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Research Article

On Oscillatory and Asymptotic Behavior of a Second-Order Nonlinear Damped Neutral Differential Equation

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This paper discusses oscillatory and asymptotic properties of solutions of a class of second-order nonlinear damped neutral differential equations. Some new sufficient conditions for any solution of the equation to be oscillatory or to converge to zero are given. The results obtained extend and improve some of the related results reported in the literature. The results are illustrated with examples.

1. Introduction

In this paper, we consider the second-order nonlinear damped neutral differential equation

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + h(t)\left(z'(t)\right)^{\alpha} + q(t) f(x(\delta(t))) = 0, \quad t \ge t_0 > 0, \tag{1}$$

where $z(t) = x(t) - p(t)x(\tau(t))$ and α is a ratio of positive odd integers. Throughout this paper and without further mention, we assume that

(i) $p : [t_0, \infty) \to \mathbb{R}$ is a real-valued continuous function with $0 \le p(t) \le p_0 < 1$;

(ii) $h : [t_0, \infty) \to \mathbb{R}$ and $q : [t_0, \infty) \to [0, \infty)$ are real-valued continuous functions, and $q(t)$ is not identically zero for all sufficiently large t;

(iii) $r : [t_0, \infty) \to \mathbb{R}^{+} = (0, \infty)$ is a real-valued continuous function such that

$$R(t, t_0) := \int_{t_0}^{t} \frac{ds}{\left(\gamma(s) r(s)\right)^{1/\alpha}} \longrightarrow \infty \quad \text{as } t \longrightarrow \infty, \tag{2}$$

where

$$\gamma(t) = \exp\left(\int_{t_0}^{t} \frac{h(s)}{r(s)} ds\right); \tag{3}$$

(iv) $f : \mathbb{R} \to \mathbb{R}$ is a real-valued continuous function with $uf(u) > 0$ for $u \ne 0$ and there exists a constant $k > 0$ such that $f(u)/u^{\beta} \ge k$ for all $u \ne 0$, where β is a ratio of positive odd integers with $\alpha \ge \beta$;

(v) $\tau, \delta : [t_0, \infty) \to \mathbb{R}$ are real-valued continuous functions such that $\tau(t) \le t$, $\lim_{t\to\infty}\tau(t) = \infty$, and $\lim_{t\to\infty}\delta(t) = \infty$.

Without further mention, we will assume throughout that every solution $x(t)$ of (1) that is under consideration here is continuable to the right and nontrivial; that is, $x(t)$ is defined on some ray $[t_x, \infty)$ for some $t_x \ge t_0$ and $\sup\{|x(t)| : t \ge t_1\} > 0$ for every $t_1 \ge t_x$, which has the properties $z(t) \in C^1([t_x, \infty), \mathbb{R})$ and $r(t)(z'(t))^{\alpha} \in C^1([t_x, \infty), \mathbb{R})$ for any $t_x \ge t_0$. We tacitly assume that (1) possesses such solutions. Such a solution is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$; otherwise, it is called *nonoscillatory*. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

Since Sturm [1] introduced the concept of oscillation when he studied the problem of the heat transmission, the oscillation theory has been a very active area of research in the qualitative theory of both ordinary and functional differential equations. Usually, a qualitative approach is concerned with the behavior of solutions of a given differential equation and does not seek explicit solutions. Since then, asymptotic and oscillatory properties of solutions to different classes of ordinary differential equations, functional differential equations, and dynamic equations have attracted the attention of many researchers; see, for example, [2–27] and the references therein (see also [28–33] for numerical methods with semianalytical methods).

Recently, neutral delay differential equations, that is, equations in which the highest order derivative of the unknown function appears both with and without delays, have received strong interest in the study of oscillation properties of their solutions. The problem of asymptotic and oscillatory behavior or solutions of neutral differential equations is of both theoretical and practical interest. One reason for this is that they arise, for example, in applications to electric networks containing lossless transmission lines. Such networks appear in high speed computers where lossless transmission lines are used to interconnect switching circuits. They also occur in problems dealing with vibrating masses attached to an elastic bar and in the solution of variational problems with time delays. Interested readers can refer to the book by Hale [34] for some applications in science and technology.

