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Research Article

Boundary Layers and Shock Profiles for the Broadwell Model

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We consider the existence of nonlinear boundary layers and the typically nonlinear problem of existence of shock profiles for the Broadwell model, which is a simplified discrete velocity model for the Boltzmann equation. We find explicit expressions for the nonlinear boundary layers and the shock profiles. In spite of the few velocities used for the Broadwell model, the solutions are (at least partly) in qualitatively good agreement with the results for the discrete Boltzmann equation, that is the general discrete velocity model, and the full Boltzmann equation.

1. Introduction

The Boltzmann equation (BE) is a fundamental equation in kinetic theory. Half-space problems for the BE are of great importance in the study of the asymptotic behavior of the solutions of boundary value problems of the BE for small Knudsen numbers [1, 2] and have been extensively studied both for the full BE [3, 4] and for the discrete Boltzmann equation (DBE) [5–8]. The half-space problems provide the boundary conditions for the fluid-dynamic-type equations and Knudsen-layer corrections to the solution of the fluid-dynamic-type equations in a neighborhood of the boundary. In [8] nonlinear boundary layers for the DBE, the general discrete velocity model (DVM) was considered. Existence of weakly nonlinear boundary layers was proved. Here we exemplify the theory in [8] for a simplified model, the Broadwell model [9], where the whole machinery is actually not really needed, even if it helps out. For the nonlinear Broadwell model, we obtain explicit expressions for boundary layers near a wall moving with a constant speed. The number of conditions, on the assigned data for the outgoing particles at the boundary, needed for the existence of a unique (in a neighborhood of the assigned Maxwellian at infinity) solution of the problem is in complete agreement with the results in [8] for the DBE and [3] for the full BE. Here we also want to mention a series of papers studying initial boundary value problems for the Broadwell model using Green's functions [10–16].

We also consider the question of existence of shock profiles [17, 18] for the same model [9, 19]. The shock profiles can then be seen as heteroclinic orbits connecting two singular points (Maxwellians) [20]. In [20] existence of shock profiles for the DBE in the case of weak shocks was proved. We exemplify the theory in [20] for the Broadwell model, where again the whole machinery is not really needed, even if it helps out. In this way we, in a new way, obtain similar explicit solutions, not only for weak shocks, as the ones obtained in [19] for the same problem.

The paper is organized as follows. In Section 2 we introduce the Broadwell model and find explicit expressions for the nonlinear boundary layers near a wall moving with a constant speed, and in Section 3 we find explicit expressions for the shock profiles for the Broadwell model.

2. Nonlinear Boundary Layers for the Broadwell Model Near a Moving Wall

In this section we study boundary layers for the nonlinear Broadwell model near a wall moving with a constant speed b. In [8] the nonlinear boundary layers for the DBE, the general discrete velocity model (DVM) was considered. Existence of nonlinear boundary layers was proved. Here we exemplify the theory in [8] in the case of a simplified model, where the whole machinery is actually not really needed, even if it helps out. The same problem was considered in [21] for a mixture model, where one of the two species was modelled by the Broadwell model.

We consider the classical Broadwell model [9] in space (with velocities $\xi_1 = (1,0,0)$, $\xi_2 = (-1,0,0)$, $\xi_3 = (0,1,0)$, $\xi_4 = (0,-1,0)$, $\xi_5 = (0,0,1)$, and $\xi_6 = (0,0,-1)$)

$$\begin{aligned}
\frac{\partial \tilde{f}_1}{\partial t} + \frac{\partial \tilde{f}_1}{\partial x} &= \frac{2\sigma}{3}\left(\tilde{f}_3\tilde{f}_4 + \tilde{f}_5\tilde{f}_6 - 2\tilde{f}_1\tilde{f}_2\right),\\
\frac{\partial \tilde{f}_2}{\partial t} - \frac{\partial \tilde{f}_2}{\partial x} &= \frac{2\sigma}{3}\left(\tilde{f}_3\tilde{f}_4 + \tilde{f}_5\tilde{f}_6 - 2\tilde{f}_1\tilde{f}_2\right),\\
\frac{\partial \tilde{f}_3}{\partial t} &= -\frac{2\sigma}{3}\left(\tilde{f}_3\tilde{f}_4 - \tilde{f}_1\tilde{f}_2\right),\\
\frac{\partial \tilde{f}_4}{\partial t} &= -\frac{2\sigma}{3}\left(\tilde{f}_3\tilde{f}_4 - \tilde{f}_1\tilde{f}_2\right),\\
\frac{\partial \tilde{f}_5}{\partial t} &= -\frac{2\sigma}{3}\left(\tilde{f}_5\tilde{f}_6 - \tilde{f}_1\tilde{f}_2\right),\\
\frac{\partial \tilde{f}_6}{\partial t} &= -\frac{2\sigma}{3}\left(\tilde{f}_5\tilde{f}_6 - \tilde{f}_1\tilde{f}_2\right),
\end{aligned} \tag{1}$$

