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Research Article

Existence of Infinitely Many Periodic Solutions for Perturbed Semilinear Fourth-Order Impulsive Differential Inclusions

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This paper discusses the existence of infinitely many periodic solutions for a semilinear fourth-order impulsive differential inclusion with a perturbed nonlinearity and two parameters. The approach is based on a critical point theorem for nonsmooth functionals.

1. Introduction

The goal of this paper is to establish the existence of infinitely many periodic solutions for the following perturbed semilinear fourth-order impulsive differential inclusion:

$$\begin{aligned} &u^{(\mathrm{iv})}(x) - pu''(x) + a(x)u(x) \\ &\quad \in \lambda \partial F(u(x)) + \mu \partial G_u(x, u(x)), \quad x \in [0,T] \setminus Q, \\ &\Delta\left(u''(x_k)\right) = I_{1k}\left(u'(x_k)\right), \\ &-\Delta\left(u'''(x_k)\right) = I_{2k}\left(u(x_k)\right), \\ &\qquad\qquad k = 1, 2, \ldots, m, \\ &u(0) = u''(0) = u(T) = u''(T) = 0, \end{aligned} \tag{1}$$

where p is a positive constant, a is continuous positive even $2T$-periodic function on $\mathbb{R}$, $T > 0$, $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = T$, $Q = \{x_1, x_2, \ldots, x_m\}$, $\partial F(u)$ and $\partial G_u(x,u)$ are generalized gradients of F and G_u, respectively, the operator Δ is defined as $\Delta(u(x_k)) := u(x_k^+) - u(x_k^-)$, with $u(x_k^+)$ and $u(x_k^-)$ denoting the right and left limits, respectively, of $u(x)$ at $x = x_k$, $I_{1k}, I_{2k} \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \ldots, m$, $\lambda > 0$, $\mu \geq 0$, and $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function satisfying the following:

($F1$) $F(0) = 0$ and $\partial F(s) = -\partial F(-s)$ for all $s \in \mathbb{R}$;

($F2$) there exist two constants $c > 0$ and $r \in [1, 2)$, such that

$$\left|\xi\right| \leq c\left(1 + |s|^{r-1}\right), \quad \xi \in \partial F(s), \ s \in \mathbb{R}. \tag{2}$$

Also, G is a function defined on $[0,T] \times \mathbb{R}$, satisfying the following:

($G1$) $G(\cdot, s) : \mathbb{R} \to \mathbb{R}$ is measurable for each $s \in \mathbb{R}$, $G(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz for $x \in [0,T] \setminus Q$, $G(x,0) = 0$, and $-\partial G_s(-x,-s) = \partial G_s(x,s) = \partial G_s(x+2T, s)$ for a.e. $x \in [0,T] \setminus Q$ and $s \in \mathbb{R}$;

($G2$) there exists a constant $b > 0$, such that

$$\left|\xi\right| \leq b\left(1 + |s|^{r-1}\right) \quad s \in \mathbb{R}, \ \xi \in \partial G_s(x,s), \tag{3}$$

where r is defined in ($F2$).

We study the existence of solutions, that is, absolutely continuous on every (x_k, x_{k+1}) and left continuous at x_k functions which satisfy (1) for a.e. x with (possible) jumps (impulses) at x_k.

Fourth-order ordinary differential equations act as models for the bending or deforming of elastic beams, and, therefore, they have important applications in engineering and physical sciences. Boundary value problems for fourth-order ordinary differential equations have been of great

concern in recent years (e.g., see [1–4]). On the other hand, impulsive differential equations occur in many applications such as various mathematical models including population dynamics, ecology, biological systems, biotechnology, industrial robotic, pharmacokinetics, and optimal control. For the general aspects of impulsive differential equations, we refer the reader to [5–9]. In association with this development, a theory of impulsive differential equations has been given extensive attention. Very recently, some researchers have studied the existence and multiplicity of solutions for impulsive fourth-order two-point boundary value problems; we refer the reader to [10–12] and references therein. Differential inclusions arise in models for control systems, mechanical systems, economical systems, game theory, and biological systems to name a few.

Recently, multiplicity of solutions for differential inclusions via nonsmooth variational methods and critical point theory has been considered and here we cite the papers [13–19]. For example, in [13] the existence of infinitely many antiperiodic solutions for second-order impulsive differential inclusions has been discussed. In [16], Kristály employing a nonsmooth Ricceri-type variational principle [20], developed by Marano and Motreanu [21], has established the existence of infinitely many radially symmetric solutions for a differential inclusion problem in $\mathbb{R}^N$. Also, in [17], the authors extended a recent result of Ricceri concerning the existence of three critical points of certain nonsmooth functionals. Two applications have been given, both in the theory of differential inclusions; the first one concerns a nonhomogeneous Neumann boundary value problem and the second one treats a quasilinear elliptic inclusion problem in the whole $\mathbb{R}^N$. Tian and Henderson in [18], based on a nonsmooth version of critical point theory of Ricceri due to Iannizzotto [14], have established the existence of at least three solutions for the a second-order impulsive differential inclusion with a perturbed nonlinearity and two parameters. In [19], three periodic solutions with prescribed wavelength for a class of semilinear fourth-order differential inclusions are obtained by using a nonsmooth version critical point theorem.

In the present paper, motivated by [13, 18, 19], employing an abstract critical point result (see Theorem 7 below), we are interested in ensuring the existence of infinitely many periodic solutions for problem (1); see Theorem 12. We refer to [22], in which related variational methods are used for nonhomogeneous problems.

To the best of our knowledge, no investigation has been devoted to establishing the existence of infinitely many solutions to a problem such as (1). As one reference on impulsive differential inclusions, we can refer to [23].

A special case of our main result is the following theorem.

Theorem 1. *Assume that* $(F1)$ *and* $(F2)$ *hold, and* $I_i(0) = 0$ *and* $I_i(s)s > 0$, $s \in \mathbb{R}$, $i = 1, 2, \ldots, m$. *Furthermore, suppose that*

$$\liminf_{\xi\to+\infty} \frac{\sup_{|t|\le\xi} F(t)}{\xi^2} = 0,$$
$$\limsup_{\xi\to+\infty} \frac{\int_0^1 F\left(\xi\left(x^2 - x\right)\right) dx}{(131/60)\,\xi^2 + \sum_{k=1}^m \int_0^{\xi(2x_k-1)} I_{1k}(s)\,ds + \sum_{k=1}^m \int_0^{\xi(x_k^2-x_k)} I_{2k}(s)\,ds} = +\infty. \tag{4}$$

Then, the problem

$$\begin{aligned} u^{(iv)}(x) - u''(x) + u(x) &\in \partial F(u(x)), \quad x \in [0,1] \setminus Q, \\ \Delta\left(u''(x_k)\right) &= I_{1k}\left(u'(x_k)\right), \\ -\Delta\left(u'''(x_k)\right) &= I_{2k}\left(u(x_k)\right), \\ k &= 1, 2, \ldots, m, \\ u(0) = u''(0) &= u(1) = u''(1) = 0 \end{aligned} \tag{5}$$

admits a sequence of classical solutions.

