Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2016, Article ID 3910972, 6 pages
http://dx.doi.org/10.1155/2016/3910972

Research Article

Discrete Approaches to Continuous Boundary Value Problems: Existence and Convergence of Solutions

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Received 8 June 2016; Accepted 10 July 2016

Academic Editor: Athanassios G. Bratsos

We investigate two types of first-order, two-point boundary value problems (BVPs). Firstly, we study BVPs that involve nonlinear difference equations (the "discrete" BVP); and secondly, we study BVPs involving nonlinear ordinary differential equations (the "continuous" BVP). We formulate some sufficient conditions under which the discrete BVP will admit solutions. For this, our choice of methods involves a monotone iterative technique and the method of successive approximations (a.k.a. Picard iterations) in the absence of Lipschitz conditions. Our existence results for the discrete BVP are of a constructive nature and are of independent interest in their own right. We then turn our attention to applying our existence results for the discrete BVP to the continuous BVP. We form new existence results for solutions to the continuous BVP with our methods involving linear interpolation of the data from the discrete BVP, combined with *a priori* bounds and the convergence Arzela-Ascoli theorem. Thus, our use of discrete BVPs to yield results for the continuous BVP may be considered as a discrete approach to continuous BVPs.

1. Introduction

In this paper we investigate two types of first-order, two-point boundary value problems (BVPs).

Firstly, we study BVPs that involve nonlinear difference equations (the following "discrete" BVP). Let $f : [0,1] \times D \subseteq [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and consider the discrete boundary value problem

$$\frac{\Delta x_i}{h} = f(t_i, x_i), \quad i = 0, 1, \ldots, n-1; \tag{1}$$

$$u x_0 + v x_n = w, \quad u + v \neq 0, \tag{2}$$

where $0 < h = 1/n < 1$; the grid points are denoted by $t_i = ih$ for $i = 0, \ldots, n$; $\Delta x_i := x_{i+1} - x_i$ for $i = 0, \ldots, n-1$; and u, v, and w are constants.

Secondly, we study BVPs involving nonlinear ordinary differential equations (the following "continuous" BVP):

$$x' = f(t, x), \quad t \in [0,1]; \tag{3}$$

$$u x(0) + v x(1) = w, \quad u + v \neq 0, \tag{4}$$

where $' := d/dt$.

Problem (1) and (2) may be considered as a discrete analogue of (3) and (4).

The study of discrete BVP (1) and (2) is significant for two main reasons, as these types of equations

(a) naturally arise when modelling phenomena, for example, in oscillation and control theory [1, p. 1],

(b) are of importance in the approximation of solutions to ordinary differential equations.

In this paper we discuss the existence and approximation of solutions of both sets of BVPs: (1) and (2); (3) and (4).

We formulate some sufficient conditions under which the discrete BVP (1) and (2) will admit solutions. For this, our choice of methods involves monotone iterative techniques and the method of successive approximations (a.k.a. Picard iterations). The classical method of successive approximations is powerful and constructive in nature and thus it is surprising to find that it has been significantly underutilized in the environment of discrete BVPs of the first order. Our existence results for the discrete BVP are of a constructive nature and, furthermore, some of our results bound solutions independently of the step size. These results are of independent interest of the continuous BVP (3) and (4).

We then turn our attention to applying our existence results for the discrete BVP (1) and (2) to the continuous BVP (3) and (4). We form new existence results for solutions to the continuous BVP with our methods involving linear interpolation of the data from the discrete BVP, combined with *a priori* bounds and the convergence Arzela-Ascoli theorem. Thus, our use of discrete BVPs to yield results for the continuous BVP may be considered as a discrete approach to continuous BVPs.

Several other authors have studied the existence of solutions to (1) and (2) via the method of lower and upper solutions [2, 3], [4, Sec. 2]; and by employing *a priori* bounds on solutions and Brouwer degree [5]. Mohamed et al. [6] have recently studied variations of (1) and (2) via discrete approaches.

Several authors have used the discrete approach to continuous BVPs for second-order problems, such as [7–11]. In particular, in [9–11] the boundary conditions were separated; however, in this work our boundary conditions under consideration are not separated. In addition, we employ different assumptions and different methods. For example, we use the idea of a monotonic and bounded sequence herein, rather than the maximum principles of [9] or the growth conditions and *a priori* bounds of [10, 11].

