Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2016, Article ID 4969253, 10 pages
http://dx.doi.org/10.1155/2016/4969253

Research Article

Existence of General Competitive Equilibria: A Variational Approach

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Received 30 December 2015; Revised 26 March 2016; Accepted 7 April 2016

Academic Editor: Naseer Shahzad

We study the existence of general competitive equilibria in economies with agents and goods in a finite number. We show that there exists a Walras competitive equilibrium in all ownership private economies such that, for all consumers, initial endowments do not contain free goods and utility functions are locally Lipschitz quasiconcave. The proof of the existence of competitive equilibria is based on variational methods by applying a theoretical existence result for Generalized Quasi Variational Inequalities.

1. Introduction

The proof of the existence of economic equilibria is certainly considered the first principal problem to be solved in General Equilibrium Theory (see [1]). Specifically, the question is to investigate what assumptions concerning environment and mechanism are able to guarantee the existence of one state where the aggregate demand does not excess the aggregate supply for all markets at prices endogenously determined.

In the model à-la Arrow-Debreu (see [2, 3], for major details) convexity and closure for the choice sets, price systems in the unit simplex, closure and convexity for the production plans, impossibility that two plans are able to cancel an other one, continuity and monotonicity for the utility functions, and nonsatiability and global survivability for the consumers are its main assumptions. Over the years, the assumptions' successive refinements had to contextually (1) show the properties of consequent demand (see the literature about regular economies, for instance) and (2) adopt suitable mathematical tools able either to compute or only to show the existence of a feasible equilibrium. In favour of this thesis we report the results in [2–9], obtained by Kakutani's fix point theorem in [10], by gradient's algorithm, or by the techniques known as *differential approach*, because continuity, differentiability, and concavity are guaranteed for utility functions or for the preference relations.

Given the classicism of the problem and many excellent papers, as listed above, and books on the economic equilibrium (see, e.g., [11–13]), there is however the need to justify yet an other paper on the subject.

It is well known that variational analysis introduced by Stampacchia in [14, 15] became an extremely useful tool to solve the optimization problems and so to give a solution to the competitive equilibrium problem in economics (see, e.g., [5, 16, 17] and the references therein), too. Furthermore, the importance of concavity in economics for describing the increase of the consumer's preferences and the consequential returns to scale (see, e.g., [7]) is well known. From convex analysis, clearly, strong concavity implies uniform concavity, uniform concavity implies strict concavity, strict concavity implies (weak) concavity, (weak) concavity implies quasiconcavity, and quasiconcavity does not imply differentiability. Therefore, combining all these facts, the purpose of this paper is to give a more extensive result on the existence of general competitive equilibrium by using nonsmooth analysis (see, e.g., [18, 19]) combined with variational analysis (see also [20]).

Our model describes a private ownership economy with two classes of agents, consumers and producers, and a finite number of no free goods at disposal. Consumers, as shareholders, will control the producers. The equilibrium will be realized when each consumer optimizes his utility under the budget set, each producer will realize the maximum profit according to his own production plan, and the sum of the total endowment plus the total consumption will not exceed the total production. In this economic scenery we will admit a list of assumptions which for now is the more generalized both from a mathematical and from an economic viewpoint. In detail, the assumptions are as follows:

(i) the initial endowment of any consumer will consist of at least one type of goods of minima or greater price (absence of free goods in the initial endowment) (as in [21]);

(ii) the production set Y_j of the jth producer will be a convex and compact set in $\mathbb{R}^\ell$ containing the origin;

(iii) the utility function u_i of the ith consumer will be assumed quasiconcave and locally Lipschitz continuous.

The principal novelty consists in the assumptions on the utility function. As said above, because a quasiconcave function is not always differentiable, adding the locally Lipschitz condition we can drop gradient with generalized gradient (see [19, pages 25–28]) and so, differently to what it was proved in [6, 21], we are able to advance every possibility on the return to scale in satisfaction terms for the consumer. In other words, since the quasiconcave condition for utility function is weaker than weak concave and strong concave ones, or, also, it is the most general assumption in Consumer's Theory (see [7, 9]), this fact allows us to express how marginal utility changes for any increase of consumption. Thus for the consumer's progressive satiety we could attend all the feasible consequences which go from a possible reduction (exclusively for the strong concave utility) to a possible increase (typical for the quasiconcave utility) through the constant state (typical for the weak concave utility). From this fact, we can reasonably take into account a wide range of utility functions including also the typical economic ones, in generalized form, as Cobb Douglas's and Constant Elasticity of Substitution (briefly CES) class, which, till now, have got involved in the proof of the existence of a general competitive equilibrium iff the utility is supposed to be concave (see [21]). With a more complicated type of generalized quasivariational inequality (see [20] for major details) and with the help of nonsmooth analysis (see, e.g., [18] for the continuity of set-valued map) we can yet treat the existence of a competitive economic equilibrium problem via variational method in generalized way with respect to [6, 21, 22]. The proof of the existence of competitive equilibria will be based on variational methods and, in particular, on an abstract existence result for Generalized Quasi Variational Inequalities due to Cubiotti (Theorem 3.2 in [23]).

Furthermore, an other novel result is that every competitive economic equilibrium is also a Walras competitive equilibrium (see Proposition 14).

Finally, we point out that our main result contains, as special cases, some recent existence results of competitive equilibria for pure exchange economics established in [24, 25]. Probably, our techniques also work if we consider the dynamic version of pure exchange economics introduced in [16, 17] (see also [26, 27]) and so the existing results of these papers could be improved by replacing the concavity condition with the quasiconcavity condition.

The organization of the remainder of this paper is as follows.

In Section 2 for the sake of convenience, we will recall the main notations, definitions and results that will be used in the sequel for our analysis.

In Section 3 we will describe the economic model with its environments and related restrictions and its internal mechanisms, defining the general economic equilibrium problem through constrained maximization problems system.

In Section 4 we will point out the connections between the competitive economic equilibrium problem and a suitable GQVI, proving also that in a suitable compact set any competitive equilibrium is a Walras competitive equilibrium.

In Section 5 we will prove the existence of a solution to GQVI, recalling, in particular, Theorem 3.2 of Cubiotti in [23] and, thus, concluding with the existence of at least one (Walras) competitive equilibrium for the private ownership economy considered.

In Section 6 we shall show how two well-known functions, Cobb-Douglas and CES, could be employed as utility functions in economics under the locally Lipschitz and quasiconcavity assumption.

2. Preliminaries

Throughout this paper, for each $n \in \mathbb{N}$, $\mathbb{R}^n$ denotes the Euclidean space of the real n-vectors $x = (x_1, \ldots, x_n)$ equipped with the usual inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ and norm $|x|_n = \sqrt{\langle x, x\rangle}$, for any $x, y \in \mathbb{R}^n$.

The symbols $\mathbb{R}^n_+$, $\mathbb{R}^n_{0+}$, and $\mathring{\mathbb{R}}^n_{0+}$ will indicate the cone of nonnegative, positive, and strongly positive vectors of $\mathbb{R}^n$, respectively. Furthermore, the set $\Delta_{n-1} = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$ indicates the unit simplex of $\mathbb{R}^n_+$. We adopt the usual notation for vector inequalities, that is, for any $x, y \in \mathbb{R}^n$, one has $x \geq y$ if $x - y \in \mathbb{R}^n_+$; $x > y$ if $x - y \in \mathbb{R}^n_{0+}$; and $x \gg y$ if $x - y \in \mathring{\mathbb{R}}^n_{0+}$. Let $A \subset \mathbb{R}^n$; we will write $\operatorname{int}(A)$ and $\overline{A}$ to indicate its interior and its closure, respectively. Open and closed balls of radius ε, centered at $x \in \mathbb{R}^n$, are denoted by $B_\varepsilon(x)$ and $\overline{B}_\varepsilon(x)$, respectively.

Let X be a subset of $\mathbb{R}^n$. A function $f : X \to \mathbb{R}$ is said to be *quasiconcave* iff for every $r < \sup_X f$, the set $\{x \in X : f(x) \geq r\}$ is convex. A function $f : X \to \mathbb{R}$ is said to be *locally Lipschitz continuous* near $x \in X$ if there exist constants $L > 0$ and $\varepsilon > 0$ such that $y \in B_\varepsilon(x)$ implies $|f(y) - f(x)| \leq L|y - x|_n$. L is called *Lipschitz constant* or rank of f.