2. Basic Definitions and Preliminary Results

Let $(X, \|\cdot\|_X)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X. A function $\varphi : X \to \mathbb{R}$ is called locally Lipschitz if, for all $u \in X$, there exist a neighborhood U of u and a real number $L > 0$ such that

$$\left|\varphi(v) - \varphi(w)\right| \le L\|v - w\|_X \quad \forall v, w \in U. \tag{6}$$

If φ is locally Lipschitz and $u \in X$, the generalized directional derivative of φ at u along the direction $v \in X$ is

$$\varphi^\circ(u; v) := \limsup_{w\to u,\tau\to 0^+} \frac{\varphi(w + \tau v) - \varphi(w)}{\tau}. \tag{7}$$

The generalized gradient of φ at u is the set

$$\partial\varphi(u) := \left\{u^* \in X^* : \langle u^*, v\rangle \le \varphi^\circ(u; v) \ \ \forall v \in X\right\}. \tag{8}$$

So $\partial\varphi : X \to 2^{X^*}$ is a multifunction. We say that φ has compact gradient if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*.

Lemma 2 (see [24, Proposition 1.1]). *Let $\varphi \in C^1(X)$ be a functional. Then φ is locally Lipschitz and*

$$\begin{aligned} \varphi^{\circ}(u;v) &= \left\langle \varphi'(u), v \right\rangle \quad \forall u, v \in X; \\ \partial\varphi(u) &= \left\{ \varphi'(u) \right\} \quad \forall u \in X. \end{aligned} \tag{9}$$

Lemma 3 (see [24, Proposition 1.3]). *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional. Then $\varphi^{\circ}(u;\cdot)$ is subadditive and positively homogeneous for all $u \in X$, and*

$$\varphi^{\circ}(u;v) \le L\|v\| \quad \forall u, v \in X, \tag{10}$$

with $L > 0$ being a Lipschitz constant for φ around u.

Lemma 4 (see [25]). *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional. Then $\varphi^{\circ} : X \times X \to \mathbb{R}$ is upper semicontinuous and, for all $\lambda \ge 0$, $u, v \in X$,*

$$(\lambda\varphi)^{\circ}(u;v) = \lambda\varphi^{\circ}(u;v). \tag{11}$$

Moreover, if $\varphi, \psi : X \to \mathbb{R}$ are locally Lipschitz functionals, then

$$(\varphi+\psi)^{\circ}(u;v) \le \varphi^{\circ}(u;v) + \psi^{\circ}(u;v) \quad \forall u, v \in X. \tag{12}$$

Lemma 5 (see [24, Proposition 1.6]). *Let $\varphi, \psi : X \to \mathbb{R}$ be locally Lipschitz functionals. Then*

$$\begin{aligned} \partial(\lambda\varphi)(u) &= \lambda\partial\varphi(u) \quad \forall u \in X,\ \lambda \in \mathbb{R}, \\ \partial(\varphi+\psi)(u) &\subseteq \partial\varphi(u) + \partial\psi(u) \quad \forall u \in X. \end{aligned} \tag{13}$$

Lemma 6 (see [14, Proposition 1.6]). *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional with a compact gradient. Then φ is sequentially weakly continuous.*

We say that $u \in X$ is a (generalized) critical point of a locally Lipschitz functional φ if $0 \in \partial\varphi(u)$; that is,

$$\varphi^{\circ}(u;v) \ge 0 \quad \forall v \in X. \tag{14}$$

When a nonsmooth functional, $g : X \to (-\infty, +\infty)$, is expressed as a sum of a locally Lipschitz function, $\varphi : X \to \mathbb{R}$, and a convex, proper, and lower semicontinuous function, $j : X \to (-\infty, +\infty)$, that is, $g := \varphi + j$, a (generalized) critical point of g is every $u \in X$ such that

$$\varphi^{\circ}(u; v-u) + j(v) - j(u) \ge 0 \tag{15}$$

for all $v \in X$ (see [24, Chapter 3]).

Henceforth, we assume that X is a reflexive real Banach space, $\mathcal{N} : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous functional, $\Upsilon : X \to \mathbb{R}$ is a sequentially weakly upper semicontinuous functional, λ is a positive parameter, $j : X \to (-\infty, +\infty)$ is a convex, proper, and lower semicontinuous functional, and $D(j)$ is the effective domain of j. Write

$$\begin{aligned} \mathcal{M} &:= \Upsilon - j, \\ I_\lambda &:= \mathcal{N} - \lambda\mathcal{M} = (\mathcal{N} - \lambda\Upsilon) + \lambda j. \end{aligned} \tag{16}$$

We also assume that $\mathcal{N}$ is coercive and

$$D(j) \cap \mathcal{N}^{-1}((-\infty, r)) \ne \emptyset \tag{17}$$

for all $r > \inf_X \mathcal{N}$. Moreover, owing to (17) and provided that $r > \inf_X \mathcal{N}$, we can define

$$\begin{aligned} &\varphi(r) \\ &:= \inf_{u \in \mathcal{N}^{-1}((-\infty,r))} \frac{\left(\sup_{v \in \mathcal{N}^{-1}((-\infty,r))} \mathcal{M}(v)\right) - \mathcal{M}(u)}{r - \mathcal{N}(u)}, \\ &\gamma := \liminf_{r \to +\infty} \varphi(r), \\ &\delta := \liminf_{r \to (\inf_X \mathcal{N})^+} \varphi(r). \end{aligned} \tag{18}$$

If $\mathcal{N}$ and Υ are locally Lipschitz functionals, in [22, Theorem 2.1] the following result is proved; it is a more precise version of [21, Theorem 1.1] (see also [20]).

Theorem 7. *Under the above assumption on X, $\mathcal{N}$, and $\mathcal{M}$, one has the following:*

(a) *For every $r > \inf_X \mathcal{N}$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional $I_\lambda = \mathcal{N} - \lambda\mathcal{M}$ to $\mathcal{N}^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\gamma < +\infty$, then, for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*

(b1) *I_λ possesses a global minimum or*

(b2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \mathcal{N}(u_n) = +\infty$.*

(c) *If $\delta < +\infty$, then, for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*

(c1) *there is a global minimum of $\mathcal{N}$ which is a local minimum of I_λ or*

(c2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, with $\lim_{n\to+\infty} \mathcal{N}(u_n) = \inf_X \mathcal{N}$, which converges weakly to a global minimum of $\mathcal{N}$.*

Now we recall some basic definitions and notations. We consider the reflexive Banach space $X := H^2(0,T) \cap H_0^1(0,T)$ endowed with the norm

$$\|u\| = \left(\int_0^T \left(\left|u''(x)\right|^2 + \left|u'(x)\right|^2 + \left|u(x)\right|^2 \right) dx \right)^{1/2}. \tag{19}$$

Obviously, X is a reflexive Banach space and completely embedded in $C([0,T])$. So there exists a constant C_0, such that $C_0 = \sup_{u \in X, \|u\| \ne 0} (\|u\|_\infty / \|u\|)$. From the positivity of p and a, it is easy to see that

$$\begin{aligned} \|u\|_X = \Bigg(\int_0^T \Big(&\left|u''(x)\right|^2 + p\left|u'(x)\right|^2 \\ &+ a(x)\left|u(x)\right|^2 \Big) dx \Bigg)^{1/2} \end{aligned} \tag{20}$$

is also a norm of X, which is equivalent to the usual norm. Therefore, there exist two constants C_1 and C_2 such that

$$C_1 \|u\| \le \|u\|_X \le C_2 \|u\|. \tag{21}$$

Thus,

$$\|u\|_\infty \le \frac{C_0}{C_1} \|u\|_X. \tag{22}$$

Definition 8. A function $u \in X$ is said to be a weak solution of problem (1) such that, corresponding to it, there exists a mapping $[0,T] \ni x \mapsto u^*(x)$ with $u^*(x) \in \lambda \partial F(u(x)) + \mu \partial G_u(x, u(x))$, for a.e. $x \in [0,T]$, and having the property that, for every $v \in X$, $u^* v \in L^1[0,T]$ and

$$\begin{aligned} &\int_0^T \left(u''(x) v''(x) + p u'(x) v'(x) + a(x) u(x) v(x) \right. \\ &\left. - u^*(x) v(x) \right) \mathrm{d}x + \sum_{k=1}^m I_{2k}(u(x_k)) v(x_k) \\ &+ \sum_{k=1}^m I_{1k}(u'(x_k)) v'(x_k) = 0. \end{aligned} \tag{23}$$

