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Research Article

Existence of the Solution for System of Coupled Hybrid Differential Equations with Fractional Order and Nonlocal Conditions

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This paper is motivated by some papers treating the fractional hybrid differential equations with nonlocal conditions and the system of coupled hybrid fractional differential equations; an existence theorem for fractional hybrid differential equations involving Caputo differential operators of order $1 < \alpha \leq 2$ is proved under mixed Lipschitz and Carathéodory conditions. The existence and uniqueness result is elaborated for the system of coupled hybrid fractional differential equations.

1. Introduction

Our aim in this paper is to study the existence of solution for the boundary value problems for hybrid differential equations with fractional order $1 < \alpha \leq 2$ and nonlocal condition (BVPHDEFNL for short) of the form

$$\begin{gathered} {}^{c}D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \\ \text{a.e. } t \in J = [0,T], \quad 1 < \alpha \leq 2, \\ \frac{x(0)}{f(0,x(0))} = \mathscr{L}(x), \\ \frac{x(T)}{f(T,x(T))} = x_T, \end{gathered} \tag{1}$$

where ${}^{c}D^{\alpha}$ is the Caputo fractional derivative.

$f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in \mathscr{C}(J \times \mathbb{R}, \mathbb{R})$, $\mathscr{L} : C(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function and $x_T \in \mathbb{R}$. And exploitation of results obtained to study the existence of solutions for a system of coupled hybrid fractional differential equations is as follows:

$$\begin{gathered} {}^{c}D^{\alpha}\left(\frac{x(t)}{f_1(t,x(t),y(t))}\right) = g_1(t,x(t),y(t)) \\ \text{a.e. } t \in [0,1], \quad 1 < \alpha \leq 2, \\ {}^{c}D^{\beta}\left(\frac{y(t)}{f_2(t,x(t),y(t))}\right) = g_2(t,x(t),y(t)) \\ \text{a.e. } t \in [0,1], \quad 1 < \beta \leq 2, \\ \frac{x(0)}{f_1(0,x(0),y(0))} = \mathscr{L}_1(x,y); \quad x(1) = 0, \\ \frac{y(0)}{f_2(0,x(0),y(0))} = \mathscr{L}_2(x,y); \quad y(1) = 0, \end{gathered} \tag{2}$$

where ${}^{c}D^{\alpha}$ is the Caputo fractional derivative.

$f_i \in C([0,1]\times\mathbb{R}\times\mathbb{R}, \mathbb{R}\setminus\{0\})$, $g_i \in C([0,1]\times\mathbb{R}\times\mathbb{R}, \mathbb{R})$, and $\mathscr{L}_i : C([0,1], \mathbb{R})\times C([0,1], \mathbb{R}) \to \mathbb{R}$ are continuous functions $(i = 1, 2)$.

Fractional differential equations are a generalization of ordinary differential equations and integration to arbitrary noninteger orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventeenth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy, and science. There are several concepts of fractional derivatives, some classical, such as Riemann-Liouville or Caputo definitions. For noteworthy papers dealing with the integral operator and the arbitrary fractional order differential operator, see [1–7].

The quadratic perturbations of nonlinear differential equations have attracted much attention. We call such fractional hybrid differential equations. There have been many works on the theory of hybrid differential equations, and we refer the readers to the articles [8–12].

Dhage and Lakshmikantham [11] discussed the following first order hybrid differential equation

$$\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) \quad \text{a.e. } t \in J = [0,T], \qquad x(t_0) = x_0 \in \mathbb{R}, \tag{3}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathscr{C}(J \times \mathbb{R}, \mathbb{R})$. They established the existence, uniqueness results, and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved, utilizing the theory of inequalities, the existence of extremal solutions and comparison results.

Zhao et al. [13] have discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operators:

$$D^q\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) \quad \text{a.e. } t \in J = [0,T], \qquad x(0) = 0, \tag{4}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathscr{C}(J \times \mathbb{R}, \mathbb{R})$. The authors of [13] established the existence theorem for fractional hybrid differential equation and some fundamental differential inequalities. They also established the existence of extremal solutions.

Hilal and Kajouni [14] have studied boundary fractional hybrid differential equations involving Caputo differential operators of order $0 < \alpha < 1$ as follows:

$$D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \quad \text{a.e. } t \in J = [0,T], \tag{5}$$

$$a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c,$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g \in \mathscr{C}(J \times \mathbb{R}, \mathbb{R})$ and a, b, and c are real constants with $a + b \neq 0$. They proved the existence result for boundary fractional hybrid differential equations under mixed Lipschitz and Carathéodory conditions. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions. Necessary tools are considered and the comparison principle is proved which will be useful for further study of qualitative behavior of solutions.

The nonlocal condition is a condition attached to the main equation; it replaces the classic nonlocal condition in order to model physical phenomena of the fashion nearest from reality. The nonlocal condition involves the function

$$\mathscr{L}(x) = \sum_{i=1}^{p} c_i x(t_i), \tag{6}$$

where c_i, $i = 1, 2, \ldots, p$, are given constants and $0 < t_1 < t_2 < \cdots < t_p$.