Let f be a locally Lipschitz function near $x \in X$ and let $z \in \mathbb{R}^n$. According to Clarke (see [19, page 25]), the *generalized directional derivative* of f at x in the direction z is defined by

$$f^{\circ}(x,z) := \limsup_{y\to x,\sigma\to 0^+} \frac{f(y+\sigma z)-f(y)}{\sigma}, \tag{1}$$

and the *generalized gradient*, or simply *subdifferential*, of f at x is the set-valued mapping, $x \to \partial^{\circ} f(x)$, defined as follows:

$$\partial^{\circ} f(x) := \{T \in \mathbb{R}^n : f^{\circ}(x,z) \geq \langle T,z\rangle,\ \forall z \in \mathbb{R}^n\}. \tag{2}$$

For the sequel, we recall below three results on generalized derivative and subdifferential, here, for convenience, rewritten when X is a subset of $\mathbb{R}^n$.

Proposition 1 (Proposition 2.1.1 of [19]). *Let f be a locally Lipschitz function near x of* rank L. *Then the following hold:*

(a) *the function $z \to f^{\circ}(x,z)$ is finite, positively homogeneous, and subadditive on R^n and satisfies*

$$f^{\circ}(x,z) \leq L|z|_n, \tag{3}$$

(b) $f^{\circ}(x,-z) = (-f)^{\circ}(x,z)$.

Proposition 2 (Proposition 2.1.2 of [19]). *Let f be a locally Lipschitz function near x of* rank L. *Then the following hold:*

(a) $\partial^{\circ} f(x)$ *is a nonempty, convex, and compact set of* $\mathbb{R}^n$,

(b) *for every $z \in \mathbb{R}^n$, one has*

$$f^{\circ}(x,z) = \max\{\langle T,z\rangle : T \in \partial^{\circ} f(x)\}. \tag{4}$$

Proposition 3 (Proposition 2.1.5 of [19]). *Let f be a locally Lipschitz function near x of* rank L. *Then*

(a) *the multifunction ∂f is closed,*

(b) *the multifunction ∂f is upper semicontinuous at x.*

Let X be a set; we write 2^X for the family of all nonempty subsets of X. A *correspondence* or a *multifunction* between two sets X and Y is a function $F : X \to 2^Y$. The *graph* of a multifunction $F : X \to 2^Y$ is the subset of $X \times X$ defined by $\mathrm{gr}(F) = \{(x,y) \in X \times Y : x \in X \wedge y \in F(X)\}$. Let X be a subset of $\mathbb{R}^n$ and let $\Gamma : X \to 2^X$, $\Phi : X \to 2^{\mathbb{R}^n}$ be two multifunctions. The classical *generalized quasivariational inequality problem* associated to X, Γ, Φ, denoted briefly by $\mathrm{GQVI}(X,\Gamma,\Phi)$, is to find $(\overline{x},\overline{z}) \in X \times \mathbb{R}^n$ such that

$$\begin{gathered}\overline{x} \in \Gamma(\overline{x}),\\ \overline{z} \in \Phi(\overline{x}),\\ \langle \overline{z}, \overline{x}-y\rangle \leq 0 \quad \forall y \in \Gamma(\overline{x}).\end{gathered} \tag{5}$$

For the reader convenience, we report here the statement of Theorem 3.2 of [23], which the prove of our main result is based on.

Theorem 4 (Theorem 3.2 of Cubiotti [23]). *Let X be a closed convex subset of $\mathbb{R}^n$, $K \subseteq X$ a nonempty compact set, and $\Phi : X \to 2^{\mathbb{R}^n}$ and $\Gamma : X \to 2^X$ two multifunctions. Assume the following:*

(i) *the set $\Phi(x)$ is convex for each $x \in K$, with $x \in \Gamma(x)$;*

(ii) *the set $\Phi(x)$ is nonempty and compact for each $x \in X$;*

(iii) *for each $y \in X-X$, the set $\{x \in X : \inf_{z\in\Phi(x)}\langle z,y\rangle \leq 0\}$ is closed;*

(iv) *Γ is a lower semicontinuous multifunction (i.e., $\{x \in X : \Gamma(x) \cap A \neq \emptyset\}$ is open in X, for each open set A in X) with closed graph and convex values.*

Moreover, assume that there exists an increasing sequence ϵ_k of positive real numbers, with $X \cap \overline{B}(0,\epsilon_1) \neq \emptyset$ and $\lim_{k\to\infty}\epsilon_k = +\infty$ such that if one puts $D_k = \overline{B}(0,\epsilon_k)$, for each $k \in \mathbb{N}$ one has the following:

(v) *$\Gamma(x) \cap D_k \neq \emptyset$, for all $x \in X \cap D_k$;*

(vi) *for each $x \in (X \cap D_k) \setminus K$, with $x \in \Gamma(x)$,*

$$\sup_{y\in\Gamma(x)\cap D_k} \inf_{z\in\Phi(x)} \langle z, x-y\rangle > 0. \tag{6}$$

Then, there exists at least one solution to $\mathrm{GQVI}(X,\Gamma,\Phi)$ *belonging to $K \times \mathbb{R}^n$.*

3. Economic Model

We consider a private ownership economy $\mathscr{E}$ à-la Arrow-Debreu (see [2, 3] for major details), where there are ℓ commodities, m producers, and n consumers ($\ell, n, m \in \mathbb{N}$). We index the commodities by the subscripts $h = 1,\dots,\ell$, the producers by the subscripts $j = 1,\dots,m$, and the consumers by the subscripts $i = 1,\dots,n$. We regard $\mathbb{R}^{\ell}$ as the commodity space. By assuming the vector $p = (p_1,\dots,p_{\ell}) \in \mathbb{R}^{\ell}_+$ as a price system, the value of a commodity bundle $a \in \mathbb{R}^{\ell}$ relative to the price p will be given by the inner product $\langle p,a\rangle = \sum_{h=1}^{\ell} p_h a_h$.

3.1. Environments. The n consumers are labeled by $i = 1,\dots,n$. The consumption set $X_i = \mathbb{R}^{\ell}_+$ related to the consumer i is the set of all the ℓ-uple of nonnegative real numbers. The preferences of the consumer i are expressed by utility functions $u_i : X_i \to \mathbb{R}$, endowments by vectors $e_i \in X_i$, and shares of the profits of each firm by vectors $\theta_i \in [0,1]^m$. The m producers are labeled by $j = 1,\dots,m$. The production set $Y_j \subseteq \mathbb{R}^{\ell}$ related to consumer i is a subset of the set of all the ℓ-tuple of real numbers. The aggregate endowment is $e = \sum_{i=1}^{n} e_i$. We assume that the firms are owed by someone, or in other terms, $1 = \sum_{i=1}^{n}\theta_{ij}$ with $\theta_{ij} \in [0,1]$ for all $j = 1,\dots,m$. We summarized a private ownership economy by the tuple $\mathscr{E} = (\{X_i, e_i, u_i, \theta_i\}_i, \{Y_j\}_j, e)$.