Definition 9. A solution u is called a classical solution of the impulsive differential inclusion (1) if $u \in AC^3(x_k, x_{k+1})$ and $u(x_k^-) = u(x_k)$, for $k = 1, \dots, n$, and

$$\begin{aligned} &u^{(\mathrm{iv})}(x) + p u''(x) + a(x) u(x) = u^*(x), \\ &\qquad x \in [0,T] \setminus Q, \\ &\Delta u''(x_k) = I_{1k}(u'(x_k)), \\ &-\Delta(u'''(x_k)) = I_{2k}(u(x_k)), \\ &\qquad k = 1, 2, \dots, m, \\ &u(0) = u''(0) = u(T) = u''(T) = 0, \end{aligned} \tag{24}$$

where $u^*(x) \in \lambda F(u(x)) + \mu G_u(x, u(x))$ for a.e. $x \in [0,T]$.

Lemma 10. *If $u \in X$ is a weak solution of (1), then u is a classical solution of (1).*

Proof. Let $u \in X$ be a weak solution of (1). Then there exists u^* with $u^*(x) \in \lambda \partial F(u(x)) + \mu \partial G_u(x, u(x))$ for a.e. $x \in [0,T]$, satisfying (23). Using integration by parts (23) becomes

$$\begin{aligned} &\int_0^T u''(x) v''(x) \, \mathrm{d}x \\ &= \int_0^T \left(p u''(x) - a(x) u(x) + u^*(x) \right) v(x) \, \mathrm{d}x \\ &\quad - \sum_{k=1}^m I_{2k}(u(x_k)) v(x_k) \\ &\quad - \sum_{k=1}^m I_{1k}(u'(x_k)) v'(x_k), \end{aligned} \tag{25}$$

so $u \in H^4((0,T) \setminus Q)$, and (25) holds for each $v \in X$ with $v(x_k) = v'(x_k) = 0$, $k = 1, \dots, m$. Through integration by parts, we obtain

$$\begin{aligned} &u^{(\mathrm{iv})}(x) - p u''(x) + a(x) u(x) - u^*(x) = 0, \\ &\qquad \text{a.e. } x \in [0,T] \setminus Q. \end{aligned} \tag{26}$$

Due to $u \in X$, similar to [26, Section 2], we can get $u \in AC^3(x_k, x_{k+1})$ and $u(x_k^-) = u(x_k)$ for $k = 1, \dots, n$. Now we show that the boundary conditions are satisfied. Choose any $j \in \{0, 1, 2, \dots, m-1\}$ and $v \in X$ such that $v(x) = 0$ if $x \in [x_k, x_{k+1}]$ for $k \ne j$. Then, from (25) and by means of integration by parts, we can get

$$\begin{aligned} &u''(x_{j+1}) v'(x_{j+1}) - u''(x_j) v'(x_j) \\ &\quad + \int_{x_j}^{x_{j+1}} \left(u^{(\mathrm{iv})} - p u'' + a(x) u - u^* \right) v \, \mathrm{d}x = 0. \end{aligned} \tag{27}$$

Thus,

$$\begin{aligned} &\sum_{j=0}^{m-1} \left(u''(x_{j+1}) v'(x_{j+1}) - u''(x_j) v'(x_j) \right) \\ &\quad + \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} \left(u^{(\mathrm{iv})} - p u'' + a(x) u - u^* \right) v \, \mathrm{d}x \\ &= 0, \end{aligned} \tag{28}$$

which implies that

$$\begin{aligned} &u''(T) v'(T) - u''(0) v'(0) \\ &\quad + \int_0^T \left(u^{(\mathrm{iv})}(x) - p u''(x) + a(x) u(x) - u^*(x) \right) \\ &\quad \cdot v(x) \, \mathrm{d}x = 0. \end{aligned} \tag{29}$$

Since u satisfies (26), we have that $u''(T)v'(T) - u''(0)v'(0) = 0$ holds for all $v \in X$, which implies that $u''(0) = u''(T) = 0$ and in view of $u \in X$ we get $u(0) = u(T) = 0$. Now using a technique similar to the technique of [18, Lemma 3.5] shows that the impulsive conditions are satisfied. From the equality

$$\begin{aligned} u^{(\mathrm{iv})}(x) v(x) &= \left(\left\{ \int_0^x u^{(\mathrm{iv})}(s) \, \mathrm{d}s \right\} v(x) \right)' \\ &\quad - \left(\int_0^x u^{(\mathrm{iv})}(s) \, \mathrm{d}s \right) v'(x), \end{aligned} \tag{30}$$

we have

$$\begin{aligned}
&\int_0^T u^{(\mathrm{iv})}(x)\, v(x)\,\mathrm{d}x = \int_0^T \left[\left(\left\{\int_0^x u^{(\mathrm{iv})}(s)\,\mathrm{d}s\right\} v(x)\right)'\right.\\
&\left.-\left(\int_0^x u^{(\mathrm{iv})}(s)\,\mathrm{d}s\right) v'(x)\right]\mathrm{d}x = v(T)\\
&\cdot\int_0^T u^{(\mathrm{iv})}(x)\,\mathrm{d}x - \int_0^T \left[u'''(x) - u'''(0)\right.\\
&\left.-\sum_{0\le x_k<x}\Delta u'''(x_k)\right] v'(x)\,\mathrm{d}x = \left[u'''(T)\right.\\
&\left.-u'''(0) - \sum_{k=1}^{m}\Delta u'''(x_k)\right] v(T) - \int_0^T u'''(x)\\
&\cdot v'(x)\,\mathrm{d}x + u'''(0)\,[v(T) - v(0)]\\
&+\sum_{k=1}^{m}\int_{x_k}^{x_{k+1}}\sum_{0\le x_k<x}\Delta u'''(x_k)\, v'(x)\,\mathrm{d}x = u'''(T)\, v(T)\\
&-u'''(0)\, v(0) - \sum_{k=1}^{m}\Delta u'''(x_k)\, v(T) - \int_0^T u''(x)\\
&\cdot v''(x)\,\mathrm{d}x - \sum_{k=1}^{m}\Delta u'''(x_k)\, v(x_k)\\
&+\sum_{k=1}^{m}\Delta u'''(x_k)\, v(T) = u'''(T)\, v(T) - u'''(0)\, v(0)\\
&-\sum_{k=1}^{m}\Delta u'''(x_k)\, v(x_k) - \int_0^T u''(x)\, v''(x)\,\mathrm{d}x.
\end{aligned} \tag{31}$$

Substituting (31) into (25), we have

$$\begin{aligned}
&\int_0^T \left(u^{(\mathrm{iv})}(x) - pu''(x) + a(x)\,u(x) - u^*(x)\right)\\
&\cdot v(x)\,\mathrm{d}x + u'''(T)\, v(T) - u'''(0)\, v(0)\\
&+\sum_{k=1}^{m}\left[-\Delta u'''(x_k) - I_{2k}(u(x_k))\right] v(x_k) = 0.
\end{aligned} \tag{32}$$