On reviewing the literature, it becomes apparent that most results concerning the oscillation of all solutions of (1) are for the special case when $h(t) = 0$. Regarding the oscillation of undamped neutral differential equations, that is, special cases of (1) with $h(t) = 0$, many papers have been published for different cases of $p(t)$ such as $-1 \le p(t) \le 0$, $-\infty < p_0 \le p(t) \le 0$, and $0 \le p(t) \le p_0 < 1$. We refer the reader to [4, 14, 16–20, 24–26] and the references cited therein as examples of recent results on this topic.

In 2015, Li et al. [20] considered (1) with $h(t) = 0$ in the case where $\delta(t) \le t$, $\delta'(t) > 0$, and $\alpha = \beta$ and presented some new conditions which ensure that any solution x of (1) with $h(t) = 0$ either is oscillatory or converges to zero.

Motivated by the work of Li et al. [20] and the papers mentioned above, in the present paper, by employing Riccati type transformation and the integral averaging technique involving integrals and/or weighted integrals of coefficients of a given differential equation, we establish some new sufficient conditions for all solutions of (1) to be oscillatory or to converge to zero. The results obtained improve the results of Li et al. [20] in the sense that we do not require the restrictive condition $\delta'(t) > 0$ and extend some known results in the relevant literature. It should be noted that Li et al. [20] only discussed the oscillation properties of solutions in the delay case $\delta(t) \le t$. Here, we also consider the advanced case $\delta(t) \ge t$ as well. Some examples are also considered to illustrate the main results. We also want to note that the results obtained can easily be extended to more general neutral differential equations and neutral dynamic equations on any time scales of the type (1). It is therefore hoped that the present paper will contribute to the studies on oscillatory and asymptotic behavior of solutions of neutral differential equations with damping term.

2. Main Results

For any continuous function $u(t)$, we set $u_+(t) = \max\{0, u(t)\}$, and, to simplify the formulation of our results, we will use the following notations:

$$R(t, t_1) := \int_{t_1}^{t} \frac{ds}{(\gamma(s) r(s))^{1/\alpha}} \quad \text{for } t_0 \le t_1 \le t < \infty, \tag{4}$$

and, for sufficiently large t_*,

$$\theta(t, t_*) := \frac{R(\delta(t), t_*)}{R(t, t_*)}, \qquad \phi(t, t_*) := \begin{cases} 1, & \delta(t) \ge t, \\ \theta^{\beta}(t, t_*), & \delta(t) \le t. \end{cases} \tag{5}$$

We begin with a lemma that will be used to prove our main results.

Lemma 1 (see [35]). *If X and Y are nonnegative and $\lambda > 1$, then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \le (\lambda - 1) Y^{\lambda}, \tag{6}$$

where the equality holds if and only if $X = Y$.

Theorem 2. *Assume that condition (2) is satisfied and there exists a positive function $\rho \in C^1([t_0, \infty), \mathbb{R})$ such that, for all sufficiently large T_* and for $T > T_*$,*

$$\limsup_{t\to\infty} \int_{T}^{t} \left\{ k\rho(s)\gamma(s) q(s) \phi(s, T_*) - \frac{c^* \rho_+'(s)}{R^{\beta}(s, T_*)} \right\} ds = \infty, \tag{7}$$

with $c^* > 0$; then, any solution x of (1) either oscillates or tends to zero as $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \ge t_1$. Multiplying (1) by $\gamma(t)$ (see (3)), (1) takes the form

$$\left(\gamma(t) r(t) \left(z'(t)\right)^{\alpha}\right)' + q(t)\gamma(t) f(x(\delta(t))) = 0. \tag{8}$$

From (ii), (3), (iv), and (8), we have

$$\left(\gamma(t) r(t) \left(z'(t)\right)^{\alpha}\right)' = -q(t)\gamma(t) f(x(\delta(t))) \le 0 \quad \text{for } t \ge t_1, \tag{9}$$

so $\gamma(t) r(t) (z'(t))^{\alpha}$ is eventually decreasing, say for $t \in [t_2, \infty) \subset [t_1, \infty)$. We claim that

$$z'(t) > 0 \quad \text{for } t \ge t_2. \tag{10}$$

If this is not so, then there exists $t_3 \in [t_2, \infty)$ such that $z'(t_3) \leq 0$. In view of (9), there is $t_4 \geq t_3$ such that

$$\gamma(t) r(t) \left(z'(t)\right)^\alpha \leq \gamma(t_4) r(t_4) \left(z'(t_4)\right)^\alpha := c < 0 \quad (11)$$

for $t \in [t_4, \infty)$. Hence,

$$z'(t) \leq c^{1/\alpha} \frac{1}{\left(\gamma(t) r(t)\right)^{1/\alpha}}, \quad (12)$$

from which it follows that

$$z(t) \leq z(t_4) + c^{1/\alpha} \int_{t_4}^{t} \frac{ds}{\left(\gamma(s) r(s)\right)^{1/\alpha}} \quad \text{for } t \geq t_4. \quad (13)$$