3.2. Basic Restrictions. For a private ownership economy $\mathscr{E}$ we assume the restrictions listed as follows:

(A1) any price system $p = (p^{(1)}, \ldots, p^{(\ell)}) \in \mathbb{R}_+^\ell$ is normalized and bounded below by the vector $q = (q^{(1)}, \ldots, q^{(\ell)}) \in \mathbb{R}_+^\ell$, called the minima prices, whose hth component does not exceed the value of $1/\ell$, or equivalently

$$P = \left\{p \in \Delta_{\ell-1} : q^{(h)} \leq p^{(h)},\ \forall h = 1, \ldots, \ell\right\} \tag{7}$$

is the set of the available price systems;

(A2) in any initial commodity bundle $e_i \in \mathbb{R}_+^\ell$ there exists at least one good h of positive quantity and positive minimum price:

$$\text{there exists } \overline{h} \in \{1, \ldots, \ell\} \text{ such that, } \forall i = 1, \ldots, n,\ q^{(\overline{h})} > 0,\ e_i^{(\overline{h})} > 0; \tag{8}$$

(A3) for all $j = 1, \ldots, m$, Y_j is convex and compact in R^ℓ such that $0_{\mathbb{R}^l} \in Y_j$;

(A4) there exists an open convex $A \supset \mathbb{R}^\ell$ such that, for all $i = 1, \ldots, n$, the following holds:

(a) $u_i : A \to \mathbb{R}$ is locally Lipschitz continuous and quasi-concave,

(b) $0_{\mathbb{R}^\ell} \notin \partial^\circ(-u_i)(x_i)$ for all $x_i \in K_i$,

(c) $(-u_i)^\circ(x_i, \mathbf{i}_h) < 0$, for all $x_i \in K_i$, and $h = 1, \ldots, \ell$ such that $x_i^{(h)} = 0$,

where

$$K_i = \prod_{i=1}^{n} \left(\bigcup_{h=1}^{\ell} \left\{x_i \in \mathbb{R}_+^\ell : x_i^{(h)} \leq e_i^{(h)} + M\right\} \cap \prod_{b=1}^{\ell} \left[0, \sum_{a=1}^{n} e_a^b + M\right] \right), \tag{9}$$

with $M = \max_{1\leq j\leq m} \max_{(y_j^{(1)},\ldots,y_j^{(\ell)})\in Y_j} \sum_{h=1}^{l} |y_j^{(h)}|$, and $\mathbf{i}_h$ is the unit vector of the hth axis.

Remark 5. Locally Lipschitz condition in an open convex set $A \supset \mathbb{R}_+^\ell$, listed in (A4), will be needed to consider the Clark-subdifferential of u_i at each point of the closed set $\mathbb{R}_+^\ell$. Quasiconcavity condition (b) in (A4) is weaker than the concavity condition and the strictly concave condition usually considered in the literature. Finally, the existence of the constant M comes from the compactness of the set $\bigcup_{j=1}^m Y_j$ (see assumption (A3)).

3.3. Existence Equilibrium Problem. Because consumer i owns two resources (initial endowment and profit share), indicated by $\langle p, x_i\rangle$ his expenditure and by $\langle p, y_j\rangle$ his profit derived from producer j,

$$M_i(p, y) := \left\{x_i \in \mathbb{R}_+^\ell : \langle p, x_i\rangle \leq \langle p, e_i\rangle + \max\left\{0, \left\langle p, \sum_{j=1}^{m} \theta_{ij} y_j \right\rangle\right\}\right\} \tag{10}$$

represents his budget set.

Set $Y = \prod_{j=1}^m Y_j$, and $M(p, y) = \prod_{i=1}^n M_i(p, y)$.

Definition 6. An *allocation* is a couple $(x, y) \in \mathbb{R}_+^{n\times\ell} \times Y$, where $x = (x_1, \ldots, x_n)$ represents the consumptions of all the consumers and $y = (y_1, \ldots, y_m)$ represents the productions of all the producers. In particular, an allocation (x, y) is said to be an *individual allocation* if $x \in M(p, y)$. An allocation is said to be a *weakly balanced allocation* if $\sum_{i=1}^n (x_i - e_i) - \sum_{j=1}^m y_j \leq 0$. Finally, an allocation is said to be an *available allocation* if it is both individual and weakly balanced.

Definition 7. A *state* of the economy $\mathscr{E}$ is a triple $(p, x, y) \in P \times \mathbb{R}_+^{n\times\ell} \times Y$, where p is a price vector and (x, y) is an available allocation.

Considering consumers and producers as price-takers, the above model leads to the following general economic problem.

Problem 8. Find $(\overline{p}, \overline{x}, \overline{y}) \in P \times \mathbb{R}_+^{n\times\ell} \times Y$, with $\overline{x} \in M(\overline{p}, \overline{y})$ satisfying

$$u_i(\overline{x}_i) = \max_{x_i \in M_i(\overline{p}, \overline{y})} u_i(x_i) \quad \forall i = 1, \ldots, n \tag{11a}$$

$$\left\langle \overline{p}, \overline{y}_j \right\rangle = \max_{y_j \in Y_j} \left\langle \overline{p}, y_j \right\rangle \quad \forall j = 1, \ldots, m \tag{11b}$$

$$\sum_{i=1}^{n} \left(\overline{x}_i^{(h)} - e_i^{(h)}\right) - \sum_{j=1}^{m} \overline{y}_j^{(h)} \leq 0 \quad \forall h = 1, \ldots, \ell. \tag{11c}$$

Remark 9. A solution to Problem 8 is, then, a price vector $\overline{p}$ and an available allocation $(\overline{x}, \overline{y})$ such that $\overline{x}$ maximizes the consumers' utility, $\overline{y}$ maximizes the producers' profit, and $(\overline{x}, \overline{y})$ makes the market clear.

Definition 10. A state $(\overline{p}, \overline{x}, \overline{y}) \in P \times \mathbb{R}_+^{n\times\ell} \times Y$, with $\overline{x} \in M(\overline{p}, \overline{y})$, satisfying conditions (11a), (11b), and (11c) of Problem 8 is said to be a *competitive equilibrium* or *free disposal-equilibrium* for the private ownership economy $\mathscr{E}$.

An equilibrium $(\overline{p}, \overline{x}, \overline{y}) \in P \times M(\overline{p}, \overline{y}) \times Y$ is said to be a *Walras competitive equilibrium* if in addition it satisfies the Walras' law:

$$\langle \overline{p}, \overline{x}_i \rangle = \langle \overline{p}, e_i \rangle + \max \left\{ 0, \left\langle \overline{p}, \sum_{j=1}^{m} \theta_{ij} y_j \right\rangle \right\} \quad \forall i = 1, \ldots, n. \tag{12}$$

Remark 11. If there is no production (i.e., $Y_j = 0_{\mathbb{R}^\ell_+}$ for all $j = 1, \ldots, m$), then $\mathscr{E}$ becomes a pure exchange economy. In this case Problem 8 assumes the form of the one in [24, 25].

4. Variational Method

Now, we establish a GQVI problem as follows.

Problem 12. Find $(\overline{p}, \overline{x}, \overline{y}) \in P \times \mathbb{R}^{n\times\ell}_+ \times Y$, with $\overline{x} \in M(\overline{p}, \overline{y})$, such that there exists $T = (T_1, \ldots, T_n) \in \prod_{i=1}^{n} \partial^\circ(-u_i)(\overline{x}_i)$ satisfying

$$\begin{gathered} -\sum_{i=1}^{n} \langle T_i, x_i - \overline{x}_i \rangle + \sum_{j=1}^{m} \langle \overline{p}, y_j - \overline{y}_j \rangle \\ + \left\langle \sum_{i=1}^{n} (\overline{x}_i - e_i) - \sum_{j=1}^{m} \overline{y}_j, p - \overline{p} \right\rangle \leq 0, \\ \forall (p, x, y) \in P \times M(\overline{p}, \overline{y}) \times Y. \end{gathered} \tag{GQVI}$$

Let $K = \prod_{i=1}^{n} K_i$, where K_i is as in condition (A4c).