where σ is the mutual collision cross section. For a flow axially symmetric around the x-axis we can reduce system (1) to (with $f_1 = \tilde{f}_1$, $f_2 = \tilde{f}_3 = \tilde{f}_4 = \tilde{f}_5 = \tilde{f}_6$, and $f_3 = \tilde{f}_2$) [9]

$$\begin{aligned}
\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x} &= \frac{4\sigma}{3}\left(f_2^2 - f_1 f_3\right),\\
4\frac{\partial f_2}{\partial t} &= -\frac{8\sigma}{3}\left(f_2^2 - f_1 f_3\right),\\
\frac{\partial f_3}{\partial t} - \frac{\partial f_3}{\partial x} &= \frac{4\sigma}{3}\left(f_2^2 - f_1 f_3\right).
\end{aligned} \tag{2}$$

The collision invariants are

$$\phi = \alpha(1,1,1) + \beta(1,0,-1), \quad \alpha,\beta \in \mathbb{R}, \tag{3}$$

and the Maxwellians (equilibrium distributions) are

$$\begin{aligned}
&M = s^4\left(a^4, a^2, 1\right),\\
&\quad \text{with } s = e^{(\alpha-\beta)/4} > 0,\ a = e^{\beta/4} > 0,\ \alpha,\beta \in \mathbb{R}.
\end{aligned} \tag{4}$$

The density, momentum, and internal energy can be obtained by

$$\begin{aligned}
\rho &= f_1 + 4f_2 + f_3,\\
\rho u &= f_1 - f_3,\\
2\rho e &= f_1 + f_3.
\end{aligned} \tag{5}$$

Let b, the speed of the wall, be a real number such that

$$b \notin \{-1,0,1\}. \tag{6}$$

We define the projections $R_+ : \mathbb{R}^3 \to \mathbb{R}^{n^+}$ and $R_- : \mathbb{R}^3 \to \mathbb{R}^{n^-}$, $n^- = 3 - n^+$, by

$$\begin{aligned}
R_+ h &= h^+ = \left(h_1, \ldots, h_{n^+}\right),\\
R_- h &= h^- = \left(h_{n^+ +1}, \ldots, h_3\right),
\end{aligned} \tag{7}$$

where

$$n^+ = \begin{cases} 0 & \text{if } b > 1\\ 1 & \text{if } 0 < b < 1\\ 2 & \text{if } -1 < b < 0\\ 3 & \text{if } b < -1, \end{cases} \qquad h = \left(h_1, h_2, h_3\right). \tag{8}$$

Consider the problem

$$\begin{aligned}
D\frac{\partial f}{\partial t} + B_0\frac{\partial f}{\partial x} &= Q(f,f), \quad x > bt,\ t > 0,\\
f^+(bt,t) &= \tilde{C} f^-(bt,t) + \tilde{\varphi}_0,\\
f(x,0) &= f_0(x),\\
f_0(x) &\longrightarrow M \quad \text{as } x \longrightarrow \infty,
\end{aligned} \tag{9}$$

where $f = (f_1, f_2, f_3)$, $D = (1,4,1)$, $B_0 = (1,0,-1)$, $M = s^4(a^4, a^2, 1)$, $\tilde{C}$ is a given $n^+ \times n^-$ matrix, $\tilde{\varphi}_0 \in \mathbb{R}^{n^+}$, and $Q(f,f)$ is defined by the bilinear expression:

$$Q(f,g) = \frac{2\sigma}{3}\left(2f_2g_2 - f_1g_3 - g_1f_3\right)(1,-2,1). \tag{10}$$

