Research Article
Antinormal Weighted Composition Operators

Dilip Kumar and Harish Chandra

Department of Mathematics, Banaras Hindu University, Varanasi 221005, India

Correspondence should be addressed to Harish Chandra; harishc@bhu.ac.in

Let $l^2 = L^2(\mathbb{N}, \mu)$, where $\mathbb{N}$ is set of all positive integers and μ is the counting measure whose σ-algebra is the power set of $\mathbb{N}$. In this paper, we obtain necessary and sufficient conditions for a weighted composition operator to be antinormal on the Hilbert space l^2. We also determine a class of antinormal weighted composition operators on Hardy space $H^2(\mathbb{D})$.

1. Introduction

A deep and interesting problem in operator theory is to determine the distance of an operator from a given class of bounded linear operators on a Hilbert space. In particular, the distance between an operator and the set of Hermitian, positive, compact, and unitary operators has been investigated in [1–4]. In 1974, Holmes [5] discovered that there are operators for which their largest possible distance from the set of normal operators can be achieved. He named such operators as antinormal operators and showed that no compact operator is antinormal. Thus, this extremality of operators is a consequence of infinite dimensional phenomenon of the underlying space. Subsequently, this class has been extensively studied by several authors in [6–9]. For the first time, the same problem has been studied in context of composition operators on the Hilbert space l^2 in [10] by Tripathi and Lal. In this paper, we investigate antinormality of weighted composition operators on l^2 and $H^2(\mathbb{D})$.

Notation and Terminology. In this paper, $\mathbb{N}$ and $\mathbb{C}$ denote the set of all positive integers and the set of all complex numbers, respectively. Also, let $\chi_n : \mathbb{N} \to \{0, 1\}$ be defined as

$$\chi_n(m) = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Suppose H is a separable complex Hilbert space and $B(H)$ denotes the algebra of all bounded linear operators on H. Further, for $T \in B(H)$, let $N(T)$ and $R(T)$, respectively, denote the null space and the range space of T.

We now record certain definitions and results which are useful in our context.

Definition 1. A function $u : X \to \mathbb{C}$ is said to be bounded away from zero if there exists a positive real number, say r, such that $0 < 1/|u(x)| \le r$ for all $x \in X$.

Definition 2 (see [11]). An operator $T \in B(H)$ is said to be Fredholm operator if dimension of $N(T)$ and the dimension of the quotient space $H/R(T)$ are both finite.

We also note here that, equivalently, T is Fredholm if both $N(T)$ and $N(T^*)$ are finite dimensional.

Definition 3. Essential spectrum of an operator $T \in B(H)$ is defined as $\sigma_e(T) = \{\alpha \in \mathbb{C} : T - \alpha I \text{ is not Fredholm}\}$.

Since every invertible operator is Fredholm operator, therefore $\sigma_e(T) \subseteq \sigma(T)$; see [12].

Definition 4. Minimum modulus of an operator $T \in B(H)$ is defined as $m(T) = \inf\{\|Tx\| : \|x\| = 1\}$.

Definition 5. Essential minimum modulus of an operator $T \in B(H)$ is defined as $m_e(T) = \inf\{\alpha \ge 0 : \alpha \in \sigma_e(|T|)\}$, where $|T| = (T^*T)^{1/2}$.

Definition 6. An operator $T \in B(H)$ is said to be antinormal if $d(T, \mathcal{N}) = \inf_{N \in \mathcal{N}} \|T - N\| = \|T\|$, where $\mathcal{N}$ is the class of all normal operators in $B(H)$.

Remark 7. An operator $T \in B(H)$ is antinormal if and only if its adjoint T^* is antinormal.

Definition 8. For an operator T in $B(H)$, index of T is defined as

$$\text{index}(T) = \begin{cases} \dim(N(T)) - \dim(N(T^*)), & \text{if } \dim(N(T)) \text{ or } \dim(N(T^*)) < \infty \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Remark 9. Observe that $\text{index}(T) = -\text{index}(T^*)$.

The following results will be used in the later part of the paper.