Proposition 13. *Let $(\overline{p}, \overline{x}, \overline{y}) \in P \times \mathbb{R}^{n\times\ell}_+ \times Y$, with $\overline{x} \in M(\overline{p}, \overline{y})$, satisfying condition (11c) of Problem 8. Then $\overline{x} \in K$.*

Proof. From (11c) of Problem 8 and the definition of the constant M, it promptly follows that

$$\overline{x}_i \in \prod_{b=1}^{\ell} \left[0, \sum_{a=1}^{n} e_a^{(b)} + M \right], \tag{13}$$

for all $i = 1, \ldots, n$. Now, suppose that $\overline{x} \notin K$. Then,

$$\overline{x}_i \notin \bigcup_{h=1}^{\ell} \left\{ x_i \in X_i : x_i^{(h)} \leq e_i^{(h)} + M \right\} \tag{14}$$

for some $i \in \{1, \ldots, n\}$. This means that, for all $h = 1, \ldots, \ell$, we should have

$$\overline{x}_i^h > e_i^{(h)} + M \geq e_i^{(h)} + \sum_{j=1}^{m} \left| \overline{y}_j^{(h)} \right|. \tag{15}$$

Hence,

$$\begin{aligned} \langle \overline{p}, \overline{x}_i \rangle &> \langle \overline{p}, e_i \rangle + \sum_{j=1}^{m} \sum_{h=1}^{\ell} p^{(h)} \left| \overline{y}_j^{(h)} \right| \\ &\geq \langle \overline{p}, e_i \rangle + \max \left\{ 0, \left\langle \overline{p}, \sum_{j=1}^{m} \theta_{ij} \overline{y}_j \right\rangle \right\}, \end{aligned} \tag{16}$$

which is in contradiction with $\overline{x}_i \in M_i(\overline{p}, \overline{y})$. Thus, $\overline{x} \in K$. □

Proposition 14. *Let assumption (A4) be entirely satisfied. Then, any competitive equilibrium is a Walras competitive equilibrium.*

Proof. Let $(\overline{p}, \overline{x}, \overline{y}) \in P \times M(\overline{p}, \overline{y}) \times Y$ be a competitive equilibrium and fix $i \in \{1, \ldots, n\}$. From Proposition 13, we have $\overline{x}_i \in K_i$. So, by assumption (A4b) and condition (11a) of Problem 8 it cannot be that $\overline{x}_i \in \operatorname{int}(M_i(\overline{p}, \overline{y}))$.

Moreover, assumption (A4c) and again condition (11a) of Problem 8 imply that $x_i^{(h)} > 0$ for all $h = 1, \ldots, l$. Therefore, (12) is verified, which means that $(\overline{p}, \overline{x}, \overline{y})$ is a Walras competitive equilibrium. □

The next proposition will be needed for the main theorem of this section.

Proposition 15. *Let assumption (A4a) be satisfied. Let $i \in \{1, \ldots, n\}$ and let $x_i, z_i \in \mathbb{R}^\ell_+$ be such that $u_i(x_i) < u_i(z_i)$. Then, $(-u_i)^\circ(x_i, z_i - x_i) \leq 0$.*

Proof. Let $\{(y_n, t_n)\}$ be a sequence in $A \times (0, 1)$ (the set A is as in (A4a)) such that $(y_n, t_n) \to (x_i, 0)$. In force of locally Lipschitz continuity (and thus continuity) and $u_i(x_i) < u_i(z_i)$, we can suppose that $u_i(y_n) < u_i(z_i)$ for all $n \in \mathbb{N}$. In force of quasiconcavity, one has $u_i(y_n) \leq u_i(y_n + t_n(z_i - y_n))$ for all $n \in \mathbb{N}$. Consequently,

$$\limsup_{n\to\infty} \frac{-u_i(y_n + t_n(z_i - y_n)) + u_i(y_n)}{t_n} \leq 0. \tag{17}$$

Taking into account the arbitrariness of the sequence $\{(y_n, t_n)\}_{n\in N}$, conclusion follows. □

The next theorem will state that under the above assumptions any competitive equilibrium is a solution to Problem 12.

Theorem 16. *Let assumptions (A1), (A2), (A3), and (A4c) be satisfied. Moreover, let $(\overline{p}, \overline{x}, \overline{y}) \in P \times \mathbb{R}^\ell_+ \times Y$, with $\overline{x} \in M(\overline{p}, \overline{y})$. Assume that $(\overline{p}, \overline{x}, \overline{y})$ is a solution to Problem 12. Then, $(\overline{p}, \overline{x}, \overline{y})$ is a solution to Problem 8.*

Proof. Let $(\overline{p}, \overline{x}, \overline{y})$ be a solution to Problem 12, and let $T \in \prod_{i=1}^{n} \partial^\circ(-u_i)(\overline{x}_i)$ satisfying inequality (GQVI).

Testing (GQVI) with $(p, \overline{x}, \overline{y})$, $p \in P$, one has

$$\left\langle \sum_{i=1}^{n} (\overline{x}_i - e_i) - \sum_{j=1}^{m} \overline{y}_j, p - \overline{p} \right\rangle \leq 0 \quad \forall p \in P. \tag{18}$$

Moreover, from $\overline{x} \in M(\overline{p}, \overline{y})$, we promptly obtain

$$\left\langle \sum_{i=1}^{n} (\overline{x}_i - e_i) - \sum_{j=1}^{m} \overline{y}_j, \overline{p} \right\rangle \leq 0 \quad \forall p \in P. \tag{19}$$

Hence,

$$\left\langle \sum_{i=1}^{n} (\overline{x}_i - e_i) - \sum_{j=1}^{m} \overline{y}_j, p \right\rangle = \left\langle \sum_{i=1}^{n} (\overline{x}_i - e_i) - \sum_{j=1}^{m} \overline{y}_j, p - \overline{p} \right\rangle + \left\langle \sum_{i=1}^{n} (\overline{x}_i - e_i) - \sum_{j=1}^{m} \overline{y}_j, \overline{p} \right\rangle \le 0 \quad \forall p \in P. \tag{20}$$

Now, choosing $p = (0, \dots, 0, 1, 0, \dots, 0) \in P$ (the fact is possible in force of assumption (A1)), with 1 at the hth position, we obtain condition (11c) of Problem 8. Furthermore, for each fixed $j \in \{1, \dots, m\}$, testing (GQVI) with $(\overline{p}, \overline{x}, (\overline{y}_1, \dots, \overline{y}_{j-1}, y_j, \overline{y}_{j+1}, \dots, \overline{y}_m))$, $y_j \in Y_j$, we obtain $\langle \overline{p}, y_j - \overline{y}_j \rangle \le 0$, for all $y_j \in Y_j$, which is condition (11b) of Problem 8.

At this point, condition (11a) of Problem 8 remains the only one to be proved. Fix $i \in \{1, \dots, n\}$. Testing (GQVI) with $(\overline{p}, (\overline{x}_1, \dots, \overline{x}_{i-1}, x_i, \overline{x}_{i+1}, \dots, \overline{x}_n), \overline{y})$, $x_i \in M_i(\overline{p}, \overline{y})$, we obtain

$$\langle T_i, x_i - \overline{x}_i \rangle \ge 0. \tag{21}$$

Testing, for each $h \in \{1, \dots, \ell\}$, inequality (21) with

$$x_i = \left(\overline{x}_i^{(1)}, \dots, \overline{x}_i^{(h-1)}, 0, \overline{x}_i^{(h+1)}, \dots, \overline{x}_i^{(\ell)}\right), \tag{22}$$

one has

$$T_i^{(h)} \overline{x}_i^{(h)} \le 0, \quad \text{for each } h \in \{1, \dots, \ell\}, \tag{23}$$

which clearly implies $\langle T_i, \overline{x}_i \rangle \le 0$. We claim that

$$\langle T_i, \overline{x}_i \rangle < 0. \tag{24}$$

Indeed, assume, on the contrary, that

$$\langle T_i, \overline{x}_i \rangle = 0. \tag{25}$$

Then, for each $h = 1, \dots, \ell$, one has $x_i^{(h)} > 0$. Indeed, if $x_i^{(h)} = 0$ for some $h \in \{1, \dots, \ell\}$, by condition (A4c), we should have

$$T_i^{(h)} \cdot \rho \le \rho \cdot (-u_i)^{\circ} (\overline{x}_i, \mathbf{i}_h) < 0, \quad \text{for each } \rho > 0. \tag{26}$$