Let us observe that Cauchy problems with nonlocal conditions were initiated by Byszewski and Lakshmikantham [2] and, since then, such problems have also attracted several authors including A. Aizicovici, K. Ezzinbi, Z. Fan, J. Liu, J. Liang, Y. Lin, T.-J. Xiao, G. N'Guérékata, E. Hernàndez, and H. Lee (see [2, 15]).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $X = C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from $J = [0, T]$ into $\mathbb{R}$ with the norm

$$\|y\| = \sup\{|y(t)|, t \in J\}. \tag{7}$$

And let $\mathscr{C}(J \times \mathbb{R}, \mathbb{R})$ denote the class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ such that

(i) the map $t \mapsto g(t, x)$ is measurable for each $x \in \mathbb{R}$,

(ii) the map $x \mapsto g(t, x)$ is continuous for each $t \in J$.

The class $\mathscr{C}(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.

By $L^1(J; \mathbb{R})$ we denote the space of Lebesgue integrable real-valued functions on J equipped with the norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^T |x(s)|\, ds. \tag{8}$$

Definition 1. The fractional integral of the function $h \in L^1([a,b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds, \tag{9}$$

where Γ is the gamma function.

Definition 2. For a function h given on the interval $[a, b]$, the Caputo fractional order derivative of h is defined by

$${}^cD_{a^+}^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\, ds, \tag{10}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 3 (see [16]). *Let $\alpha > 0$. Then the fractional differential equation*

$$^{c}D^{\alpha}h(t) = 0 \tag{11}$$

has solutions

$$h(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}, \quad c_i \in \mathbb{R},\ i = 1,2,\ldots,n-1,\ n = [\alpha]+1. \tag{12}$$

Lemma 4 (see [16]). *Let $\alpha > 0$. Then*

$$I^{\alpha}D^{\alpha}h(t) = h(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}, \tag{13}$$

for some $c_i \in \mathbb{R},\ i = 1,2,\ldots,n-1,\ n = [\alpha]+1$.

Definition 5. By a solution of the BVPHDEFNL (1) we mean a function $x \in C(J,\mathbb{R})$ such that

(i) $d^2u/dt^2 \in L^1(J,\mathbb{R})$, where $u : t \mapsto x/f(t,x)$ for each $x \in \mathbb{R}$,

(ii) x satisfies the equations in (1).

3. Existence Result

In this section, we prove the existence results for the boundary value problems for hybrid differential equations with fractional order (1) on the closed and bounded interval $J = [0,T]$ under mixed Lipschitz and Carathéodory conditions on the nonlinearities involved in it.

We defined the multiplication in X by

$$(xy)(t) = x(t)\,y(t), \quad \text{for } x, y \in X. \tag{14}$$

Clearly, $X = C(J;\mathbb{R})$ is a Banach algebra with respect to above norm and multiplication in it.

We prove the existence of solution for the BVPHDEFNL (1) by a fixed point theorem in Banach algebra due to Dhage [10].

Lemma 6 (see [10]). *Let S be a nonempty, closed convex, and bounded subset of the Banach algebra X and let $A : X \to X$ and $B : X \to X$ be two operators such that*

(a) *A is Lipschitzian with a Lipschitz constant α,*

(b) *B is completely continuous,*

(c) *$x = AxBy \Rightarrow x \in S$ for all $y \in S$,*

(d) *$\alpha M < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.*

Then the operator equation $AxBx = x$ has a solution in S.

We make the following assumptions:

(H_0): the function $x \mapsto x/f(t,x)$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.

(H_1): there exists a constant $L > 0$ such that

$$\left|f(t,x) - f(t,y)\right| \le L\left|x - y\right|, \tag{15}$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

(H_2): there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$\left|g(t,x)\right| \le h(t) \quad \text{a.e. } t \in J, \tag{16}$$

for all $x \in \mathbb{R}$.

(H_3): there exists a constant $M > 0$ such that $|\mathscr{L}(y)| \le M$, for each $y \in C(J;\mathbb{R})$.

As a consequence of Lemmas 3 and 4 we have the following result which is useful in what follows.

Lemma 7. *Assume that hypothesis (H_0) holds. Then for any $h \in L^1(J;\mathbb{R})$, the function $x \in C(J;\mathbb{R})$ is a solution of the BVPHDEFNL:*

$$\begin{aligned} &{}^{c}D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = h(t) \quad \text{a.e. } t \in J = [0,T],\ 1 < \alpha \le 2, \\ &\frac{x(0)}{f(0,x(0))} = \mathscr{L}(x), \\ &\frac{x(T)}{f(T,x(T))} = x_T \end{aligned} \tag{17}$$

if and only if x satisfies the hybrid integral equation

$$\begin{aligned} x(t) = f(t,x(t))\Bigg[&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds \\ &- \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds - \left(\frac{t}{T}-1\right)\mathscr{L}(x) \\ &+ \frac{t}{T}x_T\Bigg]. \end{aligned} \tag{18}$$

Proof. Assume that x is a solution of the problem (18). Applying the Caputo fractional operator of the order α, we obtain the first equation in (17). Again, substituting $t = 0$ and $t = T$ in (18) we will have the second equation in (17).