Theorem 10 (see [7]). *Let $T \in B(H)$.*

(i) *If $\text{index}(T) = 0$, then $d(T, \mathcal{N}) \leq (\|T\| - m(T))/2$.*

(ii) *If $\text{index}(T) < 0$, then $m_e(T) \leq d(T, \mathcal{N}) \leq (\|T\| + m_e(T))/2$.*

Remark 11. If $\text{index}(T) = 0$, then T is not antinormal.

Theorem 12 (see [7]). *Let $T \in B(H)$ with $\text{index}(T) < 0$. Then, following conditions are equivalent:*

(i) *T is antinormal.*

(ii) *$m_e(T) = \|T\|$.*

(iii) *$d(T, \mathcal{U}) = 1 + \|T\|$, where $\mathcal{U}$ is the class of all unitary operators in $B(H)$.*

(iv) *$T = \alpha W(1 - K)$ for some $\alpha > 0$, isometry W and positive compact contraction K.*

2. Antinormal Weighted Composition Operators on l^2

The Hilbert space l^2 is the space of all square summable sequences of complex numbers. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be self-map and $\varphi^{-1}(n)$ denote the inverse image of n under φ. Let $|\varphi^{-1}(n)|$ denote the cardinality of $\varphi^{-1}(n)$. For $u \in l^\infty$, a weighted composition transformation induced by u and φ is defined as

$$\left(uC_\varphi\right)(f)(n) = u(n) f(\varphi(n)) \quad \forall f \in l^2, \ \forall n \in \mathbb{N}. \quad (3)$$

If uC_φ is bounded, then it is called weighted composition operator induced by u and φ. The following result gives a necessary and sufficient condition for boundedness of uC_φ.

Theorem 13. *Weighted composition operator uC_φ on l^2 is bounded if and only if $\sup_{n \in \mathbb{N}} \{|u(n)|^2 |\varphi^{-1}(n)|\} < \infty$. In this case, $\|uC_\varphi\| = \{\sup_{n \in \mathbb{N}} \{|u(n)|^2 |\varphi^{-1}(n)|\}\}^{1/2}$.*

Proof. Let $f = \sum_{n \in \mathbb{N}} f(n)\chi_n$ be an element in l^2. Then,

$$\begin{aligned} \|uC_\varphi f\|^2 &= \left\| uC_\varphi \left(\sum_{n \in \mathbb{N}} f(n)\chi_n \right) \right\|^2 \\ &= \left\| \sum_{n \in \mathbb{N}} u(n) f(n) \chi_{\varphi^{-1}(n)} \right\|^2 \\ &= \sum_{n \in \mathbb{N}} |u(n)|^2 |f(n)|^2 |\varphi^{-1}(n)| \\ &\leq \sup_{n \in \mathbb{N}} \{|u(n)|^2 |\varphi^{-1}(n)|\} \sum_{n \in \mathbb{N}} |f(n)|^2 \\ &= \sup_{n \in \mathbb{N}} \{|u(n)|^2 |\varphi^{-1}(n)|\} \|f\|^2. \end{aligned} \quad (4)$$

Thus,

$$\sup_{n \in \mathbb{N}} \{|u(n)|^2 |\varphi^{-1}(n)|\} < \infty \quad (5)$$

implies that uC_φ is bounded and

$$\|uC_\varphi\| \leq \left\{ \sup_{n \in \mathbb{N}} \{|u(n)|^2 |\varphi^{-1}(n)|\} \right\}^{1/2}. \quad (6)$$

Conversely, if uC_φ is bounded, then we have

$$\|uC_\phi \chi_n\|^2 = |u(n)|^2 |\varphi^{-1}(n)| \quad \forall n \in \mathbb{N}. \quad (7)$$

Therefore,

$$\left\{ \sup_{n \in \mathbb{N}} \{|u(n)|^2 |\varphi^{-1}(n)|\} \right\}^{1/2} \leq \|uC_\varphi\|, \quad (8)$$

hence the proof. □

In the following result, we determined weighted composition operators on l^2 which are not antinormal.