Since u satisfies (26), we have

$$\begin{aligned}
&u'''(T)\, v(T) - u'''(0)\, v(0) - \sum_{k=1}^{m}\Delta\left(u'''(x_k)\right) v(x_k)\\
&-\sum_{k=1}^{m} I_{2k}(u(x_k))\, v(x_k) = 0
\end{aligned} \tag{33}$$

for all $v \in X$. Thus, $-\Delta u'''(x_k) = I_{2k}(u(x_k))$, $k = 1,\ldots,m$. Similarly from the equality

$$\begin{aligned}
u''(x)\, v(x) &= \left(\left\{\int_0^x u''(s)\,\mathrm{d}s\right\} v(x)\right)'\\
&-\left(\int_0^x u''(s)\,\mathrm{d}s\right) v'(x),
\end{aligned} \tag{34}$$

we have $\Delta u''(x_k) = I_{1k}(u'(x_k))$. So u is a classical solution of (1). □

Now we introduce the functionals $\mathcal{N}$, $\mathcal{F}$, $\mathcal{G}$, and I_λ by

$$\begin{aligned}
\mathcal{N}(u) &= \frac{1}{2}\|u\|_X + \sum_{k=1}^{m}\int_0^{u'(t_k)} I_{1k}(s)\,\mathrm{d}s\\
&+\sum_{k=1}^{m}\int_0^{u(t_k)} I_{2k}(s)\,\mathrm{d}s, \quad u \in X,\\
\mathcal{F}(u) &= \int_0^T F(u(x))\,\mathrm{d}x,\\
\mathcal{G}(u) &= \int_0^T G(x, u(x))\,\mathrm{d}x,\\
&\qquad u \in X,\\
I_{\lambda,\mu}(u) &= \mathcal{N}(u) - \lambda\mathcal{F}(u) - \mu\mathcal{G}(u) \quad u \in X.
\end{aligned} \tag{35}$$

Thus, $\mathcal{N} \in C^1(X,\mathbb{R})$ and

$$\begin{aligned}
\left\langle \mathcal{N}'(u), v\right\rangle &= \int_0^T \left(u''(x)\, v''(x) + pu'(x)\, v'(x)\right.\\
&\left.+ a(x)\,u(x)\, v(x)\right)\mathrm{d}x + \sum_{k=1}^{m} I_{2k}(u(t_k))\, v(t_k)\\
&+\sum_{k=1}^{m} I_{1k}\left(u'(t_k)\right) v'(t_k)
\end{aligned} \tag{36}$$

for all $u, v \in X$.

Lemma 11. *Assume that* (*F*1), (*F*2), (*G*1), *and* (*G*2) *hold. Then the functional* $I_{\lambda,\mu} : X \to \mathbb{R}$ *is locally Lipschitz. Moreover, each critical point* $u \in X$ *of* $I_{\lambda,\mu}$ *is a weak solution of (1).*

Proof. Let $I_{\lambda,\mu} = \mathcal{N}(u) + I_1(u)$, where $I_1(u) = -\lambda\mathcal{F}(u) - \mu\mathcal{G}(u)$. Since $\mathcal{N} \in C^1(X,\mathbb{R})$ by Lemma 2, $\mathcal{N}$ is locally Lipschitz on X. From (*F*2) and (*G*2), we know that I_1 is locally Lipschitz on $L^r([0,T])$. Moreover, X is compactly embedded into $L^r([0,T])$. Thus, I_1 is locally Lipschitz on X [27, Theorem 2.2]. According to Lemma 5, we get

$$\begin{aligned}
\partial I_1(u) \subset &-\lambda\int_0^T \partial F(u(x))\,\mathrm{d}x\\
&-\mu\int_0^T \partial G_u(x, u(x))\,\mathrm{d}x.
\end{aligned} \tag{37}$$

The explanation of (37) is as follows: for every $u^* \in \partial I_1(u)$, there is a corresponding mapping $u^*(x) \in -\lambda\partial F(u(x)) - \mu\partial G_u(x,u(x))$ for a.e. $x \in [0,T]$ having the property that, for every $v \in X$, the function $u^*v \in L^1[0,T]$ and $\langle u^*, v\rangle = \int_0^T u^*(x)v(x)\mathrm{d}x$. Therefore, $I_{\lambda,\mu}$ is locally Lipschitz on X.

Now we show that each critical point of $I_{\lambda,\mu}$ is a weak solution of (1). Assume that $u \in X$ is a critical point of $I_{\lambda,\mu}$. So

$$0 \in \partial I_{\lambda,\mu}(u) = \left\{u^* \in X^* : \langle u^*, v\rangle \le I^\circ_{\lambda,\mu}(u;v) \ \text{ for } v \in X\right\}. \tag{38}$$

So, by Lemma 2 and (38), we have

$$\mathcal{N}'(u) + u^* = 0 \quad \text{with } u^* \in \partial I_1(u) \tag{39}$$

and hence $u^*(x) \in -\lambda\partial F(u(x)) - \mu\partial G_u(x,u(x))$ a.e. on $[0,T]$. It follows from (37) and (39) that for every $v \in X$ we have

$$\begin{aligned}\int_0^T &\left(u''(x)v''(x) + pu'(x)v'(x) + a(x)u(x)v(x)\right.\\ &\left. + u^*(x)v(x)\right)\mathrm{d}x + \sum_{k=1}^m I_{2k}(u(t_k))v(t_k)\\ &+ \sum_{k=1}^m I_{1k}\left(u'(t_k)\right)v'(t_k) = 0\end{aligned} \tag{40}$$

for all $v \in X$. Thus, by Definition 8, u is a weak solution of (1). □

3. Main Results

First for every $\xi \in \mathbb{R}^+$ we set

$$\begin{aligned}D(\xi) = \frac{C_2^2}{2}\xi^2\left(4T + \frac{T^3}{3} + \frac{T^5}{30}\right)\\ + \sum_{k=1}^m \int_0^{\xi(2x_k - T)} I_{1k}(s)\,\mathrm{d}s\\ + \sum_{k=1}^m \int_0^{\xi(x_k^2 - Tx_k)} I_{2k}(s)\,\mathrm{d}s.\end{aligned} \tag{41}$$

Now we formulate our main result using the following assumptions:

$(F3)$ $\liminf_{\xi\to+\infty}(\sup_{|t|\le\xi}F(t)/\xi^2) < (1/2T)(C_1/C_0)^2 \limsup_{\xi\to+\infty}(\int_0^T F(\xi(x^2 - Tx))\mathrm{d}x/D(\xi))$;

$(I1)$ $I_{jk}(0) = 0$, $I_{jk}(s)s > 0$, $s \in \mathbb{R}$, $j = 1,2$ and $k = 1,2,\ldots,m$.

Theorem 12. *Assume that* $(F1)$–$(F3)$ *and* $(I1)$ *hold. Let*

$$\begin{aligned}\lambda_1 &:= \frac{1}{\limsup_{\xi\to+\infty}\left(\int_0^T F\left(\xi\left(x^2 - Tx\right)\right)\mathrm{d}x/D(\xi)\right)},\\ \lambda_2 &:= \frac{C_1^2}{2TC_0^2\liminf_{\xi\to+\infty}\left(\sup_{|t|\le\xi}F(t)/\xi^2\right)}.\end{aligned} \tag{42}$$

Then, for every $\lambda \in (\lambda_1,\lambda_2)$ *and every nonnegative function* G *satisfying* $(G1)$, $(G2)$, *and the assumption*

$(G3)$ $G_\infty := \lim_{\xi\to+\infty}(\int_0^T \sup_{|t|\le\xi}G(x,t)\mathrm{d}x/\xi^2) < +\infty$,

putting

$$\begin{aligned}\mu_{G,\lambda} := \frac{1}{2G_\infty}\left(\frac{C_1}{C_0}\right)^2\\ \cdot\left(1 - 2T\lambda\left(\frac{C_0}{C_1}\right)^2\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}F(t)}{\xi^2}\right),\end{aligned} \tag{43}$$

where $\mu_{G,\lambda} = +\infty$ *when* $G_\infty = 0$, *for every* $\mu \in [0,\mu_{G,\lambda})$ *problem (1) admits an unbounded sequence of classical solutions in* X.