Conversely, ${}^{c}D^{\alpha}(x(t)/f(t,x(t))) = h(t)$, so we get

$$\frac{x(t)}{f(t,x(t))} + c_0 + c_1 t = I^{\alpha}h(t). \tag{19}$$

Then $x(0)/f(0,x(0)) + c_0 = 0$ and $c_0 = -\mathscr{L}(x)$, and even

$$\frac{x(T)}{f(T,x(T))} + c_0 + c_1 T = \int_0^T (T-s)^{\alpha-1}h(s)\,ds. \tag{20}$$

Thus,

$$c_1 = \frac{1}{T}\left(\mathscr{L}(x) - x_T + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds\right) \tag{21}$$

implies that

$$\begin{aligned} \frac{x(t)}{f(t,x(t))} = {}&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds \\ &- \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)\,ds \\ &- \left(\frac{t}{T}-1\right)\mathscr{L}(x) + \frac{t}{T}x_T. \end{aligned} \tag{22}$$

□

Theorem 8. *Assume hypotheses (H_0)–(H_3). Further, if*

$$L\left(\frac{2T^{\alpha}}{\Gamma(\alpha+1)}\|h\|_{L^1}+M+|x_T|\right)<1, \quad (23)$$

then the hybrid fractional order differential equation (1) has a solution defined on J.

Proof. We defined a subset S of X by

$$S=\left\{x\in\frac{X}{\|x\|}\leq N\right\}, \quad (24)$$

where $N = F_0((2T^{\alpha}/\Gamma(\alpha+1))\|h\|_{L^1}+M+|x_T|)/(1-L((2T^{\alpha}/\Gamma(\alpha+1))\|h\|_{L^1}+M+|x_T|))$ and $F_0=\sup_{t\in J}|f(t,0)|$.

It is clear that S satisfies hypothesis of Lemma 6. By an application of Lemma 7, (1) is equivalent to the nonlinear hybrid integral equation

$$\begin{aligned} x(t) = f(t,x(t))\Bigg[&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s))\,ds \\ &-\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s,x(s))\,ds \\ &-\left(\frac{t}{T}-1\right)\mathscr{L}(x)+\frac{t}{T}x_T\Bigg]. \end{aligned} \quad (25)$$

Define two operators $A: X\to X$ and $B: S\to X$ by

$$\begin{aligned} Ax(t) &= f(t,x(t)), \quad t\in J, \\ Bx(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s))\,ds \\ &\quad -\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s,x(s))\,ds \\ &\quad -\left(\frac{t}{T}-1\right)\mathscr{L}(x)+\frac{t}{T}x_T. \end{aligned} \quad (26)$$

Then the hybrid integral equation (25) is transformed into the operator equation as

$$x(t) = Ax(t)\,Bx(t), \quad t\in J. \quad (27)$$

We will show that the operators A and B satisfy all the conditions of Lemma 6.

Claim 1. Let $x,y\in X$. Then by hypothesis (H_1),

$$\begin{aligned} |Ax(t)-Ay(t)| &= |f(t,x(t))-f(t,y(t))| \\ &\leq L|x(t)-y(t)|\leq L\|x-y\|, \end{aligned} \quad (28)$$

for all $t\in J$. Taking supremum over t, we obtain

$$\|Ax-Ay\|\leq L\|x-y\|, \quad (29)$$

for all $x,y\in X$.

Claim 2 (we show that B is continuous in S). Let (x_n) be a sequence in S converging to a point $x\in S$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x_n(s))\,ds \\ &\quad = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lim_{n\to\infty} g(s,x_n(s))\,ds, \\ &\lim_{n\to\infty}\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s,x_n(s))\,ds \\ &\quad = \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\lim_{n\to\infty} g(s,x_n(s))\,ds. \end{aligned} \quad (30)$$

And since $\mathscr{L}$ is a continuous function

$$\lim_{n\to\infty}\mathscr{L}(x_n)=\mathscr{L}(x), \quad (31)$$

then

$$\begin{aligned} \lim_{n\to\infty}Bx_n(t) &= \lim_{n\to\infty}\Bigg[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x_n(s))\,ds \\ &\quad -\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s,x_n(s))\,ds \\ &\quad -\left(\frac{t}{T}-1\right)\mathscr{L}(x_n)+\frac{t}{T}x_T\Bigg] = \lim_{n\to\infty}\frac{1}{\Gamma(\alpha)} \\ &\quad \cdot\int_0^t (t-s)^{\alpha-1} g(s,x_n(s))\,ds - \lim_{n\to\infty}\frac{t}{T\Gamma(\alpha)} \\ &\quad \cdot\int_0^T (T-s)^{\alpha-1} g(s,x_n(s))\,ds - \lim_{n\to\infty}\left(\frac{t}{T}-1\right) \\ &\quad \cdot\mathscr{L}(x_n)+\frac{t}{T}x_T = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s))\,ds \\ &\quad -\frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s,x(s))\,ds - \left(\frac{t}{T}-1\right) \\ &\quad \cdot\mathscr{L}(x)+\frac{t}{T}x_T = Bx(t), \end{aligned} \quad (32)$$

for all $t\in J$. This shows that B is a continuous operator on S.

Claim 3 (B is compact operator on S). First, we show that $B(S)$ is a uniformly bounded set in X.

Let $x\in S$. Then by hypothesis (H_2), for all $t\in J$,

$$\begin{aligned} |Bx(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1} g(s,x(s))\right| ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^T \left|(T-s)^{\alpha-1} g(s,x(s))\right| ds \\ &\quad + |\mathscr{L}(x)| + |x_T| \\ &\leq \frac{T^{\alpha}}{\alpha\Gamma(\alpha)}\|h\|_{L^1} + \frac{T^{\alpha}}{\alpha\Gamma(\alpha)}\|h\|_{L^1} + |\mathscr{L}(x)| \\ &\quad + |x_T| \leq \frac{2T^{\alpha}}{\Gamma(\alpha+1)}\|h\|_{L^1}+M+|x_T|. \end{aligned} \quad (33)$$

Thus, $\|Bx\|\leq (2T^{\alpha}/\Gamma(\alpha+1))\|h\|_{L^1}+M+|x_T|$, for all $x\in S$.