After the change of variables $y = x - bt$ and the transformation

$$f = M + M^{1/2}h, \quad \text{with } M = s^4\left(a^4, a^2, 1\right), \tag{11}$$

we obtain the new system

$$\begin{aligned}
\frac{\partial h}{\partial t} + B\frac{\partial h}{\partial y} + Lh &= S(h,h), \quad y > 0,\ t > 0,\\
h^+(0,t) &= Ch^-(0,t) + \varphi_0,\\
h(y,0) &= h_0(y),\\
h_0(y) &\longrightarrow 0 \quad \text{as } y \longrightarrow \infty,
\end{aligned} \tag{12}$$

where $h = (h_1, h_2, h_3)$, $B = B_0 - bD$, C is an $n^+ \times n^-$ matrix, $\varphi_0 \in \mathbb{R}^{n^+}$, and

$$\begin{aligned}
Lh &= -2M^{-1/2}Q\left(M, M^{1/2}h\right)\\
&= \frac{4s^4\sigma}{3}\left(h_1 - 2ah_2 + a^2h_3\right)\left(1,-2a,a^2\right),\\
S(h,g) &= M^{-1/2}Q\left(M^{1/2}h, M^{1/2}g\right)\\
&= \frac{2s^2\sigma}{3}\left(2h_2g_2 - h_1g_3 - g_1h_3\right)\left(1,-2a,a^2\right).
\end{aligned} \tag{13}$$

Similar initial boundary value problems have been studied in a series of papers using Green's functions (with $s^4 = 1/6$, $a = 1$) for $1/\sqrt{3} < b < 1$ in [10] (with $C = (c_1 \ \ c_2)$) and [14], for $-1 < b < -1/\sqrt{3}$ in [11], for $0 < b < 1/\sqrt{3}$, with $C = 0$ in [12, 13, 15], and for diffuse boundary conditions in [16].

Here we consider the stationary nonlinear system

$$\begin{gathered} B\frac{dh}{dy} + Lh = S(h,h), \quad h = h(y), \ y > 0, \\ h^+(0) = Ch^-(0) + \varphi_0, \\ h(y) \longrightarrow 0 \quad \text{as } y \longrightarrow \infty. \end{gathered} \tag{14}$$

The linearized collision operator

$$L = \frac{4s^4\sigma}{3}\begin{pmatrix} 1 & -2a & a^2 \\ -2a & 4a^2 & -2a^3 \\ a^2 & -2a^3 & a^4 \end{pmatrix} \tag{15}$$

is symmetric and semipositive and have the null-space

$$N(L) = \operatorname{span}\left(\left(a^2, a, 1\right), \left(a^2, 0, -1\right)\right) = \operatorname{span}(e_1, e_2), \tag{16}$$

for some e_1 and e_2, such that

$$\langle e_1, e_2 \rangle_B = \gamma_i \delta_{ij}, \quad i, j = 1, 2. \tag{17}$$

Note also that

$$\langle S(h,h), e \rangle = 0, \quad \forall e \in N(L). \tag{18}$$

Here and below, $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product and we denote $\langle \cdot, \cdot \rangle_B = \langle \cdot, B\cdot \rangle$. If $b \neq a^2/(1+a^2)$ we can choose

$$\begin{aligned} e_1 &= \left(a, \frac{1}{2}, 0\right), \\ e_2 &= \left(a^2 b, \frac{a^3}{2}(1-b), a^2(1-b) - b\right), \end{aligned} \tag{19}$$

and then

$$\begin{aligned} \gamma_1 &= a^2 - b\left(1 + a^2\right), \\ \gamma_2 &= \left(a^2 - b\left(1 + a^2\right)\right) \\ &\quad \cdot \left(-a^2 + \left(1 - a^4\right) b + \left(1 + a^2 + a^4\right) b^2\right). \end{aligned} \tag{20}$$

We let $m^\pm$ denote the number of the positive and negative eigenvalues of the matrix $B^{-1}L$. The numbers $n^\pm$, with $n^+ + n^- = 3$, defined above, denote the numbers of the positive and negative eigenvalues of the matrix B. Moreover, we let k^+, k^-, and l denote the number of positive, negative, and zero eigenvalues of the 2×2-matrix

$$K = \begin{pmatrix} \langle y_1, y_1 \rangle_B & \langle y_1, y_2 \rangle_B \\ \langle y_2, y_1 \rangle_B & \langle y_2, y_2 \rangle_B \end{pmatrix}, \tag{21}$$

where $y_1 = (a^2, a, 1)$ and $y_2 = (a^2, 0, -1)$. Then $m^\pm = n^\pm - k^\pm - l$ [22, 23]. The eigenvalues of K are

$$\lambda_\pm = \eta \pm \sqrt{\eta^2 - 4a^2\kappa}, \tag{22}$$

where

$$\begin{aligned} \eta &= \left(1 + a^2\right)\left[a^2 - 1 - b\left(1 + a^2\right)\right], \\ \kappa &= -a^2 + \left(1 - a^4\right) b + \left(1 + a^2 + a^4\right) b^2. \end{aligned} \tag{23}$$