In view of (2), (13) implies that

$$\lim_{t\to\infty} z(t) = -\infty. \quad (14)$$

Thus, there are two cases to consider.

Case 1. If $x(t)$ is unbounded, then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = \infty$, where

$$x(t_k) = \max\{x(s) : t_0 \leq s \leq t_k\}. \quad (15)$$

Since $\lim_{t\to\infty} \tau(t) = \infty$, we can choose a large k such that $\tau(t_k) > t_0$. Thus, by (15) and the fact that $\tau(t) \leq t$, we have

$$\begin{aligned} x(\tau(t_k)) &= \max\{x(s) : t_0 \leq s \leq \tau(t_k)\} \\ &\leq \max\{x(s) : t_0 \leq s \leq t_k\} = x(t_k), \end{aligned} \quad (16)$$

so, from the definition of z, we see that

$$\begin{aligned} z(t_k) &= x(t_k) - p(t_k) x(\tau(t_k)) \geq (1 - p(t_k)) x(t_k) \\ &\geq (1 - p_0) x(t_k) > 0, \end{aligned} \quad (17)$$

which contradicts (14).

Case 2. If $x(t)$ is bounded, then, in view of the definition of z and the fact that $0 \leq p(t) \leq p_0 < 1$, it follows that z is also bounded, which again contradicts (14). Thus, in view of Cases 1 and 2, we conclude that (10) holds.

Hence, from (9) and (10) and the definition of z, we conclude that there exists $t_2 \geq t_1$ such that, for $t \geq t_2$, either

$$\begin{gathered} z(t) > 0, \\ z'(t) > 0, \\ \left(\gamma(t) r(t) \left(z'(t)\right)^\alpha\right)' \leq 0 \end{gathered} \quad (18)$$

or

$$\begin{gathered} z(t) < 0, \\ z'(t) > 0, \\ \left(\gamma(t) r(t) \left(z'(t)\right)^\alpha\right)' \leq 0. \end{gathered} \quad (19)$$

Assume that (18) holds. We note that $x(t) \geq z(t)$ and set

$$w(t) = \rho(t) \frac{\gamma(t) r(t) \left(z'(t)\right)^\alpha}{z^\beta(t)} \quad \text{for } t \geq t_2. \quad (20)$$

Then, $w(t) > 0$ for $t \geq t_2$ and, from (1), (3), (iv), and (20), we obtain

$$\begin{aligned} w'(t) &= \rho'(t) \frac{\gamma(t) r(t) \left(z'(t)\right)^\alpha}{z^\beta(t)} + \rho(t) \\ &\quad \cdot \left[\frac{\left(\gamma(t) r(t) \left(z'(t)\right)^\alpha\right)'}{z^\beta(t)} \right. \\ &\quad \left. - \beta \frac{\gamma(t) r(t) \left(z'(t)\right)^\alpha z^{\beta-1}(t) z'(t)}{z^{2\beta}(t)} \right] = \rho'(t) \\ &\quad \cdot \frac{\gamma(t) r(t) \left(z'(t)\right)^\alpha}{z^\beta(t)} + \rho(t) \\ &\quad \cdot \frac{\gamma(t) \left[h(t) \left(z'(t)\right)^\alpha + \left(r(t) \left(z'(t)\right)^\alpha\right)'\right]}{z^\beta(t)} - \beta\rho(t) \\ &\quad \cdot \gamma(t) r(t) \frac{\left(z'(t)\right)^{\alpha+1}}{z^{\beta+1}(t)} = \rho'(t) \frac{\gamma(t) r(t) \left(z'(t)\right)^\alpha}{z^\beta(t)} \\ &\quad - \rho(t) \gamma(t) q(t) \frac{f(x(\delta(t)))}{z^\beta(t)} - \beta\rho(t) \gamma(t) r(t) \\ &\quad \cdot \frac{\left(z'(t)\right)^{\alpha+1}}{z^{\beta+1}(t)} \leq \rho'_+(t) \frac{\gamma(t) r(t) \left(z'(t)\right)^\alpha}{z^\beta(t)} - k\rho(t) \\ &\quad \cdot \gamma(t) q(t) \frac{x^\beta(\delta(t))}{z^\beta(t)} - \beta\rho(t) \gamma(t) r(t) \frac{\left(z'(t)\right)^{\alpha+1}}{z^{\beta+1}(t)} \\ &\leq \rho'_+(t) \frac{\gamma(t) r(t) \left(z'(t)\right)^\alpha}{z^\beta(t)} - k\rho(t) \gamma(t) q(t) \\ &\quad \cdot \frac{z^\beta(\delta(t))}{z^\beta(t)} - \beta\rho(t) \gamma(t) r(t) \frac{\left(z'(t)\right)^{\alpha+1}}{z^{\beta+1}(t)}. \end{aligned} \quad (21)$$