This shows that B is uniformly bounded on S.

Next, we show that $B(S)$ is an equicontinuous set on X.

We set $p(t) = \int_0^t h(s)ds$.

Let $t_1, t_2 \in J$. Then for any $x \in S$,

$$\begin{aligned}
&\left|Bx(t_1) - Bx(t_2)\right| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} g(s, x(s))\, ds \right. \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} g(s, x(s))\, ds \\
&\quad - (t_1 - t_2) \frac{1}{T\Gamma(\alpha)} \int_0^T \left|(T - s)^{\alpha-1} g(s, x(s))\right| ds \\
&\quad \left. - \frac{1}{T}(t_1 - t_2)\mathscr{L}(x) + \frac{1}{T}(t_1 - t_2) x_T \right| \\
&\le \frac{T^{\alpha-1}}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} \left|g(s, x(s))\right| ds \right| + \frac{T^{\alpha-1}}{\Gamma(\alpha)} \left|t_1 - t_2\right| \\
&\quad \cdot \int_0^T \left|g(s, x(s))\right| ds + \frac{M + \left|x_T\right|}{T} \left|t_1 - t_2\right| \\
&\le \frac{T^{\alpha-1}}{\Gamma(\alpha)} \left|p(t_1) - p(t_2)\right| + \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{M + \left|x_T\right|}{T} \right) \\
&\quad \cdot \left|t_1 - t_2\right|.
\end{aligned} \tag{34}$$

Since p is continuous on compact J, it is uniformly continuous. Hence,

$$\begin{gathered}
\left|t_1 - t_2\right| < \eta \Longrightarrow \left|Bx(t_1) - Bx(t_2)\right| < \varepsilon, \\
\forall \varepsilon > 0,\ \exists \eta > 0
\end{gathered} \tag{35}$$

for all $t_1, t_2 \in J$ and for all $x \in X$.

This shows that $B(S)$ is an equicontinuous set in X.

Then by Arzelá-Ascoli theorem, B is a continuous and compact operator on S.

Claim 4 (hypothesis (c) of Lemma 6 is satisfied). Let $x \in X$ and $y \in S$ be arbitrary such that $x = AxBy$. Then,

$$\begin{aligned}
&|x(t)| = |Ax(t)|\left|By(t)\right| \le \left|f(t, x(t))\right| \\
&\quad \cdot \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(s, x(s))\, ds \right. \\
&\quad - \frac{t}{T\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} g(s, x(s))\, ds \\
&\quad \left. - \left(\frac{t}{T} - 1\right)\mathscr{L}(x) + \frac{t}{T} x_T \right| \\
&\le \left[\left|f(t, x(t)) - f(t, 0)\right| + \left|f(t, 0)\right|\right] \\
&\quad \cdot \left(\frac{2T^\alpha}{\Gamma(\alpha+1)} \|h\|_{L^1} + M + \left|x_T\right| \right) \le \left(L|x(t)| + F_0\right) \\
&\quad \cdot \left(\frac{2T^\alpha}{\Gamma(\alpha+1)} \|h\|_{L^1} + M + \left|x_T\right| \right),
\end{aligned} \tag{36}$$

and so,

$$\begin{aligned}
&|x(t)| - L\left(\frac{2T^\alpha}{\Gamma(\alpha+1)} \|h\|_{L^1} + M + \left|x_T\right| \right)|x(t)| \\
&\quad \le F_0 \left(\frac{2T^\alpha}{(\alpha+1)} \|h\|_{L^1} + M + \left|x_T\right| \right),
\end{aligned} \tag{37}$$

which implies

$$|x(t)| \le \frac{F_0\left(\left(2T^\alpha/\Gamma(\alpha+1)\right)\|h\|_{L^1} + M + \left|x_T\right|\right)}{1 - L\left(\left(2T^\alpha/\Gamma(\alpha+1)\right)\|h\|_{L^1} + M + \left|x_T\right|\right)}. \tag{38}$$

Taking supremum over t,

$$\begin{aligned}
\|x\| &\le \frac{F_0\left(\left(2T^\alpha/\Gamma(\alpha+1)\right)\|h\|_{L^1} + M + \left|x_T\right|\right)}{1 - L\left(\left(2T^\alpha/\Gamma(\alpha+1)\right)\|h\|_{L^1} + M + \left|x_T\right|\right)} \\
&= N.
\end{aligned} \tag{39}$$

Then $x \in S$, and hypothesis (c) of Lemma 6 is satisfied.