We find that $\operatorname{sgn}(\lambda_\pm) = \operatorname{sgn}(\eta)$ if $\kappa > 0$, $\operatorname{sgn}(\lambda_\pm) = \pm 1$ if $\kappa < 0$, $\lambda_+ = 2\eta$, and $\lambda_- = 0$ if $\kappa = 0$, but $\kappa < 0$ if $\eta = 0$, $\operatorname{sgn}(\eta) = -\operatorname{sgn}(b)$ if $\kappa > 0$, and

$$\begin{aligned} &\kappa = 0 \Longleftrightarrow \\ &b = b_\pm = \frac{a^4 - 1 \pm \sqrt{1 + 4a^2 + 2a^4 + 4a^6 + a^8}}{2\left(1 + a^2 + a^4\right)}, \\ &\kappa > 0 \quad \text{if } b > b_+, \text{ or } b < b_-, \\ &\kappa < 0 \quad \text{if } b_- < b < b_+. \end{aligned} \tag{24}$$

Hence, we obtain the following number of positive and negative eigenvalues for different values of b

b		-1		b_-		0		b_+		1	
n^+	3		2	2	2		1	1	1		0
n^-	0		1	1	1		2	2	2		3
k^+	2		2	1	1		1	0	0		0
k^-	0		0	0	1		1	1	2		2
l	0		0	1	0		0	1	0		0
m^+	1		0	0	1		0	0	1		0
m^-	0		1	0	0		1	0	0		1

(25)

for

$$b_\pm = \frac{a^4 - 1 \pm \sqrt{1 + 4a^2 + 2a^4 + 4a^6 + a^8}}{2\left(1 + a^2 + a^4\right)}. \tag{26}$$

Particularly, if $a = 1$ then $b_\pm = \pm 1/\sqrt{3}$.

Explicitly, the eigenvalues of the matrix $B^{-1}L$ are 0 (of multiplicity 2) and

$$\begin{aligned} \lambda &= \frac{4s^4\sigma\kappa}{3\left(b - b^3\right)}, \\ &\text{with } \kappa = -a^2 + \left(1 - a^4\right) b + \left(1 + a^2 + a^4\right) b^2. \end{aligned} \tag{27}$$

For $b \neq b_\pm$ an eigenvector corresponding to the nonzero eigenvalue λ is

$$v = \left(b + b^2, \frac{a}{2}\left(1 - b^2\right), a^2\left(b^2 - b\right)\right); \tag{28}$$

that is

$$\begin{aligned} B^{-1}Lv &= \lambda v, \\ \langle v, e \rangle_B &= 0, \quad \forall e \in N(L). \end{aligned} \tag{29}$$

Furthermore,

$$B^{-1}S(v,v)=kv,\quad \text{with } k=\frac{s^2a^2\sigma}{3b}\left(1+3b^2\right). \tag{30}$$

Example 1. If $a=1$, corresponding to a nondrifting Maxwellian M, then we get that

$$\begin{aligned} b_\pm &= \pm\frac{1}{\sqrt{3}},\\ \lambda &= \frac{4s^4\sigma\left(3b^2-1\right)}{3\left(b-b^3\right)},\\ v &= \left(b+b^2,\frac{1-b^2}{2},b^2-b\right),\\ k &= \frac{s^2\sigma}{3b}\left(1+3b^2\right). \end{aligned} \tag{31}$$

Note that if $\varphi_0=0$, then we always have the trivial solution $h=0$, and if $-1<b\le b_-$ ($\varphi_0=(\varphi_{01},\varphi_{02})$, where both φ_{01} and φ_{02} must be zero, i.e., $\varphi_{01}=\varphi_{02}=0$), $0<b\le b_+$ ($\varphi_0\in\mathbb{R}$, where φ_0 must be zero, i.e., $\varphi_0=0$), or $1<b$ (no boundary conditions at all at the wall), then we have no other solutions. Otherwise, we have solutions if and only if $h_0\in\operatorname{span}(v^+-Cv^-)$.

Below we consider the remaining different cases.