Our ideas complement those of [2, 3, 5] and [4, Sec. 2] and appear to be of a more constructive nature as solutions to (1) and (2) obtained by the theorems herein may be computed (or approximated) via an iterative process. Our results herein improve some of the results in [6] and our techniques and methods contrast with theirs; for example, we do not rely on Lipschitz conditions in our theorems.

Our results are innovative for two main reasons: (i) they are new for the discrete BVP; (ii) they form new connections to the continuous BVP. Furthermore, we believe that the discrete approach to continuous BVPs that we present open up several lines of inquiry for first-order BVPs.

A solution to the discrete BVP (1) and (2) is a vector $\tilde{x} := (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ having components x_i that

(a) satisfy $(t_i, x_i) \in [0,1] \times D$ for $i = 0, \ldots, n$,

(b) satisfy (1) for $i = 0, \ldots, n-1$ and also satisfy (2).

A solution to the continuous BVP (3) and (4) is differentiable function $x = x(t)$ that

(a) satisfies $(t, x(t)) \in [0,1] \times D$ for $t \in [0,1]$,

(b) satisfies (3) for $t \in [0,1]$ and also satisfies (4).

We now present a simple result showing the equivalence between (1) and (2) and a particular summation equation that will be used throughout this work.

Lemma 1. *The discrete BVP (1) and (2) and the summation equation*

$$x_i = h\sum_{j=0}^{n-1} G(t_i, t_j) f(t_j, x_j) + \frac{w}{u+v}, \quad i = 0, \ldots, n; \tag{5}$$

are equivalent, with

$$G(t_i, t_j) := \begin{cases} \dfrac{u}{u+v}, & \text{for } 0 \le j \le i-1, \\ -\dfrac{v}{u+v}, & \text{for } i \le j \le n-1. \end{cases} \tag{6}$$

Proof. For completeness we provide a proof. Let $\tilde{x}$ be a solution to (1) and (2). If we sum (1) from 0 to $i-1$ then we obtain

$$x_i = h\sum_{j=0}^{i-1} f(t_j, x_j) + x_0, \quad i = 0, \ldots, n; \tag{7}$$

and so for $i = n$ we obtain

$$x_n = h\sum_{j=0}^{n-1} f(t_j, x_j) + x_0. \tag{8}$$

Using boundary conditions (2) we can eliminate x_n in (8) to obtain

$$x_0 = \frac{w}{u+v} - \frac{v}{u+v} h\sum_{j=0}^{n-1} f(t_j, x_j). \tag{9}$$

Thus, substitution of (9) into (7) yields

$$x_i = h\sum_{j=0}^{i-1} f(t_j, x_j) - \frac{v}{u+v} h\sum_{j=0}^{n-1} f(t_j, x_j) + \frac{w}{u+v}, \quad i = 0, \ldots, n, \tag{10}$$

which can then be recast into form (5) by splitting the second term to sum from $j = 0$ to $i-1$ and from $j = i$ to $n-1$.

Now let $\tilde{x}$ be a solution to (10). It can be directly verified that (1) and (2) hold. □

2. Monotone Sequential Approach

In this section we formulate some existence results for solutions to (1) and (2) by generating a monotone and bounded sequence of vectors whose limit will be a solution to (1) and (2).

Throughout this section the domain $[0,1] \times D$ of f will be the rectangle

$$R_b := \left\{(t, p) \in [0,1] \times \mathbb{R} : \left|p - \frac{w}{u+v}\right| \le b\right\} \tag{11}$$

for some positive number b.

Since f is continuous on the compact set R_b we may define a number $M \ge 0$ such that

$$M \ge \max_{(t,p)\in R_b} |f(t, p)|. \tag{12}$$

The main result of this section is the following.

Theorem 2. *Let $f : R_b \to \mathbb{R}$ be continuous and let*

$$\frac{M}{|u+v|}\max\{|u|,|v|\} \le b; \tag{13}$$

$$\frac{u}{u+v} > 0, \qquad -\frac{v}{u+v} > 0. \tag{14}$$

If

$$f(t,y) \le f(t,z), \quad \forall y \le z,\ (y,z) \in R_b^2; \tag{15}$$

$$f\left(t, \frac{w}{u+v}\right) \ge 0, \quad \forall t \in [0,1], \tag{16}$$

then problem (1) and (2) has at least one solution $\widetilde{x} \in \mathbb{R}^{n+1}$ for each $h \in (0,1)$ such that $(t_i, x_i) \in R_b$ for $i = 0,\dots,n$.