In view of the fact that $z'(t) > 0$ for $t \geq t_2$, it follows from (21) that

$$\begin{aligned} w'(t) &\leq \rho'_+(t) \gamma(t) r(t) \left(\frac{z'(t)}{z(t)}\right)^\alpha z^{\alpha-\beta}(t) \\ &\quad - k\rho(t) \gamma(t) q(t) \left(\frac{z(\delta(t))}{z(t)}\right)^\beta. \end{aligned} \quad (22)$$

Since $\gamma(t)r(t)(z'(t))^\alpha$ is nonincreasing, we see that

$$z(t) \geq z(t) - z(t_2) = \int_{t_2}^{t} \frac{\left(\gamma(s) r(s) \left(z'(s)\right)^\alpha\right)^{1/\alpha}}{\left(\gamma(s) r(s)\right)^{1/\alpha}} ds \geq \left(\gamma(t) r(t)\right)^{1/\alpha} z'(t) \int_{t_2}^{t} \frac{1}{\left(\gamma(s) r(s)\right)^{1/\alpha}} ds, \tag{23}$$

from which it follows that

$$\frac{z'(t)}{z(t)} \leq \frac{1}{\left(\gamma(t) r(t)\right)^{1/\alpha} R(t, t_2)}. \tag{24}$$

Using again the fact that $\gamma(t)r(t)(z'(t))^\alpha$ is eventually decreasing for $t \geq t_2$, we get

$$0 < \gamma(t) r(t) \left(z'(t)\right)^\alpha \leq \gamma(t_2) r(t_2) \left(z'(t_2)\right)^\alpha := c_1 \quad \forall t \geq t_2, \tag{25}$$

and thus we have, for all $t \geq t_3 := t_2 + 1$, that

$$\begin{aligned} z(t) &\leq z(t_2) + c_1^{1/\alpha} \int_{t_2}^{t} \frac{ds}{\left(\gamma(s) r(s)\right)^{1/\alpha}} \\ &= z(t_2) + c_1^{1/\alpha} R(t, t_2) \\ &= \left[\frac{z(t_2)}{R(t, t_2)} + c_1^{1/\alpha}\right] R(t, t_2) \\ &\leq \left[\frac{z(t_2)}{R(t_3, t_2)} + c_1^{1/\alpha}\right] R(t, t_2) = c_2 R(t, t_2) \end{aligned} \tag{26}$$

holds, where $c_2 = [z(t_2)/R(t_3, t_2) + c_1^{1/\alpha}]$.

Using (24) and (26) in (22), we obtain

$$w'(t) \leq \frac{c_3 \rho'_+(t)}{R^\beta(t, t_2)} - k\rho(t) \gamma(t) q(t) \left(\frac{z(\delta(t))}{z(t)}\right)^\beta \quad \text{for } t \geq t_2, \tag{27}$$

where $c_3 = c_2^{\alpha-\beta}$.