(i) If $b_+<b<1$ then $C=(c_1\ \ c_2)$ and $\varphi_0\in\mathbb{R}$. Hence, if $\varphi_0\neq 0$ we obtain the unique solution

$$h(y)=\frac{\lambda}{k+De^{\lambda y}}\left(b+b^2,\frac{a}{2}\left(1-b^2\right),a^2\left(b^2-b\right)\right), \tag{32}$$

with

$$D=\frac{2\left(b+b^2\right)-c_1a\left(1-b^2\right)-2c_2a^2\left(b^2-b\right)}{2h_0}\lambda-k. \tag{33}$$

(ii) If $b_-<b<0$ then $C=\binom{c_1}{c_2}$ and $\varphi_0=(\varphi_{01},\varphi_{02})\in\mathbb{R}^2$. Hence, if $\varphi_{01}\neq 0$, $c_1\neq 2b/a(1-b)$, and

$$\varphi_{02}=\frac{a\left(1-b^2\right)-2c_2a^2\left(b^2-b\right)}{2\left(b+b^2\right)-2c_1a^2\left(b^2-b\right)}\varphi_{01}, \tag{34}$$

then we obtain the unique solution (32) with

$$D=\frac{b+b^2-c_1^2a\left(b^2-b\right)}{\varphi_{01}}\lambda-k, \tag{35}$$

and if $c_1=2b/a(1-b)$, $c_2\neq-(1+b)/2ab$, $\varphi_{01}=0$, and $\varphi_{02}\neq 0$, then we obtain the unique solution (32) with

$$D=\frac{a\left(1-b^2\right)-2c_2a^2\left(b^2-b\right)}{2\varphi_{02}}\lambda-k. \tag{36}$$

(iii) If $b<-1$ then $C=0$ and $\varphi_0=(\varphi_{01},\varphi_{02},\varphi_{03})\in\mathbb{R}^3$. Hence, if $\varphi_{01}\neq 0$ and

$$\begin{aligned} \varphi_{02} &= \frac{a(1-b)}{2b}\varphi_{01},\\ \varphi_{03} &= \frac{a^2(b-1)}{1+b}\varphi_{01}, \end{aligned} \tag{37}$$

then we obtain the unique solution (32), where

$$D=\frac{b+b^2}{\varphi_{01}}\lambda-k. \tag{38}$$

We note that in each of the above cases k^+ conditions on the assigned data φ_0 are implied to have a unique solution. This is in good agreement with the results for the DBE in [8] and for the continuous BE in [3].

Remark 2. Similar results can be obtained for the (reduced) plane Broadwell model

$$\begin{aligned} (1-b)\frac{df_1}{dy} &= \sigma\left(f_2^2-f_1f_3\right),\\ -2b\frac{df_2}{dy} &= -2\sigma\left(f_2^2-f_1f_3\right),\\ -(1+b)\frac{df_3}{dy} &= \sigma\left(f_2^2-f_1f_3\right). \end{aligned} \tag{39}$$

Particularly, with $a=1$ we have

$$\begin{aligned} b_\pm &= \pm\frac{1}{\sqrt{2}},\\ \lambda &= \frac{2s^4\sigma\left(2b^2-1\right)}{3\left(b-b^3\right)},\\ v &= \left(b+b^2,1-b^2,b^2-b\right),\\ k &= \frac{s^2\sigma}{b}. \end{aligned} \tag{40}$$

3. Shock Profiles

In this section we are concerned with the existence of shock profiles [17, 18]

$$F=F\left(x^1,\xi,t\right)=f\left(x^1-bt,\xi\right) \tag{41}$$

for the Boltzmann equation

$$\frac{\partial F}{\partial t}+\xi\cdot\nabla_{\mathbf{x}}F=Q(F,F). \tag{42}$$

Here $\mathbf{x}=(x^1,\ldots,x^d)\in\mathbb{R}^d$, $\xi=(\xi^1,\ldots,\xi^d)\in\mathbb{R}^d$, and $t\in\mathbb{R}_+$ denote position, velocity, and time, respectively. Furthermore, b denotes the speed of the wave. The profiles are assumed to approach two given Maxwellians

$$M_\pm=\frac{\rho_\pm}{\left(2\pi T_\pm\right)^{d/2}}e^{-|\xi-\mathbf{u}_\pm|^2/(2T_\pm)} \tag{43}$$

(ρ, $\mathbf{u}$, and T denote density, bulk velocity, and temperature, resp.) as $x \rightarrow \pm\infty$, which are related through the Rankine-Hugoniot conditions.