Theorem 14. *Let $u \in l^\infty$ be bounded away from zero and φ be a bijective self-map on $\mathbb{N}$. Then, uC_φ is not antinormal.*

Proof. Since φ is bijective and u is bounded away from zero, uC_φ is invertible on l^2. Hence, $\text{index}(uC_\varphi) = 0$. Consequently, uC_φ is not antinormal by Remark 11. □

The following remark shows that the condition that "u is bounded away from zero" is not essential for the validity of the above theorem.

Remark 15. Let $u \in l^\infty$ be such that $u(n) \to 0$ as $n \to \infty$. Then, it is well known that uC_φ is compact ([13], p. 89). Hence, uC_φ is not antinormal.

We now prove the following lemma.

Lemma 16. *Suppose φ is injective but not surjective and $u(n) \neq 0$ for each $n \in \mathbb{N}$. Then, uC_φ is not injective but $(uC_\varphi)^*$ is injective.*

Proof. Since φ is not surjective, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{-1}(n_0)$ is empty. Therefore,

$$\left(uC_\varphi\right)\chi_{n_0} = u\left(n_0\right)\chi_{\varphi^{-1}(n_0)} = 0. \tag{9}$$

Consequently, uC_φ is not injective.

Now, for $f = \sum_{n\in\mathbb{N}} f(n)\chi_n$, we have

$$\left(uC_\varphi\right)^* f = \sum_{n\in\mathbb{N}} \overline{u(n)} f(n) \chi_{\varphi(n)}. \tag{10}$$

Now, a simple computation using the above expression and injectivity of φ shows that whenever $(uC_\varphi)^* f = 0$, then $\overline{u(n)} f(n) = 0 \ \forall n \in \mathbb{N}$. Since $u(n) \neq 0 \ \forall n \in \mathbb{N}$, therefore $f = 0$. Hence, $(uC_\varphi)^*$ is injective. $\square$

The following result gives a necessary and sufficient condition for antinormality of uC_φ when φ is injective but not surjective.

Theorem 17. *Suppose $u(n) \neq 0$ for each $n \in \mathbb{N}$ and φ is injective but not surjective. Then, uC_φ is antinormal if and only if the following conditions hold:*

(a) *For each $0 \le \alpha < \|uC_\varphi\|^2$, $\overline{u(n)}u(\varphi(n)) \neq \alpha$, except for finitely many $n \in \mathbb{N}$.*

(b) *$\|uC_\varphi\|^2 = \overline{u(n)}u(\varphi(n))$ for infinitely many $n \in \mathbb{N}$.*

Proof. Since φ is injective but not surjective and $u(n) \neq 0$ for each $n \in \mathbb{N}$, hence by above lemma index$((uC_\varphi)^*) < 0$. Now, for $f = \sum_{n\in\mathbb{N}} f(n)\chi_n \in l^2$ and for given complex number α, we have

$$\left(\left(uC_\varphi\right)\left(uC_\varphi\right)^* - \alpha I\right) f$$
$$= \sum_{n\in\mathbb{N}} \left(\overline{u(n)}u\left(\varphi(n)\right) - \alpha\right) f(n) \chi_n. \tag{11}$$

From the above equation and condition (a) it follows that dimker$((uC_\varphi)(uC_\varphi)^* - \alpha I) = $ dimker$((uC_\varphi)(uC_\varphi)^* - \alpha I)^*$ is finite whenever $0 \le \alpha < \|uC_\varphi\|^2$. Hence, $((uC_\varphi)(uC_\varphi)^* - \alpha I)$ is Fredholm for each $0 \le \alpha < \|uC_\varphi\|^2$. Thus, $\sqrt{\alpha} \notin \sigma_e(|(uC_\varphi)^*|)$ for each $0 \le \alpha < \|uC_\varphi\|^2$. Again by condition (b) $\|uC_\varphi\|^2 = \overline{u(n)}u(\varphi(n))$ for infinitely many $n \in \mathbb{N}$. Hence, dimker$((uC_\varphi)(uC_\varphi)^* - \|uC_\varphi\|^2 I)$ is infinite. Therefore,