Finally, we have

$$\begin{aligned}
M &= \|B(S)\| = \sup\{\|Bx\| : x \in S\} \\
&\le \frac{2T^\alpha}{\Gamma(\alpha+1)} \|h\|_{L^1} + M + \left|x_T\right|.
\end{aligned} \tag{40}$$

So,

$$\alpha M \le L\left(\frac{2T^\alpha}{\Gamma(\alpha+1)} \|h\|_{L^1} + M + \left|x_T\right| \right) < 1. \tag{41}$$

Thus, all the conditions of Lemma 6 are satisfied and hence the operator equation $AxBx = x$ has a solution in S. As a result, BVPHDEFNL (1) has a solution defined on J. This completes the proof. □

4. An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem:

$$\begin{aligned}
&{}^cD^\alpha \left(\frac{(2 + \ln(t+1))\left(x(t) + x^2(t)\right)}{e^{1-t}} \right) \\
&\quad = \frac{e^{-x^2(t)}}{x^2(t) + t^2 + 1} \qquad \text{a.e. } t \in J = [0, 1],\ 1 < \alpha \le 2, \\
&\frac{x(0)}{f(0, x(0))} = \sum_{i=1}^n c_i x(t_i); \\
&\frac{x(1)}{f(1, x(1))} = 1,
\end{aligned} \tag{42}$$

where $0 < t_1 < t_2 < \cdots < t_n < 1,\ c_i,\ i = 1, 2, \ldots, n$, are given positive constants.

And $\sum_{i=1}^{n} c_i < (1 - \pi)/2\overline{M}$, where $\overline{M} = \max_{1 \le i \le n} \mathscr{L}(t_i)$.

We set

$$f(t, x) = \frac{e^{t-1}}{(2 + \ln(1 + t))(1 + x)}, \quad (t, x) \in [0, 1] \times [0, +\infty),$$
$$g(t, x) = \frac{e^{-x^2}}{x^2 + t^2 + 1}, \tag{43}$$
$$\mathscr{L}(x(t)) = \sum_{i=1}^{n} c_i x(t_i).$$

Let $x, y \in [0, +\infty)$ and $t \in J$. We have

$$\begin{aligned} \left| f(t, x) - f(t, y) \right| &= \frac{e^{t-1}}{2 + \ln(1 + t)} \left| \frac{1}{1 + x} - \frac{1}{1 + y} \right| \\ &\le \frac{1}{2} \left| \frac{x - y}{(1 + x)(1 + y)} \right| \\ &\le \frac{1}{2} \left| x - y \right|. \end{aligned} \tag{44}$$

Hence, condition (H_1) holds with $L = 1/2$. Also we have

$$\begin{aligned} \left| g(t, x) \right| &= \frac{e^{-x^2}}{x^2 + t^2 + 1} \\ &\le h(t), \end{aligned} \tag{45}$$

where $h(t) = 1/(1 + t^2)$. We have

$$\int_0^1 h(t)\, dt = \int_0^1 \frac{1}{1 + t^2} = \frac{\pi}{4}. \tag{46}$$

Then condition (H_2) holds.

Furthermore, since $\mathscr{L} \in C(J, \mathbb{R})$, then we set $\overline{M} = \max_{1 \le i \le n} \mathscr{L}(t_i)$ and we have

$$\left| \mathscr{L}(t_i) \right| = \left| \sum_{i=1}^{n} c_i \mathscr{L}(t_i) \right| \le \overline{M} \sum_{i=1}^{n} c_i. \tag{47}$$

We will check that condition (23) is satisfied with $T = 1$.

Since $\sum_{i=1}^{n} c_i < (1 - \pi)/2\overline{M}$, then $\pi + 2\overline{M} \sum_{i=1}^{n} c_i < 1$.

Thus,

$$\frac{\pi}{\Gamma(\alpha + 1)} + 2\overline{M} \sum_{i=1}^{n} c_i < 1, \tag{48}$$

which is satisfied for each $\alpha \in (1, 2]$. Then by Theorem 8 problem (42) has a solution on $[0, 1]$.

5. System of Coupled Hybrid Fractional Differential Equations

The aim of this section is to obtain the existence results, by means of Banach's fixed point theorem, for the problem of coupled hybrid fractional differential equations for (1). Consider

$$\begin{aligned} {}^cD^{\alpha} \left(\frac{x(t)}{f_1(t, x(t), y(t))} \right) &= g_1(t, x(t), y(t)) \\ &\quad \text{a.e. } t \in [0, 1],\ 1 < \alpha \le 2, \\ {}^cD^{\beta} \left(\frac{y(t)}{f_2(t, x(t), y(t))} \right) &= g_2(t, x(t), y(t)) \\ &\quad \text{a.e. } t \in [0, 1],\ 1 < \beta \le 2, \\ \frac{x(0)}{f_1(0, x(0), y(0))} &= \mathscr{L}_1(x, y), \quad x(1) = 0, \\ \frac{y(0)}{f_2(0, x(0), y(0))} &= \mathscr{L}_2(x, y), \quad y(1) = 0, \end{aligned} \tag{49}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative.

$f_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g_i \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $\mathscr{L}_i : C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R}) \to \mathbb{R}$ are continuous functions $(i = 1, 2)$.

Main Results. Let $\Omega = \{\omega(t) \setminus \omega(t) \in C^1([0, 1])\}$ denote a Banach space equipped with the norm $\|\omega\| = \sup\{|\omega(t)|, t \in [0, 1]\}$, where $\Omega = \mathscr{K} \times \mathscr{R}$. Notice that the product space $(\mathscr{K} \times \mathscr{R}, \|(x, y)\|)$ with the norm $\|(x, y)\| = \|x\| + \|y\|$, $(x, y) \in \mathscr{K} \times \mathscr{R}$ is also a Banach space.