The (shock wave) problem is to find a solution $f = f(y,\xi)$ ($y = x^1 - bt$) of the equation

$$\left(\xi^1 - b\right)\frac{\partial f}{\partial y} = Q\left(f, f\right), \tag{44}$$

such that

$$f \longrightarrow M_{\pm} \quad \text{as } y \longrightarrow \pm\infty. \tag{45}$$

In [20] the shock wave problem (44), (45) for the DBE was considered. Existence of shock profiles in the case of weak shocks was proved. Here we exemplify the theory in [20] in the case of a simplified model, where the whole machinery is actually not really needed, even if it helps out. In this way we, in a different way, obtain similar results as is obtained in [19] for the same problem.

We study the reduced system (2) of the classical Broadwell model in (1) [9] in space. The collision invariants are given by (3) and the Maxwellians (equilibrium distributions) by (4).

The shock wave problem for the Broadwell model reads

$$B\frac{df}{dy} = Q\left(f, f\right), \quad \text{where } f \longrightarrow M_{\pm} \text{ as } y \longrightarrow \pm\infty, \tag{46}$$

where $B = B(b) = \operatorname{diag}(1-b, -4b, -(1+b))$, $f = (f_1, f_2, f_3)$, and $Q(f, f)$ is defined by the bilinear expression (10).

The density ρ, momentum ρu, and internal energy $2\rho E$ can be obtained by (5). The Maxwellians $M_- = s_-^4(a_-^4, a_-^2, 1)$ and $M_+ = s_+^4(a_+^4, a_+^2, 1)$ must fulfill the Rankine-Hugoniot conditions

$$\begin{aligned} \rho_+\left(u_+ - b\right) &= \rho_-\left(u_- - b\right), \\ \rho_+\left(2E_+ - bu_+\right) &= \rho_-\left(2E_- - bu_-\right), \end{aligned} \tag{47}$$

with

$$\begin{aligned} \rho_{\pm} &= s_{\pm}^4\left(1 + 4a_{\pm}^2 + a_{\pm}^4\right), \\ \rho_{\pm}u_{\pm} &= s_{\pm}^4\left(a_{\pm}^4 - 1\right), \\ 2\rho_{\pm}E_{\pm} &= s_{\pm}^4\left(1 + a_{\pm}^4\right). \end{aligned} \tag{48}$$

After some manipulations we obtain that

$$2E_{\pm} = \frac{1}{3}\left(2\sqrt{1 + 3u_{\pm}^2} - 1\right). \tag{49}$$

We consider

$$B\frac{df}{dy} = Q\left(f, f\right), \quad \text{where } f \longrightarrow M_+ \text{ as } y \longrightarrow \infty \tag{50}$$

and denote

$$\begin{aligned} &F = M + M^{1/2}h, \\ &\quad \text{with } M = M_+ = s_+^4\left(a_+^4, a_+^2, 1\right) = s^4\left(a^4, a^2, 1\right). \end{aligned} \tag{51}$$

Then we obtain

$$B\frac{dh}{dy} + Lh = S\left(h, h\right), \quad \text{where } h \longrightarrow 0 \text{ as } y \longrightarrow \infty, \tag{52}$$

with the linearized operator L and the quadratic part $S(h,h)$ given by (13). The linearized collision operator is given by (15) and then fulfills properties (16)–(20).

We assume that B is nonsingular; that is $b \notin \{-1, 0, 1\}$. Then by (52) we obtain the system

$$\frac{dh}{dy} + B^{-1}Lh = B^{-1}S\left(h, h\right). \tag{53}$$

In (25) we obtain that

$$\begin{aligned} b_{\pm} &= \frac{a^4 - 1 \pm \sqrt{1 + 4a^2 + 2a^4 + 4a^6 + a^8}}{2\left(1 + a^2 + a^4\right)} \\ &= \frac{2u_+ \pm 2\sqrt{2E_+}}{1 + 6E_+}, \end{aligned} \tag{54}$$

and the eigenvalues of the matrix $B^{-1}L$ are 0 (of multiplicity 2) and

$$\begin{aligned} \lambda &= \frac{4s^4\sigma\kappa}{3\left(b - b^3\right)} \\ &= \frac{\rho_+\sigma}{3\left(b - b^3\right)}\left(2E_+ - 1 - 4u_+b + \left(1 + 6E_+\right)b^2\right), \\ &\quad \text{with } \kappa = -a^2 + \left(1 - a^4\right)b + \left(1 + a^2 + a^4\right)b^2. \end{aligned} \tag{55}$$