$\|uC_\varphi\| \in \sigma_e(|(uC_\varphi)^*|)$. Thus, $m_e((uC_\varphi)^*) = \|uC_\varphi\|$. Hence, $(uC_\varphi)^*$ is antinormal by Theorem 12. Now, since adjoint of an antinormal operator is antinormal, therefore uC_φ is antinormal. Conversely, if either of the conditions fail, then we claim that uC_φ is not antinormal. For, if condition (a) fails, there exists an α_0 with $0 \le \alpha_0 < \|uC_\varphi\|^2$ such that $\sqrt{\alpha_0} \in \sigma_e(|(uC_\varphi)^*|)$. Therefore, $m_e((uC_\varphi)^*) \le \sqrt{\alpha_0} < \|uC_\varphi\|$. Hence, $(uC_\varphi)^*$ is not antinormal. Now, suppose condition (b) fails. Then, by (11), $\|(uC_\varphi)^*\| \notin \sigma_e(|(uC_\varphi)^*|)$. Therefore, $m_e((uC_\varphi)^*) < \|uC_\varphi\|$. Hence, uC_φ is not antinormal. $\square$

We now prove the following lemma.

Lemma 18. *Suppose φ is surjective but not injective and $u(n) \neq 0$ for each $n \in \mathbb{N}$. Then, uC_φ is injective but $(uC_\varphi)^*$ is not injective.*

Proof. Let $f \in l^2$ be such that $uC_\varphi(f) = 0$. Therefore, $u(n)f(\varphi(n)) = 0 \ \forall n \in \mathbb{N}$. This implies $f(\varphi(n)) = 0 \ \forall n \in \mathbb{N}$ as $u(n) \neq 0 \ \forall n \in \mathbb{N}$. Now, subjectivity φ implies $f = 0$. Further, Since φ is not injective, there exist $m, n \in \mathbb{N}$ with $m \neq n$ such that $\varphi(m) = \varphi(n)$. Now, define $f = \overline{u(m)}\chi_n$ and $g = \overline{u(n)}\chi_m$. It is easy to see that $f, g \in l^2$ and $f \neq g$ but $(uC_\varphi)^*(f) = (uC_\varphi)^*(g)$. Therefore, $(uC_\varphi)^*$ is not injective. $\square$

Using above lemma we prove the following result.

Theorem 19. *Suppose $u(n) \neq 0$ for each $n \in \mathbb{N}$ and φ is surjective but not injective. Then, uC_φ is antinormal if and only if the following conditions hold:*

(a) *For each $0 \le \alpha < \|uC_\varphi\|^2$, $u(n) \sum_{m\in\varphi^{-1}(n)} \overline{u(m)} \neq \alpha$, except for finitely many $n \in \mathbb{N}$,*

(b) *$\|uC_\varphi\|^2 = u(n) \sum_{m\in\varphi^{-1}(n)} \overline{u(m)}$ for infinitely many $n \in \mathbb{N}$.*

Proof. Since φ is surjective but not injective and $u(n) \neq 0$ for each $n \in \mathbb{N}$, hence by above lemma index$((uC_\varphi)^*) < 0$. For $f \in l^2$ and $\alpha \in \mathbb{C}$, we have

$$\left(\left(uC_\varphi\right)^* \left(uC_\varphi\right) - \alpha I\right) f$$
$$= \sum_{n\in\mathbb{N}} \left(u(n) \sum_{m\in\varphi^{-1}(n)} \overline{u(m)} - \alpha\right) f(n) \chi_n. \tag{12}$$

Therefore, dimker$((uC_\varphi)^*(uC_\varphi) - \alpha I) = $ dimker$((uC_\varphi)^*(uC_\varphi) - \alpha I)^*$ is finite for each $0 \le \alpha < \|uC_\varphi\|^2$ by condition (a). Hence, $\sqrt{\alpha} \notin \sigma_e(|uC_\varphi|)$ for each $0 \le \alpha < \|uC_\varphi\|^2$. Also, $u(n) \sum_{m\in\varphi^{-1}(n)} \overline{u(m)} = \|uC_\varphi\|^2$ for infinitely many $n \in \mathbb{N}$, so $\|uC_\varphi\| \in \sigma_e(|uC_\varphi|)$. Thus, $m_e(uC_\varphi) = \|uC_\varphi\|$. Consequently, uC_φ is antinormal. Conversely, if either of the conditions is not true, then we claim that uC_φ is not antinormal. If condition (a) fails, then there exists an α_0 with $0 \le \alpha_0 < \|uC_\varphi\|^2$, such that $\sqrt{\alpha_0} \in \sigma_e(|uC_\varphi|)$. Therefore, $m_e(uC_\varphi) \le \sqrt{\alpha_0} < \|uC_\varphi\|$ and so uC_φ is not antinormal. Now, suppose condition (b) fails. Then, by (12), $\|uC_\varphi\| \notin \sigma_e(|uC_\varphi|)$. Therefore,