In view of Lemma 7, we define an operator $\Phi : \mathscr{K} \times \mathscr{R} \to \mathscr{K} \times \mathscr{R}$ by

$$\Phi(x, y)(t) = (\Phi_1(x, y)(t), \Phi_2(x, y)(t)), \tag{50}$$

where

$$\begin{aligned} \Phi_1(x, y)(t) &= f_1(t, x(t), y(t)) \\ &\quad \cdot \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} g_1(s, x(s), y(s))\, ds \right. \\ &\quad - \frac{t}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 1} g_1(s, x(s), y(s))\, ds \\ &\quad \left. - (t - 1) \mathscr{L}_1(x, y) \right], \\ \Phi_2(x, y)(t) &= f_2(t, x(t), y(t)) \\ &\quad \cdot \left[\frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} g_2(s, x(s), y(s))\, ds \right. \\ &\quad - \frac{t}{\Gamma(\beta)} \int_0^1 (1 - s)^{\beta - 1} g_2(s, x(s), y(s))\, ds \\ &\quad \left. - (t - 1) \mathscr{L}_2(x, y) \right]. \end{aligned} \tag{51}$$

In the sequel, we need the following assumptions:

(H_1'): the functions f_i are continuous and bounded; that is, there exist positive numbers L_i such that $|f_i(t,u,v)| \le L_i$ for all $(t,u,v) \in [0,1] \times \mathbb{R} \times \mathbb{R}$ $(i = 1,2)$.

(H_2'): there exist real constants $\rho_0, \delta_0 > 0$ and $\rho_i, \delta_i \ge 0$ $(i = 1,2)$ such that $|g_1(t,x,y)| \le \rho_0 + \rho_1|x| + \rho_2|y|$ and $|g_2(t,x,y)| \le \delta_0 + \delta_1|x| + \delta_2|y|$ for all $x, y \in \mathbb{R}$ $(i = 1,2)$.

(H_3'): there exist real constants $M_1, M_2 > 0$ $|\mathcal{L}_1(x,y)| \le M_1$ and $|\mathcal{L}_2(x,y)| \le M_2$ for each $x, y \in C([0,1])$.

(H_4'): there exist real constants $\gamma_1, \gamma, \gamma_1', \gamma_2'$ such that $|\mathcal{L}_1(x_1,y_1) - \mathcal{L}_1(x_2,y_2)| \le \gamma_1|x_1 - x_2| + \gamma_2|y_1 - y_2|$ and $|\mathcal{L}_2(x_1,y_1) - \mathcal{L}_2(x_2,y_2)| \le \gamma_1'|x_1 - x_2| + \gamma_2'|y_1 - y_2|$.

For brevity, let us set

$$\mu_1 = \frac{2L_1}{\Gamma(\alpha+1)}, \qquad \mu_2 = \frac{2L_2}{\Gamma(\beta+1)}, \tag{52}$$

$$\mu_0 = \min\{1 - (\mu_1\rho_1 + \mu_2\delta_1), 1 - (\mu_1\rho_2 + \mu_2\delta_2)\}. \tag{53}$$

Now we present our result for the existence and uniqueness of solutions for problem (49). This result is based on Banach's contraction mapping principle.

Theorem 9. *Suppose that conditions (H_1'), (H_3'), and (H_4') hold and that $g_1, g_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. In addition, there exist positive constants η_i, ξ_i, $i = 1,2$, such that*

$$\begin{aligned} &|g_1(t,x_1,y_1) - g_1(t,x_2,y_2)| \\ &\quad = \eta_1|x_1 - x_2| + \eta_2|y_1 - y_2|, \\ &|g_2(t,x_1,y_1) - g_2(t,x_2,y_2)| \\ &\quad = \xi_1|x_1 - x_2| + \xi_2|y_1 - y_2|, \\ &\qquad \forall t \in [0,1], \ x_1, x_2, y_1, y_2 \in \mathbb{R}. \end{aligned} \tag{54}$$

If $\mu_1(\eta_1+\eta_2) + \mu_2(\xi_1+\xi_2) + L_1(\gamma_1+\gamma_2) + L_2(\gamma_1'+\gamma_2') < 1$, then problem (49) has a unique solution.

Proof. Let us set $\sup_{t\in[0,1]} g_1(t,0,0) = \kappa_1 < \infty$ and $\sup_{t\in[0,1]} g_2(t,0,0) = \kappa_2 < \infty$ and define a closed ball: $B_r = \{(x,y) \in \mathcal{K} \times \mathcal{R} : \|(x,y)\| \le r\}$, where

$$r \ge \frac{\mu_1\kappa_1 + \mu_2\kappa_2 + M_1L_1 + M_2L_2}{1 - \mu_1(\eta_1+\eta_2) - \mu_2(\xi_1+\xi_2)}. \tag{55}$$

Claim 5 (we show that $\Phi B_r \subset B_r$). Let $(x,y) \in B_r$. We have

$$\begin{aligned} |\Phi_1(x,y)(t)| &\le M_1\left[\sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \cdot |g_1(s,x(s),y(s))|\,ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \cdot |g_1(s,x(s),y(s))|\,ds\right\} + |\mathcal{L}_1(x,y)|\right] \\ &= M_1\left[\sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \cdot (|g_1(s,x(s),y(s)) - g_1(s,0,0)| + |g_1(s,0,0)|)\,ds \right.\right. \\ &\quad \left.\left. + \frac{1}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1} \cdot (|g_1(s,x(s),y(s)) - g_1(s,0,0)| + |g_1(s,0,0)|)\,ds\right\} + |\mathcal{L}_1(x,y)|\right] \\ &\le M_1\left[\frac{2}{\Gamma(\alpha+1)}(\eta_1\|x\| + \eta_2\|y\| + \kappa_1) + L_1\right] \le M_1\left[\frac{2}{\Gamma(\alpha+1)}((\eta_1+\eta_2)r + \kappa_1) + L_1\right] \\ &\le \mu_1[(\eta_1+\eta_2)r + \kappa_1] + M_1L_1. \end{aligned} \tag{56}$$