Let

$$h = \vartheta e_1 + \chi e_2 + \mu v, \tag{56}$$

where e_1 and e_2 are eigenvectors (19) corresponding to the zero eigenvalue and v is eigenvector (28) corresponding to the nonzero eigenvalue λ. Then

$$\frac{d\vartheta}{dy} = \frac{d\chi}{dy} = 0, \tag{57}$$

which implies that

$$\vartheta = \chi = 0, \tag{58}$$

since $\lim_{y\to\infty}\vartheta = \lim_{y\to\infty}\chi = 0$. Therefore

$$\frac{d\mu}{dy} + \lambda\mu = k\mu^2, \tag{59}$$

where k is given in (30). We obtain that

$$\mu = \frac{\lambda}{k + De^{\lambda y}}. \tag{60}$$

Assume that $D \neq 0$ and let

$$\begin{aligned} &b_+ < b < 1, \\ &\text{or } b_- < b \leq -\frac{1}{1 + 2a^2}. \end{aligned} \tag{61}$$

Then

$$\lim_{y\to\infty}\mu = 0,$$
$$\lim_{y\to-\infty}\mu = \frac{\lambda}{k}, \tag{62}$$

and therefore

$$h(y) = \frac{\lambda}{k + De^{\lambda y}}v, \quad D \neq 0. \tag{63}$$

We conclude that the solution of system (50) is of the form

$$f(y) = M_{+} + \frac{\lambda}{k + De^{\lambda y}}M_{+}^{1/2}v,$$
$$\text{where } \lambda = \frac{4s^4\sigma}{3(b-b^3)}\left(-a^2 + \left(1-a^4\right)b + \left(1+a^2+a^4\right)b^2\right),\ k = \frac{s^2a^2\sigma}{3b}\left(1+3b^2\right),\ v = \left(b+b^2, \frac{a}{2}\left(1-b^2\right), a^2\left(b^2-b\right)\right),\ D \neq 0. \tag{64}$$

It follows that

$$M_{-} = M_{+} + \frac{\lambda}{k}M_{+}^{1/2}v$$
$$= \frac{s^4}{1+3b^2}\left[p^2\frac{1+b}{1-b}, pq, q^2\frac{1-b}{1+b}\right], \tag{65}$$
$$\text{with } p = \left(2+a^2\right)b - a^2,\ q = 1 + \left(1+2a^2\right)b,$$

which is a Maxwellian. Formally we can allow $b < -1$ and $-1/(1+2a^2) < b < 0$. However, then, the equilibrium distribution (65) will not be nonnegative and, hence, not a Maxwellian.

We note that

$$f(y) = \Theta(y)M_{+} + (1-\Theta(y))M_{-},$$
$$\text{with } \Theta(y) = \frac{1}{1+Ce^{-\lambda y}}, \tag{66}$$

where $C = k/D \neq 0$ is an arbitrary nonzero constant. The structure coincides with the one for the Mott-Smith approximation [24] in [25]. However, λ is obtained in different ways.

Remark 3. We can instead of system (50) consider

$$B\frac{df}{dy} = Q(f,f), \quad \text{where } f \longrightarrow M_{-} \text{ as } y \longrightarrow -\infty, \tag{67}$$

with

$$\frac{a^2}{2+a^2} < b < b_{+},$$
$$\text{or } -1 < b < b_{-}, \tag{68}$$

and in a similar way as above, we obtain

$$f(y) = M_{-} + \frac{\lambda}{k + De^{\lambda y}}M_{-}^{1/2}v,$$
$$M_{+} = M_{-} + \frac{\lambda}{k}M_{-}^{1/2}v. \tag{69}$$

Example 4. If $a = 1$ then we have

$$\rho_{+} = 6s_{+}^4,$$
$$u_{+} = 0,$$
$$E_{+} = \frac{1}{6},$$
$$b_{\pm} = \pm\frac{1}{\sqrt{3}},$$
$$\lambda = \frac{4s^4\sigma\left(3b^2-1\right)}{3\left(b-b^3\right)},$$
$$v = \left(b+b^2, \frac{1-b^2}{2}, b^2-b\right),$$
$$k = \frac{s^2\sigma}{3b}\left(1+3b^2\right). \tag{70}$$