Proof. Consider summation equation (5) that, by Lemma 1, is equivalent to (1) and (2) and define the sequence of vectors $\widetilde{\phi}^{(k)} := (\phi_0^{(k)},\dots,\phi_n^{(k)})$ for $k = 0,1,2,\dots$ recursively by

$$\phi_i^{(0)} = \frac{w}{u+v}, \quad i = 0,\dots,n; \tag{17}$$

$$\phi_i^{(k+1)} = h\sum_{j=0}^{n-1} G(t_i,t_j)\, f(t_j,\phi_j^{(k)}) + \frac{w}{u+v}, \quad i = 0,\dots,n. \tag{18}$$

Firstly we show that our sequence of vectors $\widetilde{\phi}^{(k)}$ is well defined for $k = 0,1,\dots$ by showing that each $|\phi_i^{(k)} - w/(u+v)| \le b$ for $i = 0,\dots,n$ and so $(t_i,\phi_i^{(k)}) \in R_b$ for each $i = 0,\dots,n$ and $k = 0,1,\dots$. We use proof by induction.

From the definition of $\phi_i^{(0)}$ it is easy to see that $|\phi_i^{(0)} - w/(u+v)| \le b$ for $i = 0,\dots,n$. Now assume for some $k_1 \ge 0$ we have $|\phi_i^{(k_1)} - w/(u+v)| \le b$ for $i = 0,\dots,n$. From (18) we have for $i = 0,\dots,n$

$$\begin{aligned}
\left|\phi_i^{(k_1+1)} - \frac{w}{u+v}\right| &\le h\sum_{j=0}^{n-1}\left|G(t_i,t_j)\right|\left|f(t_j,\phi_j^{(k_1)})\right| \\
&\le \frac{Mh}{|u+v|}\left[\sum_{j=0}^{i-1}|u| + \sum_{j=i}^{n-1}|v|\right] \\
&= \frac{Mh}{|u+v|}\left[|u|\, i + |v|(n-i)\right] \\
&\le \frac{Mhn}{|u+v|}\max\{|u|,|v|\} \\
&= \frac{M}{|u+v|}\max\{|u|,|v|\} \le b
\end{aligned} \tag{19}$$

from (13). Thus, by induction, we have $(t_i,\phi_i^{(k)}) \in R_b$ for each $i = 0,\dots,n$ and $k = 0,1,\dots$ and so our sequence of vectors $\widetilde{\phi}^{(k)}$ is well defined for each $k = 0,1,\dots$.

Furthermore, the above has shown that the sequence of vectors $\widetilde{\phi}^{(k)}$ is uniformly bounded for $k = 0,1,\dots$.

We now show that $\widetilde{\phi}^{(k+1)} \ge \widetilde{\phi}^{(k)}$ for $k = 0,1,\dots$, where the inequality holds in a componentwise fashion. Once again, we use induction. For $i = 0,\dots,n$ consider

$$\begin{aligned}
\phi_i^{(1)} &= h\sum_{j=0}^{i-1}\left(\frac{u}{u+v}\right) f\left(t_j, \frac{w}{u+v}\right) \\
&\quad + h\sum_{j=i}^{n-1}\left(\frac{-v}{u+v}\right) f\left(t_j, \frac{w}{u+v}\right) + \frac{w}{u+v} \\
&\ge \frac{w}{u+v} = \phi_i^{(0)},
\end{aligned} \tag{20}$$

where we have used (14) and (16). Thus, $\widetilde{\phi}^{(1)} \ge \widetilde{\phi}^{(0)}$.

Now assume that $\widetilde{\phi}^{(k_1)} \ge \widetilde{\phi}^{(k_1-1)}$ for some $k_1 \ge 1$; that is, assume $\phi_i^{(k_1)} \ge \phi_i^{(k_1-1)}$ for $i = 0,\dots,n$. For each $i = 0,\dots,n$ we have

$$\begin{aligned}
\phi_i^{(k_1+1)} &= h\sum_{j=0}^{i-1}\left(\frac{u}{u+v}\right) f(t_j,\phi_j^{(k_1)}) \\
&\quad + h\sum_{j=i}^{n-1}\left(\frac{-v}{u+v}\right) f(t_j,\phi_j^{(k_1)}) + \frac{w}{u+v} \\
&\ge h\sum_{j=0}^{i-1}\left(\frac{u}{u+v}\right) f(t_j,\phi_j^{(k_1-1)}) \\
&\quad + h\sum_{j=i}^{n-1}\left(\frac{-v}{u+v}\right) f(t_j,\phi_j^{(k_1-1)}) + \frac{w}{u+v} \\
&= \phi_i^{(k_1)},
\end{aligned} \tag{21}$$

where we have used assumptions (14) and (15). Thus, $\widetilde{\phi}^{(k+1)} \ge \widetilde{\phi}^{(k)}$ for $k = 0,1,\dots$.