Hence,

$$\begin{aligned} \|\Phi_1(x,y)\| &\le \mu_1[(\eta_1+\eta_2)r + \kappa_1] + M_1L_1, \\ \|\Phi_2(x,y)\| &\le \mu_2[(\xi_1+\xi_2)r + \kappa_2] + M_2L_2. \end{aligned} \tag{57}$$

From (57), it follows that $\|\Phi(x,y)\| \le r$.

Next, for $(x_1,y_1), (x_2,y_2) \in \mathcal{K} \times \mathcal{R}$ and for any $t \in [0,1]$, we have

$$\begin{aligned} &|\Phi_1(x_2,y_2)(t) - \Phi_1(x_1,y_1)(t)| \\ &\le L_1\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|g_1(s,x_2(s),y_2(s)) - g_1(s,x_1(s),y_1(s))|\,ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|g_1(s,x_2(s),y_2(s)) - g_1(s,x_1(s),y_1(s))|\,ds + |\mathcal{L}_1(x_2,y_2) - \mathcal{L}_1(x_1,y_1)|\right] \\ &\le L_1\left[\frac{2}{\Gamma(\alpha+1)}(\eta_1|x_2(t) - x_1(t)| + \eta_2|y_2(t) - y_1(t)|) + \gamma_1|x_2(t) - x_1(t)| + \gamma_2|y_2(t) - y_1(t)|\right] \le \mu_1(\eta_1\|x_2 - x_1\| + \eta_2\|y_2 - y_1\|) + L_1(\gamma_1\|x_2 - x_1\| + \gamma_2\|y_2 - y_1\|) \\ &\le [\mu_1(\eta_1+\eta_2) + L_1(\gamma_1+\gamma_2)](\|x_2 - x_1\| + \|y_2 - y_1\|) \end{aligned} \tag{58}$$

which yields

$$\begin{aligned}&\left\|\Phi_1(x_2, y_2) - \Phi_1(x_1, y_1)\right\| \\ &\quad \le [\mu_1(\eta_1 + \eta_2) + L_1(\gamma_1 + \gamma_2)] \\ &\quad \cdot (\|x_2 - x_1\| + \|y_2 - y_1\|).\end{aligned} \tag{59}$$

Working in a similar manner, one can find that

$$\begin{aligned}&\left\|\Phi_2(x_2, y_2) - \Phi_2(x_1, y_1)\right\| \\ &\quad \le \left[\mu_2(\xi_1 + \xi_2) + L_2\left(\gamma_1' + \gamma_2'\right)\right] \\ &\quad \cdot (\|x_2 - x_1\| + \|y_2 - y_1\|).\end{aligned} \tag{60}$$

We deduce that

$$\begin{aligned}&\left\|\Phi(x_2, y_2) - \Phi(x_1, y_1)\right\| \le \left[\mu_1(\eta_1 + \eta_2)\right. \\ &\quad \left. + \mu_2(\xi_1 + \xi_2) + L_1(\gamma_1 + \gamma_2) + L_2\left(\gamma_1' + \gamma_2'\right)\right] \\ &\quad \cdot (\|x_2 - x_1\| + \|y_2 - y_1\|).\end{aligned} \tag{61}$$

In view of condition $\mu_1(\eta_1 + \eta_2) + \mu_2(\xi_1 + \xi_2) + L_1(\gamma_1 + \gamma_2) + L_2(\gamma_1' + \gamma_2') < 1$, it follows that Φ is a contraction. So Φ has a unique fixed point. This implies that problem (49) has a unique solution on $[0, 1]$. This completes the proof. □

In our second result, we discuss the existence of solutions for problem (49) by means of Leray-Schauder alternative.

Lemma 10 (see [17]). *Let $\mathfrak{F} : \mathfrak{F} \to \mathfrak{F}$ be a completely continuous operator (i.e., a map that is restricted to any bounded set in G is compact). Let $\mathcal{P}(\mathfrak{F}) = \{x \in \mathfrak{F} : x = \lambda\mathfrak{F}x$ for some $0 < \lambda < 1\}$. Then either the set $\mathcal{P}(\mathfrak{F})$ is unbounded or $\mathfrak{F}$ has at least one fixed point.*

Theorem 11. *Assume that conditions (H_1')–(H_3') hold. Furthermore, it is assumed that $\mu_1\rho_1 + \mu_2\delta_1 < 1$ and $\mu_1\rho_2 + \mu_2\delta_2 < 1$, where μ_1 and μ_2 are given by (52). Then the boundary value problem (49) has at least one solution.*

Proof. We will show that the operator $\Phi : \mathcal{K} \times \mathcal{R} \to \mathcal{K} \times \mathcal{R}$ satisfies all the assumptions of Lemma 10. In the first step, we prove that the operator Φ is completely continuous. Clearly, it follows by the continuity of functions f_1, f_2, g_1, and g_2 that the operator Φ is continuous.