Proof. Our goal is to apply Theorem 7(b) to (1). For this purpose, we fix $\overline{\lambda} \in (\lambda_1,\lambda_2)$ and let G be a nonnegative function satisfying (G1)–(G3). Since $\overline{\lambda} < \lambda_2$, we have

$$\begin{aligned}\mu_{G,\overline{\lambda}} = \frac{1}{2G_\infty}\left(\frac{C_1}{C_0}\right)^2\\ \cdot\left(1 - 2T\overline{\lambda}\left(\frac{C_0}{C_1}\right)^2\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}F(t)}{\xi^2}\right) > 0.\end{aligned} \tag{44}$$

Now fix $\overline{\mu} \in (0,\mu_{g,\overline{\lambda}})$, put $\nu_1 := \lambda_1$, and

$$\nu_2 := \frac{\lambda_2}{1 + 2\left(C_0/C_1\right)^2\left(\overline{\mu}/\overline{\lambda}\right)\lambda_2 G_\infty}. \tag{45}$$

If $G_\infty = 0$, then $\nu_1 = \lambda_1$, $\nu_2 = \lambda_2$, and $\overline{\lambda} \in (\nu_1,\nu_2)$. If $G_\infty \ne 0$, since $\overline{\mu} < \mu_{G,\overline{\lambda}}$, we have

$$\frac{\overline{\lambda}}{\lambda_2} + 2\left(\frac{C_0}{C_1}\right)^2\overline{\mu}G_\infty < 1, \tag{46}$$

and so

$$\frac{\lambda_2}{1 + 2\left(C_0/C_1\right)^2\left(\overline{\mu}/\overline{\lambda}\right)\lambda_2 G_\infty} > \overline{\lambda}; \tag{47}$$

to wit, $\overline{\lambda} < \nu_2$. Hence, taking into account that $\overline{\lambda} > \lambda_1 = \nu_1$, one has $\overline{\lambda} \in (\nu_1,\nu_2)$. Now, set

$$J(x,s) := F(s) + \frac{\overline{\mu}}{\overline{\lambda}}G(x,s) \tag{48}$$

for all $(x,s) \in [0,T]\times\mathbb{R}$. Assume that j is exactly zero in X and for each $u \in X$ and put

$$\begin{aligned}\Upsilon(u) &:= \int_0^T J(x,u(x))\,\mathrm{d}x,\\ \mathscr{M}(u) &:= \Upsilon(u) - j(u) = \Upsilon(u),\\ I_{\overline{\lambda}}(u) &:= \mathcal{N}(u) - \overline{\lambda}\mathscr{M}(u) = \mathcal{N}(u) - \overline{\lambda}\Upsilon(u).\end{aligned} \tag{49}$$

It is easy to prove that $\mathcal{N}$ is sequentially weakly lower semicontinuous on X. Obviously, $\mathcal{N} \in C^1(X)$. By Lemma 2, $\mathcal{N}$ is locally Lipschitz on X. By Lemma 11, $\mathcal{F}$ and $\mathcal{G}$ are locally Lipschitz on $L^2([0,T])$. So, Υ is locally Lipschitz on $L^2([0,T])$, and since X is compactly embedded into $L^2([0,T])$, Υ is locally Lipschitz on X. In addition, Υ is sequentially weakly upper semicontinuous. For all $u \in X$, by $(I1)$,

$$\begin{aligned} &\int_0^{u'(x_k)} I_{1k}(s)\,\mathrm{d}s > 0, \quad k = 1,2,\ldots,m, \\ &\int_0^{u(x_k)} I_{2k}(s)\,\mathrm{d}s > 0, \quad k = 1,2,\ldots,m. \end{aligned} \tag{50}$$

So, we have

$$\begin{aligned} \mathcal{N}(u) = \frac{1}{2}\|u\|_X^2 &+ \sum_{k=1}^{m}\int_0^{u'(x_k)} I_{1k}(s)\,\mathrm{d}s \\ &+ \sum_{k=1}^{m}\int_0^{u(x_k)} I_{2k}(s)\,\mathrm{d}s > \frac{1}{2}\|u\|_X^2 \end{aligned} \tag{51}$$

for all $u \in X$. Hence, $\mathcal{N}$ is coercive and $\inf_X \mathcal{N} = \mathcal{N}(0) = 0$. Under our hypotheses, we want to show that there exists a sequence $\{\overline{u}_n\} \subset X$ of critical points for the functional $I_{\overline{\lambda}}$; that is, every element $\overline{u}_n$ satisfies

$$I_{\overline{\lambda}}^{\circ}(\overline{u}_n, v - \overline{u}_n) \geq 0, \quad \text{for every } v \in X. \tag{52}$$

Now, we prove that $\gamma < +\infty$. For this, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty}\xi_n = +\infty$ and

$$\begin{aligned} &\lim_{n\to+\infty}\frac{\int_0^T \sup_{|t|\leq\xi_n} J(x,t)\,\mathrm{d}x}{\xi_n^2} \\ &\quad = \liminf_{\xi\to+\infty}\frac{\int_0^T \sup_{|t|\leq\xi} J(x,t)\,\mathrm{d}x}{\xi^2}. \end{aligned} \tag{53}$$

Put

$$r_n := \frac{1}{2}\left(\frac{C_1\xi_n}{C_0}\right)^2, \quad \forall n \in \mathbb{N}. \tag{54}$$

Then, for all $v \in X$ with $\mathcal{N}(v) < r_n$, taking into account that $\|v\|_X^2 < 2r_n$ and $\|v\|_\infty \leq (C_0/C_1)\|v\|_X$, one has $|v(x)| \leq \xi_n$ for every $x \in [0,T]$. Therefore, for all $n \in \mathbb{N}$,

$$\begin{aligned} &\varphi(r_n) \\ &\quad = \inf_{u\in\mathcal{N}^{-1}((-\infty,r))}\frac{\left(\sup_{v\in\mathcal{N}^{-1}((-\infty,r))}\mathcal{M}(v)\right) - \mathcal{M}(u)}{r - \mathcal{N}(u)} \\ &\quad \leq \frac{\sup_{\|v\|_X^2<2r_n}\left(\mathcal{F}(v) + \left(\overline{\mu}/\overline{\lambda}\right)\mathcal{G}(v)\right)}{r_n} \\ &\quad \leq \frac{\sup_{|t|\leq\xi_n}\left(TF(t) + \left(\overline{\mu}/\overline{\lambda}\right)\int_0^T G(x,t)\,\mathrm{d}x\right)}{r_n} \\ &\quad \leq 2\left(\frac{C_0}{C_1}\right)^2 \\ &\qquad \cdot\left[\frac{T\sup_{|t|\leq\xi_n}F(t)}{\xi_n^2} + \frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_0^T \sup_{|t|\leq\xi_n}G(x,t)\,\mathrm{d}x}{\xi_n^2}\right]. \end{aligned} \tag{55}$$