From the above we conclude that $\widetilde{\phi}^{(k)}$ is a uniformly bounded and nondecreasing sequence of vectors and so must converge to a vector $\widetilde{\phi}$; that is,

$$\lim_{k\to\infty}\widetilde{\phi}^{(k)} = \widetilde{\phi} \tag{22}$$

for some $\widetilde{\phi} \in \mathbb{R}^{n+1}$.

We finally show that the above $\widetilde{\phi} = (\phi_0,\dots,\phi_n) \in \mathbb{R}^{n+1}$ is actually a solution to (1) and (2). Since each $|\phi_i^{(k)} - w/(u+v)| \le b$ we must have each $|\phi_i - w/(u+v)| \le b$ and so $(t_i,\phi_i) \in R_b$ for $i = 0,\dots,n$. Furthermore, the continuity of f on R_b ensures that

$$f(t_i,\phi_i^{(k)}) \longrightarrow f(t_i,\phi_i), \quad \text{as } k \longrightarrow \infty \tag{23}$$

for each $i = 0,\dots,n$.

If we now take limits in (18) as $k \to \infty$ then we obtain

$$\phi_i = h\sum_{j=0}^{n-1} G(t_i,t_j)\, f(t_j,\phi_j) + \frac{w}{u+v}, \quad i = 0,\dots,n, \tag{24}$$

so that our limit vector $\widetilde{\phi}$ is a solution to (1) and (2). □

Table 1

i	0	1	2	3	4
$\epsilon_i^{(2)}$	0.00206246	0.00244859	0.00262248	0.00264573	0.00259853
$\epsilon_i^{(4)}$	0.0000898399	0.000100995	0.000108355	0.000112198	0.000113191
$\epsilon_i^{(6)}$	3.5115×10^{-6}	3.9453×10^{-6}	4.2343×10^{-6}	4.3843×10^{-6}	4.4242×10^{-6}
$\epsilon_i^{(8)}$	1.3681×10^{-7}	1.5372×10^{-7}	1.6497×10^{-7}	1.7082×10^{-7}	1.7238×10^{-7}
$\epsilon_i^{(10)}$	5.3299×10^{-9}	5.9883×10^{-9}	6.4269×10^{-9}	6.6545×10^{-9}	6.7152×10^{-9}
$\epsilon_i^{(12)}$	2.0764×10^{-10}	2.3329×10^{-10}	2.5037×10^{-10}	2.5924×10^{-10}	2.6161×10^{-10}

Example 3. Consider the following discrete BVP:

$$\frac{\Delta x_i}{h} = \frac{1}{10}\left[(x_i)^{1/3} + t_i\right], \quad i = 0, \ldots, n-1; \tag{25}$$

$$x_0 - \frac{1}{2}x_n = 0, \tag{26}$$

so that we have a special case of (1) and (2) with

$$\begin{gathered} f(t, y) = \frac{1}{10}\left[y^{1/3} + t\right]; \\ u = 1; \\ v = -\frac{1}{2}; \\ w = 0. \end{gathered} \tag{27}$$

We claim that problem (25) and (26) has at least one solution $\tilde{x}$ such that $|x_i| \leq 1$ for $i = 0, \ldots, n$.

Proof. We show that all of the conditions of Theorem 2 hold. Firstly, we see that the inequalities in (14) hold. If we choose $b = 1$ to form R_b then $M = 1/5$ and so (13) holds. Furthermore, f is nondecreasing in the second variable and so (15) is satisfied. Finally, (16) holds. Thus, all of the conditions of Theorem 2 hold and the result follows. □

Remark 4. In Example 3 above, letting $n = 4$, pick $\phi_i^{(0)} = 0$ for $i = 0, \ldots, 4$ and construct the approximating iterates $\phi_i^{(k)}$ as in (18). The numbers in Table 1 signify the error

$$\epsilon_i^{(k)} := \left|\Delta\phi_i^{(k)} - \frac{h}{10}\left[\left(\phi_i^{(k)}\right)^{1/3} + ih\right]\right| \tag{28}$$

that results upon substituting the generated $\phi_i^{(k)}$ into (25).