Now, if $\delta(t) \geq t$, in view of the fact that $z(t)$ is increasing, we have

$$\frac{z(\delta(t))}{z(t)} \geq 1. \tag{28}$$

Using (28) in (27), we get

$$w'(t) \leq \frac{c_3 \rho'_+(t)}{R^\beta(t, t_2)} - k\rho(t) \gamma(t) q(t). \tag{29}$$

Next, if $\delta(t) \leq t$, in view of the fact that $\lim_{t\to\infty}\delta(t) = \infty$, we can choose $t_3 > t_2$ such that $\delta(t) \geq t_2$ for all $t \geq t_3$. Thus, from the fact that $\gamma(t)r(t)(z'(t))^\alpha$ is eventually decreasing, we have

$$z(t) - z(\delta(t)) = \int_{\delta(t)}^{t} \frac{\left(\gamma(s) r(s) \left(z'(s)\right)^\alpha\right)^{1/\alpha}}{\left(\gamma(s) r(s)\right)^{1/\alpha}} ds \leq \left(\gamma(\delta(t)) r(\delta(t)) \left(z'(\delta(t))\right)^\alpha\right)^{1/\alpha} R(t, \delta(t)); \tag{30}$$

that is,

$$\frac{z(t)}{z(\delta(t))} \leq 1 + \frac{\left(\gamma(\delta(t)) r(\delta(t)) \left(z'(\delta(t))\right)^\alpha\right)^{1/\alpha}}{z(\delta(t))} \left[R(t, t_2) - R(\delta(t), t_2)\right], \tag{31}$$

$$\begin{aligned} z(\delta(t)) &\geq z(\delta(t)) - z(t_2) \\ &= \int_{t_2}^{\delta(t)} \frac{\left(\gamma(s) r(s) \left(z'(s)\right)^\alpha\right)^{1/\alpha}}{\left(\gamma(s) r(s)\right)^{1/\alpha}} ds \\ &\geq \left(\gamma(\delta(t)) r(\delta(t)) \left(z'(\delta(t))\right)^\alpha\right)^{1/\alpha} R(\delta(t), t_2); \end{aligned} \tag{32}$$

that is,

$$\frac{\left(\gamma(\delta(t)) r(\delta(t)) \left(z'(\delta(t))\right)^\alpha\right)^{1/\alpha}}{z(\delta(t))} \leq \frac{1}{R(\delta(t), t_2)} \quad \text{for } t \geq t_3. \tag{33}$$

From (31) and (33), it is easy to see that

$$\frac{z(\delta(t))}{z(t)} \geq \frac{R(\delta(t), t_2)}{R(t, t_2)} = \theta(t, t_2) \quad \text{for } t \geq t_3. \tag{34}$$

Using (34) in (27), we find that

$$w'(t) \leq \frac{c_3 \rho'_+(t)}{R^\beta(t, t_2)} - k\rho(t) \gamma(t) q(t) \theta^\beta(t, t_2). \tag{35}$$

Combining (29) and (35), we see that

$$w'(t) \leq -k\rho(t) \gamma(t) q(t) \phi(t, t_2) + \frac{c_3 \rho'_+(t)}{R^\beta(t, t_2)} \quad \text{for } t \geq t_3. \tag{36}$$

Integrating this inequality from t_3 to t yields

$$\int_{t_3}^{t} \left\{k\rho(s) \gamma(s) q(s) \phi(s, t_2) - \frac{c_3 \rho'_+(s)}{R^\beta(s, t_2)}\right\} ds \leq w(t_3) - w(t) < w(t_3), \tag{37}$$

which contradicts condition (7).

Now, let (19) hold. Then, we claim that $\lim_{t\to\infty}x(t) = 0$. In view of $z(t) < 0$ and $z'(t) > 0$, we have

$$\lim_{t\to\infty} z(t) = l \leq 0, \tag{38}$$

where l is a constant, and so $z(t)$ is bounded for sufficiently large t. We assert that $x(t)$ is also bounded. Otherwise, if $x(t)$ is unbounded, then there exists a sequence $\{t_k\}$ such that