$m_e(uC_\varphi) < \|uC_\varphi\|$. Thus, uC_φ is not antinormal in either case. $\square$

Theorem 20. *Suppose $u(n) \neq 0 \; \forall n \in \mathbb{N}$ and φ is neither injective nor surjective.*

    (i) *If $index(uC_\varphi) = 0$, then uC_φ is not antinormal.*

    (ii) *If $index(uC_\varphi) < 0$, then uC_φ is antinormal if and only if the following conditions hold:*

        (a) *for each $0 \leq \alpha < \|uC_\varphi\|$, $u(n) \sum_{m \in \varphi^{-1}(n)} \overline{u(m)} \neq \alpha$, except for finitely many $n \in range(\varphi)$,*

        (b) *if $\|uC_\varphi\|^2 = u(n) \sum_{m \in \varphi^{-1}(n)} \overline{u(m)}$ for infinitely many $n \in range(\varphi)$.*

Proof. (i) It follows by Remark 11.
(ii) Proof is same as in case of Theorem 19. $\square$

We now give examples, one in each case, satisfying Theorems 17 and 19.

Example 21. Let $\varphi(n) = 2n$ and

$$u(n) = \begin{cases} 1, & n = 1, 2 \\ 2, & \text{otherwise.} \end{cases} \tag{13}$$

Clearly, φ is injective but not surjective and $\overline{u(n)}u(\varphi(n)) = 4 = \|uC_\varphi\|^2$ for $n \geq 2$. Hence, by Theorem 17, uC_φ is antinormal.

Example 22. Let

$$\varphi(n) = \begin{cases} \dfrac{n+1}{2}, & n \text{ is odd} \\ \dfrac{n}{2}, & n \text{ is even,} \end{cases}$$

$$u(n) = \begin{cases} 1, & n = 1 \\ 2, & \text{otherwise.} \end{cases} \tag{14}$$

Clearly, φ is surjective but not injective and $\|uC_\varphi\|^2 = 8 = u(n) \sum_{m \in \varphi^{-1}(n)} \overline{u(m)}$ for $n \geq 2$. Hence, antinormality of uC_φ follows from Theorem 19.

3. Antinormal Weighted Composition Operators on Hardy Space

In this section, we continue our line of investigation of antinormal operators. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. Hardy space $H^2(\mathbb{D})$ [14] is defined to be the space of all analytic functions f on the unit disk $\mathbb{D}$ with the property

$$\|f\|^2 = \sup_{0 \leq r < 1} \int_0^{2\pi} |f(rz)|^2 \frac{d\theta}{2\pi} < \infty. \tag{15}$$

The inner product on $H^2(\mathbb{D})$ is defined as follows:

$$\langle f, g \rangle = \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi}. \tag{16}$$

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic self-map. For $u \in H^\infty(\mathbb{D})$, weighted composition transformation uC_φ on $H^2(\mathbb{D})$ is defined by $(uC_\varphi)(f)(z) = u(z)f(\varphi(z))$ for all $f \in H^2(\mathbb{D})$ and for all $z \in \mathbb{D}$. The fact that uC_φ is bounded linear operator on $H^2(\mathbb{D})$ follows from the well-known Hardy-Littlewood subordination, Theorem [15]. In 2011, Gunatillake [16] has shown that the operator uC_φ on $H^2(\mathbb{D})$ is invertible if and only if u is both bounded and bounded away from zero on the unit disc and φ is an automorphism of the unit disc.

Following result is an easy consequence of preceding statement.