We notice that the error in terms of $\phi_i^{(k)}$ at each i decreases for this example as $\phi_i^{(k)}$ converge upward to a solution in the rectangle, so that $\phi_i^{(12)}$, for example, is a good approximation to a solution x_i of (25). The actual values of $\phi_i^{(12)} \approx x_i$ are given by

$$\begin{gathered} \phi_0^{(12)} = 0.0849743, \\ \phi_1^{(12)} = 0.0959653, \\ \phi_2^{(12)} = 0.113661, \\ \phi_3^{(12)} = 0.138271, \\ \phi_4^{(12)} = 0.169949. \end{gathered} \tag{29}$$

Note that by construction, $\phi_i^{(k)}$ satisfies boundary condition (26); namely,

$$\phi_0^{(k)} - \frac{1}{2}\phi_n^{(k)} = 0, \quad \text{for each } k \in \mathbb{N}. \tag{30}$$

The following result is a modification of the ideas in Theorem 2 and its proof.

Theorem 5. *Let $f : R_b \to \mathbb{R}$ be continuous and let*

$$M\left[1 - \frac{v}{u+v}\right] \leq b; \tag{31}$$

$$-\frac{v}{u+v} > 0. \tag{32}$$

If

$$\begin{gathered} f(t, y) \leq f(t, z), \quad \forall y \leq z, \ (y, z) \in R_b^2; \\ f\left(t, \frac{w}{u+v}\right) \geq 0; \quad \forall t \in [0, 1], \end{gathered} \tag{33}$$

then problem (1) and (2) has at least one solution $\tilde{x} \in \mathbb{R}^{n+1}$ such that $(t_i, x_i) \in R_b$ for $i = 0, \ldots, n$.

Proof. The proof is very similar to that of Theorem 2 and so is only outlined.

Consider the sequence of successive approximations defined by

$$\begin{aligned} \phi_i^{(0)} &= \frac{w}{u+v}, \quad i = 0, \ldots, n; \\ \phi_i^{(k+1)} &= h\sum_{j=0}^{i-1} f\left(t_j, \phi_j^{(k)}\right) - \frac{v}{u+v}h\sum_{j=0}^{n-1} f\left(t_j, \phi_j^{(k)}\right) \\ &\quad + \frac{w}{u+v}, \quad i = 0, \ldots, n; \end{aligned} \tag{34}$$

for $k = 0, 1, 2, \ldots$. The continuity of f and (31) ensure that the successive approximations are well defined and uniformly bounded. The assumptions (32)-(33) ensure that the successive approximations are a nondecreasing sequence with the convergence and existence following in the same way as in the proof of Theorem 2. □

Remark 6. Note that (31) is a stronger assumption than (13), while (32) is weaker than (14).

There are a number of interesting variations of Theorems 2 and 5 that we now discuss.

Remark 7. The proofs of Theorems 2 and 5 essentially rest on generating a bounded, nondecreasing sequence of vectors. The statement of each theorem can be suitably modified so as to produce a bounded, nonincreasing sequence of vectors that converge to a solution of (1) and (2). All that is required is to reverse the differential inequalities in, for example, (15) and (16).

Remark 8. For simplicity, the initial approximation $\widetilde{\phi}_0$ in the proofs of Theorems 2 and 5 was chosen to be a constant vector with components $w/(u+v)$. With suitable modifications on (16) we may use any vector $\widetilde{\phi}_0$ as our initial approximation provided $(t_i, \phi_i^{(0)}) \in R_b$ for $i = 0, \ldots, n$. For BVPs that have more than one solution, different choices in our initial approximation $\widetilde{\phi}_0$ can lead to the generation of distinct limit functions $\widetilde{\phi}$. That is, through various choices of $\widetilde{\phi}_0$ we can observe convergence of $\widetilde{\phi}^{(k)}$ to various solutions of (1) and (2).

3. A Discrete Approach to Differential Equations

In this section we form a relationship between solutions to the discrete BVP (1) and (2) and solutions to the continuous BVP (3) and (4). We generate a sequence of functions that are based on the solutions to (1) and (2) guaranteed to exist from earlier sections and present some conditions under which they will converge to a function as $h \to 0$, with the function being a solution to (3) and (4). Thus, our approach uses the discrete problem to generate new existence results for the continuous problem in a constructive manner.