$\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} x(t_k) = \infty$, where $x(t_k)$ is as in (15), and so, from the definition of z and $\tau(t) \le t$, we see that

$$z(t_k) = x(t_k) - p(t_k)\, x(\tau(t_k)) \ge (1 - p(t_k))\, x(t_k) > 0, \tag{39}$$

which contradicts the fact that $z(t) < 0$ for $t \ge t_2$, and so $x(t)$ is bounded. Therefore, we have

$$\limsup_{t\to\infty} x(t) = a, \quad 0 \le a < \infty. \tag{40}$$

If $a > 0$, then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} x(t_n) = a$. Let $\varepsilon = a(1 - p_0)/2p_0$; then, for all large n, we have $x(\tau(t_n)) < a + \varepsilon$. From this and the definition of z, we obtain

$$0 \ge \lim_{n\to\infty} z(t_n) \ge \lim_{n\to\infty} x(t_n) - p_0(a + \varepsilon) = \frac{a(1 - p_0)}{2} > 0, \tag{41}$$

which contradicts the fact that $z(t) < 0$, and hence $\limsup_{t\to\infty} x(t) = 0$. Now, in view of the fact that $x(t) > 0$, we conclude that $\lim_{t\to\infty} x(t) = 0$, which completes the proof of Theorem 2. □

Theorem 3. *Let (2) be satisfied. If there exists a positive function $\rho \in C^1([t_0, \infty), \mathbb{R})$ such that, for all sufficiently large T_* and for $T > T_*$,*

$$\limsup_{t\to\infty} \int_T^t \left\{ k\rho(s)\gamma(s) q(s)\phi(s, T_*) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{c^*\gamma(s) r(s) (\rho'(s))^{\alpha+1}}{(\beta\rho(s))^\alpha R^{\beta-\alpha}(s, T_*)} \right\} ds = \infty, \tag{42}$$

with $c^* > 0$, then any solution x of (1) either is oscillatory or converges to zero as $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \ge t_1$. Proceeding as in the proof of Theorem 2, we see that (18) or (19) holds. If (18) holds, as in the proof of Theorem 2, we obtain (21), (24), (26), (28), and (34). Using (20), (24), (26), (28), and (34) in (21), we obtain

$$w'(t) \le \frac{\rho'_+(t)}{\rho(t)} w(t) - k\rho(t)\gamma(t) q(t)\phi(t, t_2) - \beta\rho(t)\gamma(t) r(t) \frac{(z'(t))^{\alpha+1}}{z^{\beta+1}(t)}, \tag{43}$$

for $t \ge t_3$. From (20), we have

$$\left(\frac{w(t)}{\rho(t)\gamma(t) r(t)}\right)^{(\alpha+1)/\alpha} = \left(\frac{(z'(t))^\alpha}{z^\beta(t)}\right)^{(\alpha+1)/\alpha} = \frac{(z'(t))^{\alpha+1}}{z^{\beta(1+1/\alpha)}(t)} = \frac{(z'(t))^{\alpha+1}}{z^{\beta+1}(t)\, z^{\beta/\alpha-1}(t)}; \tag{44}$$

that is,

$$z^{\beta/\alpha-1}(t)\left(\frac{w(t)}{\rho(t)\gamma(t) r(t)}\right)^{(\alpha+1)/\alpha} = \frac{(z'(t))^{\alpha+1}}{z^{\beta+1}(t)}. \tag{45}$$

Substituting (45) into (43) gives

$$w'(t) \le \frac{\rho'_+(t)}{\rho(t)} w(t) - k\rho(t)\gamma(t) q(t)\phi(t, t_2) - \frac{\beta z^{\beta/\alpha-1}(t)}{(\rho(t)\gamma(t) r(t))^{1/\alpha}} w^{(\alpha+1)/\alpha}(t). \tag{46}$$

From (26) and the fact that $\beta/\alpha - 1 \le 0$, (46) yields

$$w'(t) \le -k\rho(t)\gamma(t) q(t)\phi(t, t_2) + \frac{\rho'_+(t)}{\rho(t)} w(t) - \frac{c_4\beta R^{\beta/\alpha-1}(t, t_2)}{(\rho(t)\gamma(t) r(t))^{1/\alpha}} w^{(\alpha+1)/\alpha}(t), \tag{47}$$

where $c_4 = c_2^{\beta/\alpha-1}$. Letting

$$X = \left(\frac{c_4\beta R^{\beta/\alpha-1}(t, t_2)}{(\rho(t)\gamma(t) r(t))^{1/\alpha}}\right)^{1/\lambda} w(t), \quad \lambda = \frac{\alpha+1}{\alpha}, \qquad Y = \left\{\frac{1}{\lambda}\left(\frac{(\rho(t)\gamma(t) r(t))^{1/\alpha}}{c_4\beta R^{\beta/\alpha-1}(t, t_2)}\right)^{1/\lambda} \frac{\rho'_+(t)}{\rho(t)}\right\}^\alpha, \tag{48}$$

in Lemma 1, (47) implies

$$w'(t) \le -k\rho(t)\gamma(t) q(t)\phi(t, t_2) + \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{\gamma(t) r(t) (\rho'_+(t))^{\alpha+1}}{(c_4\beta\rho(t))^\alpha R^{\beta-\alpha}(t, t_2)} \quad \text{for } t \ge t_3. \tag{49}$$