Moreover, from assumptions $(F3)$ and $(G3)$, we have

$$\begin{aligned} &\lim_{n\to+\infty}\frac{T\sup_{|t|\leq\xi_n}F(t)}{\xi_n^2} + \frac{\overline{\mu}}{\overline{\lambda}}\lim_{n\to+\infty}\frac{\int_0^T \sup_{|t|\leq\xi_n}G(x,t)\,\mathrm{d}x}{\xi_n^2} \\ &\quad < +\infty, \end{aligned} \tag{56}$$

which follows

$$\lim_{n\to+\infty}\frac{\int_0^T \sup_{|t|\leq\xi_n} J(x,t)\,\mathrm{d}x}{\xi_n^2} < +\infty. \tag{57}$$

Therefore,

$$\begin{aligned} \gamma &\leq \liminf_{n\to+\infty}\varphi(r_n) \\ &\leq 2\left(\frac{C_0}{C_1}\right)^2 \liminf_{\xi\to+\infty}\frac{\int_0^T \sup_{|t|\leq\xi} J(x,t)\,\mathrm{d}x}{\xi^2} < +\infty. \end{aligned} \tag{58}$$

Since

$$\begin{aligned} \frac{\int_0^T \sup_{|t|\leq\xi} J(x,s)\,\mathrm{d}x}{\xi^2} &\leq \frac{T\sup_{|t|\leq\xi}F(t)}{\xi^2} \\ &\quad + \frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_0^T \sup_{|t|\leq\xi}G(x,t)\,\mathrm{d}x}{\xi^2}, \end{aligned} \tag{59}$$

and taking $(G3)$ into account, we get

$$\begin{aligned} &\liminf_{\xi\to+\infty}\frac{\int_0^T \sup_{|t|\leq\xi} J(x,t)\,\mathrm{d}x}{\xi^2} \\ &\quad \leq \liminf_{\xi\to+\infty}\frac{T\sup_{|t|\leq\xi}F(t)}{\xi^2} + \frac{\overline{\mu}}{\overline{\lambda}}G_\infty. \end{aligned} \tag{60}$$

Moreover, by assumption $(G3)$ we have

$$\limsup_{\xi\to+\infty}\frac{\int_0^T J\left(x,\xi\left(x^2-Tx\right)\right)\mathrm{d}x}{D(\xi)} \geq \limsup_{\xi\to+\infty}\frac{\int_0^T F\left(\xi\left(x^2-Tx\right)\right)\mathrm{d}x}{D(\xi)}. \tag{61}$$

Therefore, from (60) and (61), we observe that

$$\begin{aligned}\overline{\lambda}\in(\nu_1,\nu_2) &\subseteq \left(\frac{1}{\limsup_{\xi\to+\infty}\left(\int_0^T J\left(x,\xi\left(x^2-Tx\right)\right)\mathrm{d}x/D(\xi)\right)},\right.\\ &\left.\frac{1}{2\left(C_0/C_1\right)^2\liminf_{\xi\to+\infty}\left(\int_0^T\sup_{|t|\leq\xi}J(x,t)\,\mathrm{d}x/\xi^2\right)}\right)\\ &\subseteq\left(0,\frac{1}{\gamma}\right).\end{aligned} \tag{62}$$

For the fixed $\overline{\lambda}$, inequality (58) ensures that condition (b) of Theorem 7 can be applied and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of problem (1) such that $\lim_{n\to\infty}\|u_n\|=+\infty$. Now we prove that for the fixed $\overline{\lambda}$ the functional $I_{\overline{\lambda}}$ has no global minimum. Let us verify that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}}<\limsup_{\xi\to+\infty}\frac{\int_0^T F\left(\xi\left(x^2-Tx\right)\right)\mathrm{d}x}{D(\xi)}, \tag{63}$$

there exist a sequence $\{\eta_n\}$ of positive numbers and a constant τ such that $\lim_{n\to+\infty}\eta_n=+\infty$ and

$$\frac{1}{\overline{\lambda}}<\tau<\frac{\int_0^T F\left(\eta_n\left(x^2-Tx\right)\right)\mathrm{d}x}{D(\eta_n)} \tag{64}$$

for each $n\in\mathbb{N}$ large enough. For all $n\in\mathbb{N}$, put

$$w_n(x):=\eta_n\left(x^2-Tx\right)\quad\text{for every } x\in[0,T]. \tag{65}$$

For any fixed $n\in\mathbb{N}$, clearly $w_n\in X$ and one has

$$\|w_n\|_X^2\leq C_2^2\|w_n\|^2=\eta_n^2C_2^2\left(4T+\frac{T^3}{3}+\frac{T^5}{30}\right), \tag{66}$$

and so

$$\begin{aligned}\mathcal{N}(w_n)&\leq\frac{C_2^2}{2}\eta_n^2\left(4T+\frac{T^3}{3}+\frac{T^5}{30}\right)\\ &\quad+\sum_{k=1}^m\int_0^{\eta_n(2x_k-T)}I_{1k}(s)\,\mathrm{d}s\\ &\quad+\sum_{k=1}^m\int_0^{\eta_n(x_k^2-Tx_k)}I_{2k}(s)\,\mathrm{d}s=D(\eta_n).\end{aligned} \tag{67}$$

By (64) and (67) and since G is nonnegative we observe that

$$\begin{aligned}I_{\overline{\lambda}}(w_n)&=\mathcal{N}(w_n)-\overline{\lambda}\mathcal{M}(w_n)\\ &\leq D(\eta_n)-\overline{\lambda}\int_0^T F\left(\eta_n\left(x^2-Tx\right)\right)\mathrm{d}x\\ &<D(\eta_n)\left(1-\overline{\lambda}\tau\right)\end{aligned} \tag{68}$$

for every $n\in\mathbb{N}$ large enough; since $\overline{\lambda}\tau>1$ and $\lim_{n\to+\infty}\eta_n=+\infty$, it follows that

$$\lim_{n\to+\infty}I_{\overline{\lambda}}(w_n)=-\infty. \tag{69}$$

So, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it shows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, from part (b) of Theorem 7, the functional $I_{\overline{\lambda}}$ admits a sequence of critical points $\{\overline{u}_n\}\subset X$ such that $\lim_{n\to+\infty}\mathcal{N}(\overline{u}_n)=+\infty$. Since $\mathcal{N}$ is bounded on bounded sets and taking into account that $\lim_{n\to+\infty}\mathcal{N}(\overline{u}_n)=+\infty$, then $\{\overline{u}_n\}$ has to be unbounded; that is,

$$\lim_{n\to+\infty}\|\overline{u}_n\|_X=+\infty. \tag{70}$$

Also, if $\overline{u}_n\in X$ is a critical point of $I_{\overline{\lambda}}$, clearly, by definition, one has

$$I_{\overline{\lambda}}^{\circ}\left(\overline{u}_n,v-\overline{u}_n\right)\geq 0,\quad\text{for every } v\in X. \tag{71}$$

Finally, by Lemma 11, the critical points of $I_{\overline{\lambda}}$ are weak solutions for problem (1), and, by Lemma 10, every weak solution of (1) is a classical solution of (1). Hence, we have the result. □

Remark 13. Under the conditions

$$\begin{aligned}\liminf_{\xi\to+\infty}\frac{\sup_{|t|\leq\xi}F(t)}{\xi^2}&=0,\\ \limsup_{\xi\to+\infty}\frac{\int_0^T F\left(\xi\left(x^2-Tx\right)\right)\mathrm{d}x}{D(\xi)}&=+\infty,\end{aligned} \tag{72}$$

from Theorem 12, we see that, for every $\lambda>0$ and for each $\mu\in[0,C_1^2/2C_0^2G_\infty)$, problem (1) admits a sequence of solutions which is unbounded in X. Moreover, if $G_\infty=0$, the result holds for every $\lambda>0$ and $\mu\geq 0$.