Our first general convergence result is in the spirit of [7, Lemma 2.4], where Gaines applies the ideas to second-order BVPs. Our result involves a bound on the solutions to (1) and (2), with the bound being independent of h.

We require the following notation. Denote the sequence $n_m \to \infty$ as $m \to \infty$; let $0 < h_m = 1/n_m < 1$; and let $t_i^m = ih_m$ for $i = 0, \ldots, n$. If problem (1) and (2) has a solution for $h = h_m$ and $m \geq m_0$ that we denote by

$$\widetilde{x}^m := (x_0^m, \ldots, x_n^m) \tag{35}$$

then we construct the following sequence of continuous functions from (35) via linear interpolation to form

$$x^m(t) := x_i^m + \frac{(x_{i+1}^m - x_i^m)}{h_m}(t - t_i^m), \quad t_i^m \leq t \leq t_{i+1}^m, \tag{36}$$

for $m \geq m_0$ and $t \in [0,1]$. Note that $x^m(t_i^m) = x_i^m$ for $i = 0, \ldots, n$.

Lemma 9. *Let $f : [0,1] \times D \subseteq [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and let $R \geq 0$ be a constant. If problem (1) and (2) has a solution for $h \leq h_m$ and $m \geq m_0$ that we denote by $\widetilde{x}^m$ with*

$$\max_{i=0,\ldots,n} |x_i^m| \leq R, \quad m \geq m_0, \tag{37}$$

then problem (3) and (4) has a solution $x = x(t)$ that is the limit of a subsequence of (36).

Proof. For $m \geq m_0$ consider the sequence of functions $x^m(t)$ for $t \in [0,1]$ in (36). We show that the sequence of functions x^m is uniformly bounded and equicontinuous on $[0,1]$. For $t \in [t_i^m, t_{i+1}^m]$ and $m \geq m_0$ we have

$$|x^m(t)| \leq |x_i^m| + \left|\frac{(x_{i+1}^m - x_i^m)}{h_m}\right| |t - t_i^m| \leq R + M_1, \tag{38}$$

where

$$M_1 \geq \max_{t\in[0,1], |p|\leq R} |f(t,p)|. \tag{39}$$

Thus, x^m is uniformly bounded on $[0,1]$.

For $\beta, \gamma \in [0,1]$ and given $\varepsilon > 0$, consider

$$\begin{aligned} |x^m(\beta) - x^m(\gamma)| &\leq \left|\frac{(x_{i+1}^m - x_i^m)}{h_m}\right| |\beta - \gamma| \\ &\leq M_1 |\beta - \gamma| < \varepsilon \end{aligned} \tag{40}$$

whenever $|\beta - \gamma| < \delta(\varepsilon) := \varepsilon/M_1$. Thus, x^m is equicontinuous on $[0,1]$.

The convergence Arzela-Ascoli theorem [12, p. 527] guarantees that the sequence of continuous functions $x^m = x^m(t)$ has a subsequence $x^{k(m)}(t)$ that converges uniformly to a continuous function $x = x(t)$ for $t \in [0,1]$. That is,

$$\max_{t\in[0,1]} \left|x^{k(m)}(t) - x(t)\right| \longrightarrow 0, \quad \text{as } m \longrightarrow \infty. \tag{41}$$

The continuity of f ensures that the above limit function will be a solution to (3) and (4). □

The next theorem, in the spirit of [7, Theorem 2.5], will require the following notation. If problem (1) and (2) has a solution $\widetilde{x}$ for $0 < h \leq h_0$ then we define the continuous function $x(t, \widetilde{x})$ by

$$x(t, \widetilde{x}) := x_i + \frac{(x_{i+1} - x_i)}{h}(t - t_i), \quad t_i \leq t \leq t_{i+1}. \tag{42}$$