Integrating the last inequality from t_3 to t leads to

$$\int_{t_3}^t \left\{ k\rho(s)\gamma(s) q(s)\phi(s, t_2) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{c_3\gamma(s) r(s) (\rho'_+(s))^{\alpha+1}}{(\beta\rho(s))^\alpha R^{\beta-\alpha}(s, t_2)} \right\} ds \le w(t_3) < \infty, \tag{50}$$

which contradicts condition (42).

Finally, if (19) holds, proceeding as in the proof of Theorem 2, we see that $\lim_{t\to\infty} x(t) = 0$, which completes the proof of Theorem 3. □

Theorem 4. *Assume that (2) holds, and $\alpha \geq \beta \geq 1$. Suppose also that there exists a positive function $\rho \in C^1([t_0,\infty),\mathbb{R})$ such that, for all sufficiently large T_* and for $T > T_*$,*

$$\limsup_{t\to\infty} \int_T^t \left\{ k\rho(s)\gamma(s)q(s)\phi(s,T_*) - \frac{c^*(\gamma(s)r(s))^{1/\alpha}(\rho'_+(s))^2}{4\beta\rho(s)R^{\beta-1}(s,T_*)} \right\} ds = \infty, \tag{51}$$

with $c^* > 0$; then, any solution x of (1) either oscillates or satisfies $x(t) \to 0$ as $t \to \infty$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorems 2 and 3, we see that (18) or (19) holds. If (18) holds, as in the proof of Theorem 3, we obtain (47) which can be rewritten as

$$w'(t) \leq -k\rho(t)\gamma(t)q(t)\phi(t,t_2) + \frac{\rho'_+(t)}{\rho(t)}w(t) - \frac{c_4\beta R^{\beta/\alpha-1}(t,t_2)}{(\rho(t)\gamma(t)r(t))^{1/\alpha}}w^{1/\alpha-1}(t)w^2(t), \tag{52}$$

for $t \geq t_3$. From (20), we get

$$\begin{aligned} w^{1/\alpha-1}(t) &= (\rho(t)\gamma(t)r(t))^{1/\alpha-1}\left(\frac{(z'(t))^\alpha}{z^\beta(t)}\right)^{1/\alpha-1} \\ &= (\rho(t)\gamma(t)r(t))^{1/\alpha-1}\frac{(z'(t))^{1-\alpha}}{z^{\beta(1/\alpha-1)}(t)} \\ &= (\rho(t)\gamma(t)r(t))^{1/\alpha-1}\frac{z^{\beta-\beta/\alpha}(t)}{(z'(t))^{\alpha-1}} \\ &= (\rho(t)\gamma(t)r(t))^{1/\alpha-1}\left(\frac{z(t)}{z'(t)}\right)^{\alpha-1} z^{(\alpha-1)(\beta/\alpha-1)}(t). \end{aligned} \tag{53}$$

From (24), we have

$$\left(\frac{z(t)}{z'(t)}\right)^{\alpha-1} \geq (\gamma(t)r(t))^{(\alpha-1)/\alpha}R^{\alpha-1}(t,t_2). \tag{54}$$

From (26) and the fact that $(\alpha-1)(\beta/\alpha-1) \leq 0$, we obtain

$$z^{(\alpha-1)(\beta/\alpha-1)}(t) \geq c_2^{(\alpha-1)(\beta/\alpha-1)}R^{(\alpha-1)(\beta/\alpha-1)}(t,t_2). \tag{55}$$

Substituting (54) and (55) into (53) gives

$$w^{1/\alpha-1}(t) \geq \rho^{1/\alpha-1}(t)\,c_2^{(\alpha-1)(\beta/\alpha-1)}R^{\beta-\beta/\alpha}(t,t_2). \tag{56}$$