Now, from assumption (A2), there exists $h \in \{1, \dots, \ell\}$ such that $\overline{p}^{(h)} > 0$ and $e_i^{(h)} > 0$. So, in particular, $\langle e_i, \overline{p} \rangle > 0$. Then, if we fix $\rho > 0$ small enough, we have $x_\rho := (0, 0, \dots, \rho, 0, \dots, 0) \in M_i(\overline{p}, \overline{y})$, with ρ at the hth position. Consequently, testing (21) with $x_i = x_\rho$ and taking (25) into account, we have

$$0 \le \langle T_i, x_\rho - \overline{x}_i \rangle = \langle T_i, x_\rho \rangle - \langle T_i, \overline{x}_i \rangle = T_i^{(h)} \cdot \rho, \tag{27}$$

which is in contradiction with (26). Therefore, $x_i^{(h)} > 0$, for each $h = 1, \dots, \ell$. In view of (23) and (25) this implies $T_i = 0_{\mathbb{R}^l}$, which is again in contradiction with (26). Then, strict inequality (24) holds. At this point, let $z_i \in M_i(\overline{p}, \overline{y})$. By observing that $z_i/2, \overline{x}_i/2 \in M_i(\overline{p}, \overline{y})$, if we put $y_i^{\theta,1} = (1-\theta)(\overline{x}_i/2) + \theta(z_i/2)$ and $y_i^{\theta,2} = (1-\theta)\overline{x}_i + \theta(\overline{x}_i/2)$, for all $\theta \in (0,1)$, by the convexity of $M_i(\overline{p}, \overline{y})$, we infer $y_i^{\theta,1}, y_i^{\theta,2} \in M_i(\overline{p}, \overline{y})$. So, taking (21) into account, one has

$$\left\langle T_i, y_i^{\theta,1} - \frac{\overline{x}_i}{2} \right\rangle = \left\langle T_i, (1-\theta)\frac{\overline{x}_i}{2} + \theta\frac{z_i}{2} - \frac{\overline{x}_i}{2} \right\rangle = \frac{\theta}{2} \left\langle T_i, \frac{z_i}{2} - \overline{x}_i \right\rangle \ge 0. \tag{28}$$

Moreover, in view of (24), one also has

$$\left\langle T_i, \frac{\overline{x}_i}{2} - y_i^{\theta,2} \right\rangle = \left\langle T_i, \frac{\overline{x}_i}{2} - (1-\theta)\overline{x}_i - \theta\frac{\overline{x}_i}{2} \right\rangle = -\frac{1}{2}(1-\theta)\langle T_i, \overline{x}_i \rangle > 0, \tag{29}$$

for all $\theta \in (0,1)$. Adding side to side the above inequalities, we obtain

$$\begin{aligned} 0 &< \left\langle T_i, y_i^{\theta,1} - y_i^{\theta,2} \right\rangle \\ &= \left\langle T_i, (1-\theta)\frac{\overline{x}_i}{2} + \theta\frac{z_i}{2} - (1-\theta)\overline{x}_i - \theta\frac{\overline{x}_i}{2} \right\rangle \\ &= \frac{1}{2}\langle T_i, \theta z_i - \overline{x}_i \rangle \le \frac{1}{2}(-u_i)^{\circ}(\overline{x}_i, \theta z_i - \overline{x}_i), \\ &\qquad \forall \theta \in (0,1). \end{aligned} \tag{30}$$

From the above inequality and Proposition 15, it follows that $u_i(\overline{x}_i) \ge u_i(\theta z_i)$, for all $\theta \in (0,1)$. By the continuity of u_i, we then obtain $u_i(\overline{x}_i) \ge u_i(z_i)$. From the arbitrariness of $z_i \in M_i(\overline{p}, \overline{y})$, condition (11a) of Problem 8 follows. □

Remark 17. Under the assumptions of Theorem 16, condition (11c) actually holds as equality. Indeed, fix $i \in \{1, \dots, n\}$ and define

$$g_i(x_i) := -\sum_{h=1}^{\ell} p^h \left(x_i^{(h)} - e_i^{(h)}\right) + \max\left\{0, \left\langle p, \sum_{j=1}^{m} \theta_{ij} y_j \right\rangle\right\} \quad \forall x_i \in M_i(\overline{p}, \overline{y}). \tag{31}$$

We claim that $g_i(\overline{x}_i) = 0$. Indeed, if not, taking in mind that $\overline{x}_i \in M_i(\overline{p}, \overline{y})$, it should be $g_i(\overline{x}_i) > 0$. Then, for each $h = 1, \dots, l$, there exists $\rho_h > 0$ such that

$$\overline{x}(\rho) := (\overline{x}_1, \dots, \overline{x}_{i-1}, \overline{x}_i + \rho\mathbf{i}_h, \overline{x}_{i+1}, \dots, \overline{x}_n) \in M(\overline{p}, \overline{y}) \quad \forall \rho \in [0, \rho_h[. \tag{32}$$

Testing (GQVI) with $(\overline{p}, \overline{x}(\rho), \overline{y})$, we obtain

$$\rho \langle T_i, \mathbf{i}_h \rangle \ge 0, \quad \forall \rho \in [0, \rho_h[. \tag{33}$$

Thus, in view of (A4c), it must be $\overline{x}_i^h > 0$ for all $h = 1, \dots, l$. This fact, together with $g_i(\overline{x}_i) > 0$, implies

$\overline{x}_i \in \operatorname{int}(M_i(\overline{p}, \overline{y}))$. Testing (GQVI) with $(\overline{p}, x, \overline{y})$, where $x = (\overline{x}_1, \ldots, \overline{x}_{i-1}, x_i, \overline{x}_{i+1}, \ldots, \overline{x}_n)$, with x_i being arbitrarily chosen in $M_i(\overline{p}, \overline{y})$, we obtain $\langle T_i, x_i - \overline{x}_i \rangle \geq 0$ for all $x_i \in M_i(\overline{p}, \overline{y})$. Since $\overline{x}_i \in \operatorname{int}(M_i(\overline{p}, \overline{y}))$, from this inequality it follows that $T_i = 0_{\mathbb{R}^\ell}$ in contradiction with assumption (A4b).

5. Main Result

At this point, it remains to prove that Problem 12 admits at least a solution.

First, we need the following proposition.

Proposition 18. *For each $i = 1, \ldots, n$, the map $\partial^\circ(-u_i) : \mathbb{R}^\ell_+ \to 2^{\mathbb{R}^\ell}$ has closed graph.*

Proof. If, for each $i = 1, \ldots, n$ the utility u_i is locally Lipschitz on $\mathbb{R}^\ell_+$, then according to (b) of Proposition 1, to (a) of Proposition 2, and to (b) of Proposition 3, for each $i = 1, \ldots, n$, the subdifferential of $-u_i$ has closed graph. □

Now, put $X = P \times \mathbb{R}^{n\times\ell}_+ \times Y$ and define

$$
\begin{aligned}
& u(x) = (u_1(x_1), \ldots, u_n(x_n)), \\
& \qquad\qquad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^{n\times\ell}_+, \\
& \Gamma(p, x, y) = P \times M(p, y) \times Y, \quad \forall (p, x, y) \in X, \\
& \Phi(p, x, y) = \Bigg(-\sum_{i=1}^{n} (x_i - e_i) \\
& \quad + \sum_{j=1}^{m} y_j, \partial^\circ(-u)(x), \underbrace{(-p, \ldots, -p)}_{m\text{-times}} \Bigg), \\
& \qquad\qquad \forall (p, x, y) \in X,
\end{aligned}
\tag{34}
$$

where $\partial^\circ(-u)(x) = (\partial^\circ(-u_1)(x_1), \ldots, \partial^\circ(-u_n)(x_n))$, for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^{n\times\ell}_+$.

By means of these notations, we can rewrite the variational inequality (GQVI) as follows: find $(\overline{p}, \overline{x}, \overline{y}) \in P \times \mathbb{R}^{n\times\ell}_+ \times Y$, with $(\overline{p}, \overline{x}, \overline{y}) \in \Gamma(\overline{p}, \overline{x}, \overline{y})$, and $(\widehat{z}, T, \widehat{w}) \in \Phi(\overline{p}, \overline{x}, \overline{y})$ such that

$$
\begin{aligned}
& \langle (\widehat{z}, T, \widehat{w}), (\overline{p}, \overline{x}, \overline{y}) - (p, x, y) \rangle \leq 0, \\
& \qquad\qquad \forall (p, x, y) \in \Gamma(\overline{p}, \overline{x}, \overline{y}).
\end{aligned}
\tag{35}
$$

Theorem 19. *Assume that conditions (A3), (A4a), and (A2) hold. Then, Problem 12 admits at least a solution in $P \times C \times Y$, where*

$$
\begin{aligned}
C \coloneqq \Bigg\{ & x = \left(x_i^{(h)}\right)_{\substack{1\leq i\leq n \\ 1\leq h\leq l}} \in \mathbb{R}^{n\times\ell}_+ : \sum_{i=1}^{n}\sum_{h=1}^{\ell} x_i^{(h)} \leq \sum_{i=1}^{n}\sum_{h=1}^{\ell} e_i^{(h)} \\
& + M \Bigg\},
\end{aligned}
\tag{36}
$$

and $M > 0$ is as in (A4c).