Let $\mathcal{S} \subset \mathcal{K} \times \mathcal{R}$ be bounded. Then we can find positive constants N_1 and N_2 such that

$$\begin{aligned}&\left|g_1(t, x(t), y(t))\right| \le N_1 \\ &\left|g_2(t, x(t), y(t))\right| \le N_2, \\ &\qquad \forall (x, y) \in \mathcal{S}.\end{aligned} \tag{62}$$

Thus, for any $x, y \in \mathcal{S}$, we can get

$$\begin{aligned}&\left|\Phi_1(x, y)(t)\right| \\ &\quad \le L_1\left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|g_1(s, x(s), y(s))\right| ds\right. \\ &\quad + t\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\left|g_1(s, x(s), y(s))\right| ds \\ &\quad \left. + (1-t)\mathcal{L}_1(x, y)\right] \le L_1 N_1 \frac{2}{\Gamma(\alpha+1)} + M_1\end{aligned} \tag{63}$$

which yields

$$\left\|\Phi_1(x, y)\right\| = N_1\mu_1 + M_1. \tag{64}$$

In a similar manner,

$$\left\|\Phi_2(x, y)\right\| = N_2\mu_2 + M_2. \tag{65}$$

We deduce that the operator Φ is uniformly bounded.

Now we show that the operator Φ is equicontinuous. We take $\tau_1, \tau_2 \in [0, 1]$ with $\tau_1 < \tau_2$ and obtain

$$\begin{aligned}&\left|\Phi_1(x(\tau_2), y(\tau_2)) - \Phi_1(x(\tau_1), y(\tau_1))\right| \\ &\quad \le L_1N_1\left|\int_0^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds - \int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right| \\ &\quad + L_1N_1\left|\tau_2-\tau_1\right|\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds + M_1\left|\tau_2-\tau_1\right| \\ &\quad \le L_1N_1\left|\int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}-(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right. \\ &\quad \left. - \int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right| + L_1N_1\left|\tau_2-\tau_1\right| \\ &\quad \cdot\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds + M_1\left|\tau_2-\tau_1\right| \longrightarrow_{\tau_1\to\tau_2} 0, \\ &\left|\Phi_2(x(\tau_2), y(\tau_2)) - \Phi_2(x(\tau_1), y(\tau_1))\right| \\ &\quad \le L_2N_2\left|\int_0^{\tau_2}\frac{(\tau_2-s)^{\beta-1}}{\Gamma(\beta)}ds - \int_0^{\tau_1}\frac{(\tau_1-s)^{\beta-1}}{\Gamma(\beta)}ds\right| \\ &\quad + L_2N_2\left|\tau_2-\tau_1\right|\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}ds + M_2\left|\tau_2-\tau_1\right| \\ &\quad \le L_2N_2\left|\int_0^{\tau_1}\frac{(\tau_1-s)^{\beta-1}-(\tau_2-s)^{\beta-1}}{\Gamma(\beta)}ds\right. \\ &\quad \left. - \int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\beta-1}}{\Gamma(\beta)}ds\right| + L_2N_2\left|\tau_2-\tau_1\right| \\ &\quad \cdot\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}ds + M_2\left|\tau_2-\tau_1\right| \longrightarrow_{\tau_1\to\tau_2} 0,\end{aligned} \tag{66}$$

which tend to 0 independently of (x, y). This implies that the operator $\Phi(x, y)$ is equicontinuous. Thus, by the above findings, the operator $\Phi(x, y)$ is completely continuous.

In the next step, it will be established that the set $\mathcal{P} = \{(x, y) \in \mathcal{K} \times \mathcal{R} / (x, y) = \lambda\Phi(x, y), 0 \leq \lambda \leq 1\}$ is bounded.

Let $(x, y) \in \mathcal{P}$. Then we have $(x, y) = \lambda\Phi(x, y)$. Thus, for any $t \in [0, 1]$, we can write

$$x(t) = \lambda\Phi_1(x, y)(t), \quad y(t) = \lambda\Phi_2(x, y)(t). \tag{67}$$

Then,

$$|x(t)| \leq \frac{2L_1}{\Gamma(\alpha+1)}(\rho_0 + \rho_1\|x\| + \rho_2\|y\|) + M_1, \quad |y(t)| \leq \frac{2L_2}{\Gamma(\beta+1)}(\delta_0 + \delta_1\|x\| + \delta_2\|y\|) + M_2, \tag{68}$$

which imply that

$$\|x\| \leq \mu_1(\rho_0 + \rho_1\|x\| + \rho_2\|y\|) + M_1, \quad \|y\| \leq \mu_2(\delta_0 + \delta_1\|x\| + \delta_2\|y\|) + M_2. \tag{69}$$

Thus,

$$\|x\| + \|y\| \leq (\mu_1\rho_0 + \mu_2\delta_0 + M_1 + M_2) + (\mu_1\rho_1 + \mu_2\delta_1)\|x\| + (\mu_1\rho_2 + \mu_2\delta_2)\|y\|, \tag{70}$$

which, in view of (55), gives

$$\|(x, y)\| \leq \frac{\mu_1\rho_0 + \mu_2\delta_0 + M_1 + M_2}{\mu_0}. \tag{71}$$

This shows that the set is bounded. Hence, all the conditions of Lemma 10 are satisfied and consequently the operator Φ has at least one fixed point, which corresponds to a solution of problem (49). This completes the proof. □

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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