The following result is a special case of Theorem 12 with $\mu=0$.

Theorem 14. *Assume that* $(F1)$–$(F3)$ *and* $(I1)$ *hold. Then, for each*

$$\lambda\in\left(\frac{1}{\limsup_{\xi\to+\infty}\left(\int_0^T F\left(\xi\left(x^2-Tx\right)\right)\mathrm{d}x/D(\xi)\right)},\ \frac{1}{2T\left(C_0/C_1\right)^2\liminf_{\xi\to+\infty}\left(\sup_{|t|\leq\xi}F(t)/\xi^2\right)}\right), \tag{73}$$

the problem

$$\begin{aligned} &u^{(iv)}(x) - pu''(x) + a(x)u(x) \in \lambda \partial F(u(x)), \\ &\qquad\qquad x \in [0,T] \setminus Q, \\ &\Delta\left(u''(x_k)\right) = I_{1k}\left(u'(x_k)\right), \\ &-\Delta\left(u'''(x_k)\right) = I_{2k}\left(u(x_k)\right), \\ &\qquad\qquad k = 1,2,\ldots,m, \\ &u(0) = u''(0) = u(T) = u''(T) = 0 \end{aligned} \tag{74}$$

has an unbounded sequence of classical solutions in X.

Now, we present the following example to illustrate Theorem 14.

Example 15. Let $T = 1$, $p = 1$, and $a(x) = 1 + \cos x$ for every $x \in \mathbb{R}$. Thus, a is a continuous positive even 2π-periodic function on $\mathbb{R}$. Define the function

$$F(s) = \begin{cases} 0, & \text{if } s \in [-1,1], \\ s^2 - 1, & \text{if } s \in [-2,-1) \cup (1,2], \\ 3, & \text{if } s \in [-\infty,-2) \cup (2,+\infty). \end{cases} \tag{75}$$

Clearly, F is a continuous convex function with $F(0) = 0$. An easy calculation shows that

$$\partial F(s) = \begin{cases} \{0\}, & \text{if } s \in (-\infty,-2), \\ [-4,0], & \text{if } s = -2, \\ (-4,-2), & \text{if } s \in (-2,-1), \\ [-2,0], & \text{if } s = -1, \\ \{0\}, & \text{if } s \in (-1,1), \\ [0,2], & \text{if } s = 1, \\ (2,4), & \text{if } s \in (1,2), \\ [0,4], & \text{if } s = 2, \\ \{0\}, & \text{if } s \in (2,+\infty). \end{cases} \tag{76}$$

Hence, assumptions $(F1)$ and $(F2)$ hold. Moreover, let $Q = \{1/3, 2/3\}$, $I_{11}(x) = I_{12}(x) = (1/8)x$, and $I_{21}(x) = I_{22}(x) = (2/5)x$ for every $x \in \mathbb{R}$. Thus, $(I1)$ is satisfied and

$$\begin{aligned} D(\xi) &= \frac{131C_2^2}{30}\xi^2 + 2\sum_{k=1}^{2}\int_0^{\xi(2x_k-1)} I_{1k}(s)\,ds \\ &\quad + 2\sum_{k=1}^{2}\int_0^{\xi(x_k^2-x_k)} I_{2k}(s)\,ds = \frac{131C_2^2}{30}\xi^2 + \frac{17}{324}. \end{aligned} \tag{77}$$

So, we have

$$\begin{aligned} &\liminf_{\xi\to+\infty}\frac{\sup_{|t|\le\xi}F(t)}{\xi^2} = 0, \\ &\limsup_{\xi\to+\infty}\frac{\int_0^1 F\left(\xi x^2 - \xi x\right)dx}{D(\xi)} \\ &\qquad \ge \lim_{\xi\to\infty}\frac{\xi^2 - 1}{\left(131C_2^2/30\right)\xi^2 + 17/324} = \frac{30}{131C_2^2}. \end{aligned} \tag{78}$$

Therefore, by applying Theorem 14, the problem

$$\begin{aligned} &u^{(iv)}(x) - u''(x) + a(x)u(x) \in \lambda \partial F(u(x)), \\ &\qquad\qquad x \in [0,1] \setminus \left\{\frac{1}{3}, \frac{2}{3}\right\}, \\ &\Delta\left(u''\left(\frac{1}{3}\right)\right) = \frac{1}{8}\left(u'\left(\frac{1}{3}\right)\right), \\ &-\Delta\left(u'''\left(\frac{1}{3}\right)\right) = \frac{2}{5}\left(u\left(\frac{1}{3}\right)\right), \\ &\Delta\left(u''\left(\frac{2}{3}\right)\right) = \frac{1}{8}\left(u'\left(\frac{2}{3}\right)\right), \\ &-\Delta\left(u'''\left(\frac{2}{3}\right)\right) = \frac{2}{5}\left(u\left(\frac{2}{3}\right)\right), \\ &u(0) = u''(0) = u(1) = u''(1) = 0 \end{aligned} \tag{79}$$

for $\lambda \in (0, 30/131C_2^2)$ has an unbounded sequence of classical solutions in $H^2(0,1) \cap H_0^1(0,1)$.

Now we state the following consequence of Theorem 14, using the following assumptions:

$(F4)$ $\liminf_{\xi\to+\infty}(\sup_{|t|\le\xi}F(t)/\xi^2) < C_1^2/2TC_0^2$;

$(F5)$ $\limsup_{\xi\to+\infty}(\int_0^T F(\xi(x^2 - Tx))dx/D(\xi)) > 1$.

Corollary 16. *Assume that $(F1)$, $(F2)$, $(F4)$, $(F5)$, and $(I1)$ hold. Then, the problem*

$$\begin{aligned} &u^{(iv)}(x) - pu''(x) + a(x)u(x) \in \partial F(u(x)), \\ &\qquad\qquad x \in [0,T] \setminus Q, \\ &\Delta\left(u''(x_k)\right) = I_{1k}\left(u'(x_k)\right), \\ &-\Delta\left(u'''(x_k)\right) = I_{2k}\left(u(x_k)\right), \\ &\qquad\qquad k = 1,2,\ldots,m, \\ &u(0) = u''(0) = u(T) = u''(T) = 0 \end{aligned} \tag{80}$$

has an unbounded sequence of classical solutions in X.

Remark 17. Theorem 1 in Introduction is an immediate consequence of Corollary 16.

Here, we give a consequence of the main result.

Corollary 18. *Let $F_1 : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that $F_1(0) = 0$, $-\partial F_1(-s) = \partial F_1(s)$, $|\xi| \le a(1 + |s|^{r_1-1})$ for all $s \in \mathbb{R}$, $\xi \in F_1(s)$, $r_1 \in [1,2)$ $(a > 0)$. Furthermore, suppose that*

$(C1)$ $\liminf_{\xi\to+\infty}(\sup_{|t|\le\xi}F_1(t)/\xi^2) < +\infty$,

$(C2)$ $\limsup_{\xi\to+\infty}(\int_0^T F_1(\xi(x^2 - Tx))dx/D(\xi)) = +\infty$.