Proof. First, note the following:

(i) the set X is nonempty closed and convex in $\mathbb{R}^\ell \times \mathbb{R}^{n\times\ell} \times \mathbb{R}^{m\times\ell}$;

(ii) the set $K \coloneqq P \times C \times Y \subset X$ is nonempty and compact in $\mathbb{R}^\ell \times \mathbb{R}^{n\times\ell} \times \mathbb{R}^{m\times\ell}$;

(iii) $\Gamma(p, x, y)$ is a nonempty convex subset of X, for all $(p, x, y) \in X$.

Moreover, recalling that $\partial^\circ(-u_i)(x_i)$ is (nonempty) convex and compact in $\mathbb{R}^\ell$, for all $i = 1, \ldots, n$ and for all $x_i \in \mathbb{R}^\ell_+$, we also have the following:

(iv) $\Phi(p, x, y)$ is a nonempty convex and compact subset of $\mathbb{R}^n \times \mathbb{R}^{n\times\ell} \times \mathbb{R}^{m\times\ell}$, for all $(p, x, y) \in X$.

Thus, to satisfy all the assumptions of Theorem 3.2 of [23], it remains to check that the following further conditions hold:

(a_1) the set

$$
\begin{aligned}
\Lambda(\rho, \tau, \omega) \coloneqq \Bigg\{ & (p, x, y) \\
& \in X : \inf_{(z,T,w)\in\Phi(p,x,y)} \langle (z, T, w), (\rho, \tau, \omega) \rangle \leq 0 \Bigg\}
\end{aligned}
\tag{37}
$$

is closed, for each $(\rho, \tau, \omega) \in X - X$;

(a_2) the map $\Gamma : X \to 2^X$ is lower semicontinuous with closed graph;

(a_3) there exists R_0 such that if for each $R \in [R_0, \infty[$ we denote by B_R the closed ball in $\mathbb{R}^n \times \mathbb{R}^{n\times\ell} \times \mathbb{R}^{m\times\ell}$ centered at 0 with radius R, one has $B_R \cap X \neq \emptyset$, and

(i) $\Gamma(p, x, y) \cap B_R \neq \emptyset$, for all $(p, x, y) \in X \cap B_R$;

(ii) $\sup_{(p',x',y')\in\Gamma(p,x,y)\cap B_R} \inf_{(z,T,w)\in\Phi(p,x,y)} \langle (z, T, w), (p, x, y) - (p', x', y') \rangle > 0$, for all $(p, x, y) \in X \cap B_R \setminus K$, with $(p, x, y) \in \Gamma(p, x, y)$.

At the end we divide the proof in several steps.

Step 1. To prove that condition (a_1) holds true, fix $(\rho, \tau, \omega) \in X - X$ and let $\{(p_k, x^k, y^k)\}$ be a sequence in $\Lambda(\rho, \tau, \omega)$ such that $(p_k, x^k, y^k) \to (p_*, x^*, y^*)$ as $k \to \infty$. Let us show that $(p_*, x^*, y^*) \in \Lambda(\rho, \tau, \omega)$. At first observe that since X is closed, one has $(p_*, x^*, y^*) \in X$. Moreover, since $\Phi(p_k, x^k, y^k)$ is compact for each $k \in \mathbb{N}$, and the function

$$
\begin{aligned}
& (z, T, w) \in \mathbb{R}^n \times \mathbb{R}^{n\times\ell} \times \mathbb{R}^{m\times\ell} \\
& \qquad \longrightarrow \langle (z, T, w), (\rho, \tau, \omega) \rangle
\end{aligned}
\tag{38}
$$

is continuous in $\mathbb{R}^n \times \mathbb{R}^{n\times\ell} \times \mathbb{R}^{m\times\ell}$, then, for each $k \in \mathbb{N}$, we can find $(z^k, T^k, w^k) \in \Phi(p_k, x^k, y^k)$ such that

$$
\left\langle \left(z^k, T^k, w^k\right), (\rho, \tau, \omega) \right\rangle \leq 0. \tag{39}
$$

Note that, from the definition of Φ, for each $k \in \mathbb{N}$, one has

$$z^k = -\sum_{i=1}^{n}\left(x_i^k - e_i\right) + \sum_{j=1}^{m} y_j^k; \tag{40}$$

$$w^k = \underbrace{\left(-p_k, \ldots, -p_k\right)}_{m\text{-times}}; \tag{41}$$

$$T^k \in \partial(-u)\left(x^k\right). \tag{42}$$

Moreover, recalling that u_i is locally Lipschitz continuous in $\mathbb{R}_+^{\ell}$ for all $i = 1, \ldots, n$, then, for each $k \in \mathbb{N}$, there exist an open neighborhood A_k of x^k in $\mathbb{R}^{n\times\ell}$ and a constant $L_k \geq 0$ such that

$$\sup_{T\in\partial^{\circ}(-u)(x)} |T| \leq L_k, \quad \text{for each } x \in A_k \cap \mathbb{R}_+^{\ell}. \tag{43}$$

Furthermore, there exist an open neighborhood A_0 of x^* in $\mathbb{R}^{n\times\ell}$ and a constant $L_0 \geq 0$ such that

$$\sup_{T\in\partial^{\circ}(-u)(x)} |T| \leq L_0, \quad \text{for each } x \in A_0 \cap \mathbb{R}_+^{\ell}. \tag{44}$$

At this point, observe that the family of open sets $\{A_k\}_{k\in\mathbb{N}\cup\{0\}}$ is a covering of the compact set $\{x^k\}_{k\in\mathbb{N}} \cup \{x^*\}$. Therefore, from (43) and (44), we infer that there exists a constant $L \geq 0$ such that

$$\sup_{T\in\partial^{\circ}(-u)(x^k)} |T| \leq L, \quad \text{for each } k \in \mathbb{N}. \tag{45}$$

Consequently, from (42), up to a subsequence, we can suppose that the sequence $\{T^k\}$ converges to some $T^* \in \mathbb{R}^{n\times\ell}$. Now, from (40) and (41), we infer that

$$\begin{aligned} z^k &\longrightarrow z^* := -\sum_{i=1}^{n}\left(x_i^* - e_i\right) + \sum_{j=1}^{m} y_j^*, \quad \text{as } k \longrightarrow \infty; \\ w^k &\longrightarrow w^* := \underbrace{\left(-p_*, \ldots, -p_*\right)}_{m\text{-times}}, \quad \text{as } k \longrightarrow \infty, \end{aligned} \tag{46}$$

and, from Proposition 18 and (42), we also infer that

$$T^* \in \partial(-u)\left(x^*\right). \tag{47}$$

Furthermore, from (39), passing to the limit as $k \to \infty$, one has

$$\left\langle \left(z^*, T^*, w^*\right), (\rho, \tau, \omega)\right\rangle \leq 0. \tag{48}$$

At this point, observe that considering together conditions (46) and (47) means that $(z^*, T^*, w^*) \in \Phi(p_*, x^*, y^*)$ and this latter condition, together with (48), gives $(p_*, x^*, y^*) \in \Lambda(\rho, \tau, \omega)$. Therefore, condition (a_1) is proved.