Theorem 23. *If $u \in H^\infty(\mathbb{D})$ is bounded away from zero and φ is automorphism of the unit disc, then uC_φ is not an antinormal composition operator.*

Proof. Since uC_φ is bijective, so $index(uC_\varphi) = 0$. Hence, C_φ is not antinormal by Remark 11. $\square$

One of the difficult issues is to find a useful description of the adjoint of uC_φ for arbitrary φ. In 2011, Matache [17] showed that if $\varphi(0) = 0$ then $C_\varphi^*(z^n) = \sum_{k=1}^{k=n} \langle z^n, \varphi^k \rangle z^k$ for each $n = 1, 2, 3, \ldots$. We have used his idea in the following theorem.

Theorem 24. *If self-map $\varphi(z) = z^n$ for $n \geq 2$ and $|u(z)| = 1$ a.e. on $\partial\mathbb{D}$, then uC_φ is antinormal.*

Proof. Let $f(z) = \sum_{r \geq 0} a_r z^r \in H^2(\mathbb{D})$. Then,

$$
\begin{aligned}
\left(uC_\varphi\right)^* \left(uC_\varphi\right) f(z) &= \sum_{r \geq 0} a_r \left(uC_\varphi\right)^* \left(uC_\varphi\right)(z^r) \\
&= \sum_{r \geq 0} a_r \left(uC_\varphi\right)^* \left(u(z)\varphi^r(z)\right) \\
&= \sum_{r \geq 0} a_r \sum_{k \geq 0} \left\langle uC_\varphi \right\rangle^* \left(u(z)\varphi^r(z), z^k\right) z^k \\
&= \sum_{r \geq 0} a_r \sum_{k \geq 0} \left\langle u(z)\varphi^r(z), uC_\varphi(z^k)\right\rangle z^k \\
&= \sum_{r \geq 0} a_r \sum_{k \geq 0} \left\langle u(z)\varphi^r(z), u(z)\varphi^k(z)\right\rangle z^k \\
&= \sum_{r \geq 0} a_r \sum_{k \geq 0} \left(\int_{\partial\mathbb{D}} u(e^{i\theta})\varphi^r(e^{i\theta})\overline{u(e^{i\theta})\varphi^k(e^{i\theta})}\frac{d\theta}{2\pi}\right) z^k \\
&= \sum_{r \geq 0} a_r \sum_{k \geq 0} \left(\int_{\partial\mathbb{D}} e^{inr\theta} e^{-ink\theta}\frac{d\theta}{2\pi}\right) z^k = \sum_{r \geq 0} a_r z^r = f(z).
\end{aligned} \tag{17}
$$

Therefore $(uC_\varphi)^*(uC_\varphi) - \alpha I = (1 - \alpha)I \; \forall \alpha \geq 0$. This implies that $(uC_\varphi)^*(uC_\varphi) - \alpha I$ is invertible for $0 \leq \alpha < 1$. Hence, $(uC_\varphi)^*(uC_\varphi) - \alpha I$ is Fredholm for $0 \leq \alpha < 1$. Thus, $\alpha \notin \sigma_e(|uC_\varphi|)$ whenever $0 \leq \alpha < 1$. Also $\dim \ker((uC_\varphi)^*(uC_\varphi) - \alpha I)$ is infinite for $\alpha = 1$. Hence, $1 \in \sigma_e(|uC_\varphi|)$. Therefore, $m_e(uC_\varphi) = \inf\{\alpha \geq 0 : \alpha \in \sigma_e(|uC_\varphi|)\} = 1$. Since $\varphi(0) = 0$ and $|u(z)| = 1$ a.e. on $\partial\mathbb{D}$, $\|uC_\phi\| = \|C_\varphi\| = 1$. Further, uC_φ is injective, and uC_φ is not surjective as φ is not

an automorphism. Consequently, $\text{index}(uC_\varphi) < 0$. Hence, the antinormality of uC_φ follows from Theorem 12. $\qquad\square$

Following corollary is an immediate consequence of the preceding theorem.

Corollary 25. *If self-map* $\varphi(z) = z^n \ \forall z \in \mathbb{D}$, $n \geq 2$, *then* C_φ *is antinormal.*

Competing Interests

The authors declare that they have no competing interests.

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