})$, then one has*

$$\begin{aligned}&\int_{\mathbb{R}^2}|\mathbf{x}|^2|f(\mathbf{x})|^2 d\mathbf{x}\int_{\mathbb{R}^2}|\boldsymbol{\omega}|^2\left|\mathscr{F}_q\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\\&\quad\geq\left(\int_{\mathbb{R}^2}|f(\mathbf{x})|^2 d\mathbf{x}\right)^2.\end{aligned}\tag{69}$$

Proceeding as in the proof of Theorem 15, we obtain the QLCT directional uncertainty principle as follows.

Theorem 17. *Suppose that $f \in L^1(\mathbb{R}^2;\mathbb{H})\cap L^2(\mathbb{R}^2;\mathbb{H})$ and $L^{\mathbb{H}}_{A_1,A_2}\{f\}$ and $|\boldsymbol{\omega}|^2L^{\mathbb{H}}_{A_1,A_2}\{f\}\in L^2(\mathbb{R}^2;\mathbb{H})$. Then the following inequality is satisfied:*

$$\begin{aligned}&\int_{\mathbb{R}^2}|\mathbf{x}|^2|f(\mathbf{x})|^2 d\mathbf{x}\int_{\mathbb{R}^2}|\boldsymbol{\omega}|^2\left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}\\&\quad\geq|\mathbf{b}|^2\left(\int_{\mathbb{R}^2}|f(\mathbf{x})|^2 d\mathbf{x}\right)^2.\end{aligned}\tag{70}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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