Step 2. Now, let us show that the map Γ is lower semicontinuous in X. To this end, it is sufficient to prove that, for every $(p_0, x_0, y_0) \in X$, every $(p, x, y) \in \Gamma(p_0, x_0, y_0)$, and every sequence $\{(\widehat{p}_k, \widehat{x}^k, \widehat{y}^k)\}$ in X such that $(\widehat{p}_k, \widehat{x}^k, \widehat{y}^k) \to (p_0, x_0, y_0)$ as $k \to +\infty$, there exists a sequence $\{(p_k, x^k, y^k)\}$ in X such that $(p_k, x^k, y^k) \in \Gamma(\widehat{p}_k, \widehat{x}^k, \widehat{y}^k)$ for all $k \in \mathbb{N}$ and $(p_k, x^k, y^k) \to (p, x, y)$ as $k \to +\infty$ (see [18] at page 39, for instance).

So, let (p_0, x_0, y_0), (p, x, y), and $\{(\widehat{p}_k, \widehat{x}^k, \widehat{y}^k)\}$ be as above. For each $i = 1, \ldots, n$, using the fact that $(p, x, y) \in \Gamma(p_0, x_0, y_0)$, we have the following two situations:

either

$$\left\langle p_0, x_i\right\rangle < \left\langle p_0, e_i\right\rangle + \max\left\{0, \left\langle p_0, \sum_{j=1}^{m}\theta_{ij} y_{0\,j}\right\rangle\right\} \tag{49}$$

or

$$\left\langle p_0, x_i\right\rangle = \left\langle p_0, e_i\right\rangle + \max\left\{0, \left\langle p_0, \sum_{j=1}^{m}\theta_{ij} y_{0\,j}\right\rangle\right\}. \tag{50}$$

Suppose that (49) holds. Then, since $(\widehat{p}_k, \widehat{y}^k) \to (p_0, y_0)$, there exists $k_0 \in \mathbb{N}$ such that

$$\begin{gathered} \left\langle \widehat{p}_k, x_i\right\rangle < \left\langle \widehat{p}_k, e_i\right\rangle + \max\left\{0, \left\langle \widehat{p}_k, \sum_{j=1}^{m}\theta_{ij} \widehat{y}_j^k\right\rangle\right\}, \\ \forall k \in \mathbb{N}, \ \text{with } k \geq k_0. \end{gathered} \tag{51}$$

So, in this case, if we put $x_i^k = x_i$ for $k \geq k_0$ and $x_i^k = 0$ for $k = 1, \ldots, k_0 - 1$, it is easy to check that $x_i^k \in M_i(\widehat{p}_k, \widehat{y}^k)$, for all $k \in \mathbb{N}$. Moreover, it is clear that $x_i^k \to x_i$ as $k \to +\infty$.

Suppose that (50) holds. Then, from the survivability condition (A2), we have

$$\begin{aligned} \left\langle p_0, x_i\right\rangle &= \left\langle p_0, e_i\right\rangle + \max\left\{0, \left\langle p_0, \sum_{j=1}^{m}\theta_{ij} y_{0\,j}\right\rangle\right\} \\ &> 0. \end{aligned} \tag{52}$$

Consequently, since $(\widehat{p}_k, \widehat{x}_k, \widehat{y}^k) \to (p_0, x_0, y_0)$, we have

$$\begin{aligned} &\lim_{k\to\infty} \frac{\left\langle \widehat{p}_k, e_i\right\rangle + \max\left\{0, \left\langle \widehat{p}_k, \sum_{j=1}^{m}\theta_{ij}\widehat{y}^k{}_j\right\rangle\right\}}{\left\langle \widehat{p}_k, x_i\right\rangle} \\ &\quad = \frac{\left\langle p_0, e_i\right\rangle + \max\left\{0, \left\langle p_0, \sum_{j=1}^{m}\theta_{ij} y_{0\,j}\right\rangle\right\}}{\left\langle p_0, x_i\right\rangle} = 1. \end{aligned} \tag{53}$$

Therefore, if we put

$$\begin{aligned} a_k &= \max\left\{0, 1 - \frac{\left\langle \widehat{p}_k, e_i\right\rangle + \max\left\{0, \left\langle \widehat{p}_k, \sum_{j=1}^{m}\theta_{ij}\widehat{y}_j^k\right\rangle\right\}}{\left\langle \widehat{p}_k, x_i\right\rangle}\right\}, \\ &\qquad \forall k \in \mathbb{N}, \end{aligned} \tag{54}$$

$$x_i^k = \left(1 - a_k\right) x_i, \quad \forall k \in \mathbb{N},$$

it is easy to check that $x_i^k \in M_i(\widehat{p}_k, \widehat{y}^k)$, for all $k \in \mathbb{N}$. Indeed, if $a_k > 0$, then

$$\begin{aligned}\left\langle \widehat{p}_k, x_i^k \right\rangle &= \frac{\langle \widehat{p}_k, e_i\rangle + \max\left\{0, \left\langle \widehat{p}_k, \sum_{j=1}^m \theta_{ij}\widehat{y}_j^k\right\rangle\right\}}{\langle \widehat{p}_k, x_i\rangle} \\ &\quad\cdot \langle \widehat{p}_k, x_i\rangle \\ &= \langle \widehat{p}_k, e_i\rangle + \max\left\{0, \left\langle \widehat{p}_k, \sum_{j=1}^m \theta_{ij}\widehat{y}_j^k \right\rangle\right\},\end{aligned} \tag{55}$$

and so $x_i^k \in M_i(\widehat{p}_k, \widehat{y}^k)$, while, if $a_k = 0$, then $x_i^k = x_i$ and

$$\begin{aligned}&\frac{\langle \widehat{p}_k, e_i\rangle + \max\left\{0, \left\langle \widehat{p}_k, \sum_{j=1}^m \theta_{ij}\widehat{y}_j^k\right\rangle\right\}}{\langle \widehat{p}_k, x_i^k\rangle} \\ &\quad= \frac{\langle \widehat{p}_k, e_i\rangle + \max\left\{0, \left\langle \widehat{p}_k, \sum_{j=1}^m \theta_{ij}\widehat{y}_j^k\right\rangle\right\}}{\langle \widehat{p}_k, x_i\rangle} \geq 1\end{aligned} \tag{56}$$

from which we again obtain $x_i^k \in M_i(\widehat{p}_k, \widehat{y}^k)$. Finally, observe that $x_i^k \to x_i$ as $k \to +\infty$.

So, for each $i = 1, \ldots, n$, in both cases (49) and (50), we can find a sequence x_i^k which converges to x_i and such that $x_i^k \in M_i(\widehat{p}_k, \widehat{y}^k)$, for all $k \in \mathbb{N}$. Consequently, if we put $x^k = (x_1^k, \ldots, x_n^k)$, for all $k \in \mathbb{N}$, the sequence $\{(p_k, x^k, y^k)\}$, where $p_k = p$ and $y^k = y$, for all $k \in \mathbb{N}$, satisfies $(p_k, x^k, y^k) \in \Gamma(\widehat{p}_k, \widehat{x}^k, \widehat{y}^k)$ for all $k \in \mathbb{N}$ and $(p_k, x^k, y^k) \to (p, x, y)$ as $k \to +\infty$, as desired. Therefore, Γ is lower semicontinuous in X.

To show that condition (a_2) hold true, it remains to prove that Γ has closed graph. To this end, let $\{(\widehat{p}_k, \widehat{x}^k, \widehat{y}^k)\}$ and $\{(p_k, x^k, y^k)\}$ be two sequences in X such that $(p_k, x^k, y^k) \in \Gamma(\widehat{p}_k, \widehat{x}^k, \widehat{y}^k)$, for all $k \in \mathbb{N}$, and suppose that $(\widehat{p}_k, \widehat{x}^k, \widehat{y}^k) \to (p_0, x_0, y_0)$, $(p_k, x^k, y^k) \to (p, x, y)$, as $k \to \infty$. Let us show that $(p, x, y) \in \Gamma(p_0, x_0, y_0)$.