Furthermore,

$$f(y) = M_{+} + \frac{\lambda}{k+De^{\lambda y}}M_{+}^{1/2}v = s^4(1,1,1)$$
$$+ \frac{4s^4\left(3b^2-1\right)}{\left(1+3b^2\right)+\widetilde{D}e^{\lambda y}}\left(\frac{b}{1-b}, \frac{1}{2}, \frac{-b}{1+b}\right)$$
$$= r\left[\left((3b-1)^2\frac{1+b}{1-b}, \left(9b^2-1\right), (3b+1)^2\frac{1-b}{1+b}\right)\right.$$
$$\left. + \widetilde{D}e^{\lambda y}(1,1,1)\right], \quad \text{where } r = \frac{s^4}{\left(1+3b^2\right)+\widetilde{D}e^{\lambda y}}, \tag{71}$$

and the other Maxwellian is

$$M_{-} = \frac{s^4}{1+3b^2}\left((3b-1)^2\frac{1+b}{1-b}, (3b+1)\right.$$
$$\left.\cdot(3b-1), (3b+1)^2\frac{1-b}{1+b}\right). \tag{72}$$

Example 5. Similar results can be obtained for the (reduced) plane Broadwell model

$$(1-b)\frac{df_1}{dy} = \sigma\left(f_2^2 - f_1 f_3\right),$$
$$-2b\frac{df_2}{dy} = -2\sigma\left(f_2^2 - f_1 f_3\right), \quad (73)$$
$$-(1+b)\frac{df_3}{dy} = \sigma\left(f_2^2 - f_1 f_3\right).$$

Particularly, with $a = 1$ we have

$$\begin{aligned} \rho_+ &= 4s_+^4, \\ u_+ &= 0, \\ E_+ &= \frac{1}{4}, \\ c_\pm &= \pm\frac{1}{\sqrt{2}}, \\ \lambda &= \frac{2s^4\sigma\left(2b^2-1\right)}{3\left(b-b^3\right)}, \\ v &= \left(b+b^2, 1-b^2, b^2-b\right), \\ k &= \frac{s^2\sigma}{b}. \end{aligned} \quad (74)$$

The other Maxwellian is then

$$M_- = s^4\left((2b-1)^2\frac{1+b}{1-b}, (2b+1)(2b-1), (2b+1)^2 \cdot \frac{1-b}{1+b}\right). \quad (75)$$

The shock strength (cf. [19]) is given by the density ratio

$$\begin{aligned} \sigma_\rho &= \frac{\rho_- - \rho_+}{\rho_+} = \frac{2a^2}{\rho_+}\frac{\lambda}{k} \\ &= \frac{8s^2}{\rho_+}\frac{b+b^2+a^2\left(b^2-1\right)+a^4\left(b^2-b\right)}{\left(1-b^2\right)\left(1+3b^2\right)}, \end{aligned} \quad (76)$$

if $b > 0$ and if $b < 0$ by

$$\begin{aligned} \sigma_\rho &= \frac{\rho_- - \rho_+}{\rho_-} = \frac{2a^2}{\rho_-}\frac{\lambda}{k} \\ &= \frac{8s^2}{\rho_-}\frac{b+b^2+a^2\left(b^2-1\right)+a^4\left(b^2-b\right)}{\left(1-b^2\right)\left(1+3b^2\right)}. \end{aligned} \quad (77)$$

Then the shock strength σ_ρ tends to infinity as b approaches 1 and to zero as b approaches $b_\pm$; that is

$$\begin{aligned} \sigma_\rho &\longrightarrow \infty \quad \text{as } b \longrightarrow 1, \\ \sigma_\rho &\longrightarrow 0 \quad \text{as } b \longrightarrow b_\pm. \end{aligned} \quad (78)$$

The shock width (cf. [19]) is given by the density ratio

$$\begin{aligned} d_\rho &= \frac{\left|\rho_- - \rho_+\right|}{\max_y \left|d\rho/dy\right|} = \frac{4}{\lambda} \\ &= \frac{3b\left(1-b^2\right)}{s^4\sigma\left(b+b^2+a^2(b-1)+a^4\left(b^2-b\right)\right)} \end{aligned} \quad (79)$$

or by the velocity ratio

$$d_u = \frac{\left|u_- - u_+\right|}{\max_y \left|du/dy\right|} = \frac{4}{\lambda}\frac{\left|1-u_-/b\right|}{\rho_+}. \quad (80)$$

We conclude that the shock widths d_ρ and d_u tend to zero as b approaches 1 and to infinity as b approaches $b_\pm$; that is

$$\begin{aligned} &d_\rho \longrightarrow 0, \\ &d_u \longrightarrow 0 \\ &\qquad \text{as } b \longrightarrow 1, \\ &d_\rho \longrightarrow \infty, \\ &d_u \longrightarrow \infty \\ &\qquad \text{as } b \longrightarrow b_\pm. \end{aligned} \quad (81)$$

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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