Theorem 10. *Let $f : [0,1] \times D \subseteq [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and let $R \geq 0$ be a constant. Assume problem (1) and (2) has a solution for $h \leq h_0$ that we denote by $\widetilde{x}$ with*

$$\max_{i=0,\ldots,n} |x_i| \leq R. \tag{43}$$

Given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that if $h \leq \delta$ then problem (3) and (4) has a solution $x = x(t)$ with

$$\max_{t\in[0,1]} |x(t, \widetilde{x}) - x(t)| \leq \varepsilon. \tag{44}$$

Proof. Suppose, for some $\varepsilon > 0$, there is a sequence h_m such that $h_m \to 0$ as $m \to \infty$ and for $h = h_m = 1/n_m$ problem (1) and (2) has a solution $\tilde{x}^m$ with every solution $x = x(t)$ to (3) and (4) satisfying

$$\max_{t\in[0,1]} |x(t,\tilde{x}) - x(t)| > \varepsilon. \tag{45}$$

By assumption, for m sufficiently large, there is $R \geq 0$ such that the solution $\tilde{x}^m$ to (1) and (2) satisfies

$$\max_{i=0,\dots,n} |x_i^m| \leq R. \tag{46}$$

Thus, the conditions of Lemma 9 are satisfied and so we obtain a subsequence $x^{k(m)}(t)$ of $x^m(t)$ that converges uniformly on $[0,1]$ to a solution x of (3) and (4). Thus, (45) cannot hold. □

We now relate the above abstract results to the ideas from earlier sections.

Theorem 11. *Let the conditions of Theorem 2 hold. Given any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon)$ such that if $h \leq \delta$ then problem (3) and (4) has a solution x that satisfies (44).*

Proof. We show that the conditions of Theorem 10 are satisfied for $R_b = [0,1] \times D$. Assumption (13) ensures that the solution $\tilde{x}$ to (1) and (2) guaranteed to exist by Theorem 2 satisfies $|x_i| \leq b$ for $i = 0,\dots,n$ and so (43) holds with $R = b$.

Thus, all of the conditions of Theorem 10 hold and the result follows. □

Remark 12. Similar results to that of Theorem 11 hold under the assumptions of Theorem 5 or Remark 7.

Disclosure

Douglas R. Anderson stated that research was carried out while being a visiting fellow at School of Mathematics and Statistics, UNSW, Sydney, NSW 2052, Australia.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

References

[1] P. L. Falb and J. L. de Jong, *Some Successive Approximation Methods in Control and Oscillation Theory*, vol. 59 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1969.

[2] R. P. Agarwal, A. Cabada, V. Otero-Espinar, and S. Dontha, "Existence and uniqueness of solutions for anti-periodic difference equations," *Archives of Inequalities and Applications*, vol. 2, no. 4, pp. 397–412, 2004.

[3] F. M. Atici, A. Cabada, and J. B. Ferreiro, "Existence and comparison results for first order periodic implicit difference equations with maxima," *Journal of Difference Equations and Applications*, vol. 8, no. 4, pp. 357–369, 2002.

[4] A. Cabada, "The method of lower and upper solutions for periodic and anti-periodic difference equations," *Electronic Transactions on Numerical Analysis*, vol. 27, pp. 13–25, 2007.

[5] C. C. Tisdell, "On first-order discrete boundary value problems," *Journal of Difference Equations and Applications*, vol. 12, no. 12, pp. 1213–1223, 2006.

[6] M. Mohamed, H. B. Thompson, M. S. Jusoh, and K. Jusoff, "Discrete first-order three point boundary value problem," *Journal of Mathematics Research*, vol. 2, no. 2, pp. 207–215, 2009.

[7] R. Gaines, "Difference equations associated with boundary value problems for second order nonlinear ordinary differential equations," *SIAM Journal on Numerical Analysis*, vol. 11, pp. 411–434, 1974.

[8] I. Rachůnková and C. C. Tisdell, "Existence of non-spurious solutions to discrete Dirichlet problems with lower and upper solutions," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 67, no. 4, pp. 1236–1245, 2007.

[9] H. B. Thompson and C. Tisdell, "Systems of difference equations associated with boundary value problems for second order systems of ordinary differential equations," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 2, pp. 333–347, 2000.

[10] H. B. Thompson and C. Tisdell, "Boundary value problems for systems of difference equations associated with systems of second-order ordinary differential equations," *Applied Mathematics Letters*, vol. 15, no. 6, pp. 761–766, 2002.

[11] H. B. Thompson and C. C. Tisdell, "The nonexistence of spurious solutions to discrete, two-point boundary value problems," *Applied Mathematics Letters*, vol. 16, no. 1, pp. 79–84, 2003.

[12] W. T. Reid, *Ordinary Differential Equations*, John Wiley & Sons, New York, NY, USA, 1971.