Since P and Y are closed, one has $p \in P$ and $y \in Y$. Moreover, with $(p_k, x^k, y^k) \in \Gamma(\widehat{p}_k, \widehat{x}^k, \widehat{y}^k)$ for all $k \in \mathbb{N}$, one has

$$\left\langle \widehat{p}_k, x_i^k \right\rangle \leq \langle \widehat{p}_k, e_i\rangle + \max\left\{0, \left\langle \widehat{p}_k, \sum_{j=1}^m \theta_{ij}\widehat{y}_j^k \right\rangle\right\}, \quad \forall k \in \mathbb{N}, i = 1, \ldots, n. \tag{57}$$

Passing to the limit as $k \to \infty$, we obtain

$$\langle p_0, x_i\rangle \leq \langle p_0, e_i\rangle + \max\left\{0, \left\langle p_0, \sum_{j=1}^m \theta_{ij} y_{0j} \right\rangle\right\}, \quad \forall i = 1, \ldots, n. \tag{58}$$

Thus,

$$x \in M(p_0, y_0) \tag{59}$$

which, together with $p \in P$ and $y \in Y$, gives $(p, x, y) \in \Gamma(p_0, x_0, y_0)$, as desired.

Step 3. To finish the proof of our Theorem, it remains to show that condition (a_3) holds true as well.

Let $R_0 > 0$ be such that the closed ball $B_{R_0} \subset \mathbb{R}^n \times \mathbb{R}^{n\times\ell} \times \mathbb{R}^{m\times\ell}$ contains the compact set K. Then, for each $R \in [R_0, \infty[$, one has $K \subset X \cap B_R$ and $P \times \{0\} \times Y \subset \Gamma(p, x, y) \cap B_R$, for all $(p, x, y) \in X$. Therefore, condition (i) of (a_3) holds. Suppose that condition (ii) of (a_3) does not hold. Then, there should exist $(\overline{p}, \overline{x}, \overline{y}) \in X \cap B_R \setminus K$, with $(\overline{p}, \overline{x}, \overline{y}) \in \Gamma(\overline{p}, \overline{x}, \overline{y})$, such that

$$\begin{aligned}&\inf_{(z,T,w)\in\Phi(\overline{p},\overline{x},\overline{y})} \left\langle (z, T, w), (\overline{p}, \overline{x}, \overline{y}) - (p', x', y') \right\rangle \leq 0, \\ &\qquad \forall (p', x', y') \in \Gamma(\overline{p}, \overline{x}, \overline{y}) \cap B_R.\end{aligned} \tag{60}$$

Now, let us put $p_* := (\underbrace{1/\ell, \ldots, 1/\ell}_{\ell\text{-times}}) \in P$. Then, $(p_*, \overline{x}, \overline{y}) \in B_R \cap X$. Moreover, from $(\overline{p}, \overline{x}, \overline{y}) \in \Gamma(\overline{p}, \overline{x}, \overline{y})$, it trivially follows that $(p_*, \overline{x}, \overline{y}) \in \Gamma(\overline{p}, \overline{x}, \overline{y})$. Thus, we can test (60) with $(p', x', y') = (p_*, \overline{x}, \overline{y})$. Doing so, we get

$$\inf_{(z,T,w)\in\Phi(\overline{p},\overline{x},\overline{y})} \left\langle (z, T, w), (\overline{p} - p_*, 0, 0) \right\rangle \leq 0. \tag{61}$$

Therefore, with $\Phi(\overline{p}, \overline{x}, \overline{y})$ being a compact set, there should exist $(\overline{z}, \overline{T}, \overline{w}) \in \Phi(\overline{p}, \overline{x}, \overline{y})$ such that

$$\left\langle (\overline{z}, \overline{T}, \overline{w}), (\overline{p} - p_*, 0, 0) \right\rangle \leq 0. \tag{62}$$

From the definition of Φ, the previous inequality is equivalent to

$$\left\langle \sum_{i=1}^n (\overline{x}_i - e_i) - \sum_{j=1}^m y_j, p_* - \overline{p} \right\rangle \leq 0 \tag{63}$$

which, taking in mind that $\overline{x} \in M(\overline{p}, \overline{y})$, implies

$$\left\langle \sum_{i=1}^n (\overline{x}_i - e_i) - \sum_{j=1}^m y_j, p_* \right\rangle \leq 0. \tag{64}$$

Consequently,

$$\begin{aligned}&\sum_{i=1}^n \sum_{h=1}^\ell (\overline{x}_i^h - e_i^h) - \sum_{j=1}^m \sum_{h=1}^\ell y_j^\ell \\ &\quad= l \left\langle \sum_{i=1}^n (\overline{x}_i - e_i) - \sum_{j=1}^m y_j, p_* \right\rangle \leq 0.\end{aligned} \tag{65}$$

Therefore, $\overline{x} \in C$. But this contradicts the fact that $(\overline{p}, \overline{x}, \overline{y}) \in X \cap B_R \setminus K = (P \times \mathbb{R}_+^{n\times\ell} \times Y) \cap B_R \setminus (P \times C \times Y)$.

The proof is now complete. □

6. Applications

The well-known Cobb-Douglas utility function $u_i : \mathbb{R}_+^\ell \to \mathbb{R}_+$ defined by

$$u_i\left(x_i^{(1)}, \ldots, x_i^{(\ell)}\right) = A \prod_{h=1}^{\ell} \left(x_i^{(j)}\right)^{\lambda_h}, \tag{66}$$

where $\lambda_h > 0$ for all $h = 1, \ldots, \ell$, is concave if $\sum_{h=1}^l \lambda_h < 1$, but it is only quasiconcave (and not necessarily concave) regardless of the sum of exponents λ_h. Following the same arguments at page 178 of [25], Theorems 16 and 19 can be applied to prove the existence of competitive equilibrium for Problem 8 with u_i given by (66) for arbitrary positive exponents λ_h. The same result also applies to CES utility function defined by

$$u_i\left(x_i^{(1)},\ldots,x_i^{(\ell)}\right)=A\left(\sum_{h=1}^{\ell}\lambda_h\left(x_i^{(h)}\right)^{\rho}\right)^{1/\rho} \quad \forall i=1,\ldots,n, \tag{67}$$

where $A, \alpha_h > 0$, for all $h = 1, \ldots, \ell$, and $\rho \leq 1$.

It is also to be noted that, as we require only a local Lipschitz condition on the utility functions, our main theorems can be applied to generalized Cobb-Douglas utility functions of the type

$$u_i\left(x_i^{(1)},\ldots,x_i^{(\ell)}\right)=\prod_{j=1}^{\ell}g_j\left(x_i^{(j)}\right), \tag{68}$$

where $g_j : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing concave or convex functions, not necessarily of class C^1.

7. Conclusions

In [6, 21] under concavity assumption, a demand function $x_i(p)$ was defined and a competitive economic equilibrium was characterized as a solution of generalized quasivariational inequalities or variational inequality involving the Lagrange multipliers, respectively. In this paper we have weakened the concavity assumptions by requiring only the quasiconcavity on the utility functions. Moreover, we dot not require the differentiability of the utility functions but only a local Lipschitz condition.

In Theorem 16 we have shown that any solution of particular GQVI problem is a competitive equilibrium. In Theorem 19, using an abstract result due to Cubiotti ([23]), we have proved the existence of solution to this GQVI problem. Under the same assumptions, we finally have shown that any competitive equilibrium is a Walras competitive equilibrium too (Proposition 14).

As examples of utility functions to which our main results apply we have exhibited two typical well-known utility functions as follows:

(i) the Cobb-Douglas type utility so defined by

$$u_i\left(x_i^{(1)},\ldots,x_i^{(\ell)}\right)=A\prod_{h=1}^{\ell}\left(x_i^{(h)}\right)^{\lambda_h} \quad \forall i=1,\ldots,n, \tag{69}$$

where $A, \lambda_h > 0$, for all $h = 1, \ldots, \ell$;

(ii) the CES type utility so defined by

$$u_i\left(x_i^{(1)},\ldots,x_i^{(\ell)}\right)=A\left(\sum_{h=1}^{\ell}\lambda_h\left(x_i^{(h)}\right)^{\rho}\right)^{1/\rho} \quad \forall i=1,\ldots,n, \tag{70}$$

where $A, \lambda_h > 0$, for all $h = 1, \ldots, \ell$ and $\rho \leq 1$.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

The authors wish to thank the anonymous reviewers for their valuable suggestions and comments that improved the paper.

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