Then, for every function $F_2 : \mathbb{R} \to \mathbb{R}$ which is locally Lipschitz function such that $F_2(0) = 0$, $-\partial F_2(-s) = \partial F_2(s)$, and $|\xi| \le a(1 + |s|^{r_1-1})$ for all $s \in \mathbb{R}$, $\xi \in F_2(s)$, $r_1 \in [1,2)$ $(a > 0)$ and satisfies the conditions

$$\sup_{t\in\mathbb{R}} F_2(t) \le 0, \qquad \liminf_{\xi\to+\infty} \frac{\int_0^T F_2\left(\xi\left(x^2 - Tx\right)\right)dx}{D(\xi)} > -\infty, \tag{81}$$

for each

$$\lambda \in \left(0, \frac{1}{2T\left(C_0/C_1\right)^2 \liminf_{\xi\to+\infty}\left(\sup_{|t|\le\xi}F_1(t)/\xi^2\right)}\right), \tag{82}$$

the problem

$$\begin{aligned} &u^{(iv)}(x) - pu''(x) + a(x)u(x) \\ &\quad \in \lambda\left(\partial F_1(u(x)) + \partial F_2(u(x))\right), \quad x \in [0,T]\setminus Q, \\ &\Delta\left(u''(x_k)\right) = I_{1k}\left(u'(x_k)\right), \\ &-\Delta\left(u'''(x_k)\right) = I_{2k}\left(u(x_k)\right), \\ &\qquad\qquad k = 1,2,\ldots,m, \\ &u(0) = u''(0) = u(T) = u''(T) = 0 \end{aligned} \tag{83}$$

has an unbounded sequence of classical solutions in X.

Proof. Set $F(t) = F_1(t) + F_2(t)$ for all $t \in \mathbb{R}$. Assumption $(C2)$ along with the condition

$$\liminf_{\xi\to+\infty} \frac{\int_0^T F_2\left(\xi\left(x^2 - Tx\right)\right)dx}{D(\xi)} > -\infty \tag{84}$$

yields

$$\begin{aligned} &\limsup_{\xi\to+\infty} \frac{\int_0^T F\left(\xi\left(x^2 - Tx\right)\right)dx}{D(\xi)} \\ &= \limsup_{\xi\to+\infty} \frac{\int_0^T F_1\left(\xi\left(x^2 - Tx\right)\right)dx + \int_0^T F_2\left(\xi\left(x^2 - Tx\right)\right)dx}{D(\xi)} \\ &= +\infty. \end{aligned} \tag{85}$$

Moreover, assumption $(C1)$ together with the condition

$$\sup_{t\in\mathbb{R}} F_2(t) \le 0 \tag{86}$$

ensures

$$\liminf_{\xi\to+\infty} \frac{\sup_{|t|\le\xi}F(t)}{\xi^2} \le \liminf_{\xi\to+\infty} \frac{\sup_{|t|\le\xi}F_1(t)}{\xi^2} < +\infty. \tag{87}$$

Since

$$\begin{aligned} &\frac{1}{\liminf_{\xi\to+\infty}\left(\sup_{|t|\le\xi}F(t)/\xi^2\right)} \\ &\qquad \ge \frac{1}{\liminf_{\xi\to+\infty}\left(\sup_{|t|\le\xi}F_1(t)/\xi^2\right)}, \end{aligned} \tag{88}$$

by applying Theorem 14, we have the desired conclusion. □

Remark 19. We observe that in Theorem 12 we can replace $\xi \to +\infty$ with $\xi \to 0^+$, and then by the same argument as in the proof of Theorem 12, but using conclusion (c) of Theorem 7 instead of (b), problem (1) has a sequence of pairwise distinct classical solutions, which strongly converges to 0 in X.

We end this paper by presenting the following example.

Example 20. Let $T = \pi$, $p = 1$, and $a(x) = |\sin x| + 1$ for every $x \in \mathbb{R}$. Thus, a is a continuous positive even 2π-periodic function on $\mathbb{R}$. Now let $F(s) = -|s|$ for all $s \in \mathbb{R}$ and $G(x,s) = x(1 - \cos s)$ for all $(x,s) \in [0,\pi] \times \mathbb{R}$. Hence, F is locally Lipschitz function and

$$\partial F(s) = \begin{cases} \{1\}, & \text{if } s < 0, \\ [-1,1], & \text{if } s = 0, \\ \{-1\}, & \text{if } s > 0. \end{cases} \tag{89}$$

Therefore, $(F1)$ and $(F2)$ hold, and also we can simply see that $(G1)$ and $(G2)$ hold. Moreover, let $Q = \{\pi/3, 2\pi/3\}$, $I_{11}(x) = I_{12}(x) = x(1 + x)$, and $I_{21}(x) = I_{22}(x) = x^5$ for every $x \in \mathbb{R}$. Thus, $(I1)$ is satisfied. An easy computation shows that

$$\begin{aligned} \frac{\int_0^\pi F\left(\xi\left(x^2 - x\right)\right)dx}{D(\xi)} &= \frac{\xi\left(\int_0^1\left(x - x^2\right)dx + \int_1^\pi\left(x^2 - x\right)dx\right)}{\left(C_2^2/2\right)\xi^2\left(4\pi + \pi^3/3 + \pi^5/30\right) + 2\sum_{k=1}^2\left(\int_0^{\xi(2x_k-\pi)} I_{1k}(s) + \int_0^{\xi(x_k^2-\pi x_k)} I_{2k}(s)\,ds\right)} \\ &= \frac{\xi\left(\pi^3/3 - \pi^2/2 + 1/3\right)}{\left(C_2^2/2\right)\xi^2\left(4\pi + \pi^3/3 + \pi^5/30\right) + \left(2^7/9^6\right)\pi^{12}\xi^6} \end{aligned} \tag{90}$$

for some $\xi \in \mathbb{R}$. So,

$$\liminf_{\xi\to 0^+} \frac{\sup_{|t|\le\xi} F(t)}{\xi^2} = 0,$$
$$\limsup_{\xi\to 0^+} \frac{\int_0^\pi F\left(\xi x^2 - \xi x\right) dx}{D(\xi)} = +\infty. \tag{91}$$

Also

$$\int_0^\pi x(1-\cos t)\,dx = \frac{\pi^2}{2}(1-\cos t) \ge 0 \quad \forall t \in \mathbb{R},$$
$$\lim_{\xi\to+0^+} \frac{\sup_{|t|\le\xi} \int_0^\pi x(1-\cos t)\,dx}{\xi^2} = \frac{\pi^2}{2} \lim_{\xi\to+0^+} \frac{\sup_{|t|\le\xi}(1-\cos t)}{\xi^2} = \frac{\pi^2}{4}. \tag{92}$$

Hence, taking Remark 19 into account, the problem

$$\begin{aligned}
&u^{(iv)}(x) - u''(x) + (2+\sin x)\,u(x) \\
&\quad \in \lambda \partial F(u(x)) + \mu \partial G_u(x,u), \\
&\qquad\qquad x \in [0,\pi] \setminus \left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}, \\
&\Delta\left(u''\left(\frac{\pi}{3}\right)\right) = u'\left(\frac{\pi}{3}\right)\left(1+u'\left(\frac{\pi}{3}\right)\right), \\
&-\Delta\left(u'''\left(\frac{\pi}{3}\right)\right) = u^5\left(\frac{\pi}{3}\right), \\
&\Delta\left(u''\left(\frac{2\pi}{3}\right)\right) = u'\left(\frac{2\pi}{3}\right)\left(1+u'\left(\frac{2\pi}{3}\right)\right), \\
&-\Delta\left(u'''\left(\frac{2\pi}{3}\right)\right) = u^5\left(\frac{2\pi}{3}\right), \\
&u(0) = u''(0) = u(\pi) = u''(\pi) = 0
\end{aligned} \tag{93}$$

for $\lambda \in (0,\infty)$ and $\mu \in [0, 2C_1^2/C_0^2\pi^3)$ has an unbounded sequence of pairwise distinct classical solutions in $H^2(0,\pi) \cap H_0^1(0,\pi)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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