Using (56) in (52), we obtain

$$w'(t) \leq -k\rho(t)\gamma(t)q(t)\phi(t,t_2) + \frac{\rho'_+(t)}{\rho(t)}w(t) - \frac{\beta R^{\beta-1}(t,t_2)}{c_3\rho(t)(\gamma(t)r(t))^{1/\alpha}}w^2(t). \tag{57}$$

Completing square with respect to w, it follows from (57) that

$$w'(t) \leq -k\rho(t)\gamma(t)q(t)\phi(t,t_2) + \frac{c_3(\gamma(t)r(t))^{1/\alpha}(\rho'_+(t))^2}{4\beta\rho(t)R^{\beta-1}(t,t_2)}. \tag{58}$$

Integrating the last inequality from t_3 to t leads to

$$\int_{t_3}^t \left\{ k\rho(s)\gamma(s)q(s)\phi(s,t_2) - \frac{c_3(\gamma(s)r(s))^{1/\alpha}(\rho'_+(s))^2}{4\beta\rho(s)R^{\beta-1}(s,t_2)} \right\} ds \leq w(t_3) < \infty, \tag{59}$$

which contradicts condition (51).

Finally, if (19) holds, proceeding as in the proof of Theorem 2, we see that $\lim_{t\to\infty} x(t) = 0$, which completes the proof of Theorem 4. □

Remark 5. If $\alpha = \beta$, then we have $c^* = 1$ in Theorems 2–4.

Example 6. Consider the neutral differential equation

$$\left(x(t) - \frac{1}{2}x\left(t-\frac{\pi}{2}\right)\right)'' + \cos 4t\left(x(t) - \frac{1}{2}x\left(t-\frac{\pi}{2}\right)\right)' + (8+2\sin 4t)\,x\left(t+\frac{\pi}{2}\right) = 0, \tag{60}$$

for $t \geq 1$. Here, we have $r(t) = 1$, $h(t) = \cos 4t$, $q(t) = 8 + 2\sin 4t$, $\tau(t) = t-\pi/2$, $\delta(t) = t+\pi/2$, $f(u) = u$, and $\alpha = \beta = 1$. Then,

$$\begin{aligned} \gamma(t) &= \exp\left(\int_1^t \cos 4s\,ds\right) = e^{(1/4)\sin 4t-(1/4)\sin 4}, \\ \int_1^t \frac{ds}{(\gamma(s)r(s))^{1/\alpha}} &= e^{(1/4)\sin 4}\int_1^t \frac{ds}{e^{(1/4)\sin 4s}} \geq \frac{e^{(1/4)\sin 4}}{e^{1/4}}\int_1^t ds \longrightarrow \infty \quad \text{as } t\longrightarrow\infty, \end{aligned} \tag{61}$$

so (2) holds. Since

$$\phi(t,T_*)=\phi(t,1)=1,$$
$$R^\beta(t,T_*)=R^\beta(t,1)=e^{(1/4)\sin 4}\int_1^t \frac{ds}{e^{(1/4)\sin 4s}} \geq \frac{e^{(1/4)\sin 4}}{e^{1/4}}\int_1^t ds, \tag{62}$$

condition (7) with $\rho(t)=t$ and $T>T_*=1$ becomes

$$\begin{aligned}
&\limsup_{t\to\infty}\int_T^t\left\{k\rho(s)\gamma(s)q(s)\phi(s,T_*) - \frac{c^*\rho'_+(s)}{R^\beta(s,T_*)}\right\}ds \\
&\geq \limsup_{t\to\infty}\int_T^t\left\{se^{(1/4)\sin 4s-(1/4)\sin 4}(8+2\sin 4s) - \frac{e^{-(1/4)\sin 4}e^{1/4}}{(s-1)}\right\}ds \geq e^{-(1/4)\sin 4} \\
&\cdot \limsup_{t\to\infty}\int_T^t\left\{6se^{-1/4}-\frac{e^{1/4}}{(s-1)}\right\}ds = e^{-(1/4)\sin 4} \\
&\cdot \limsup_{t\to\infty}\left(3e^{-1/4}s^2-e^{1/4}\ln(s-1)\right)\Big|_T^t=\infty;
\end{aligned} \tag{63}$$

that is, condition (7) holds. So every solution of (60) either is oscillatory or satisfies $x(t)\to 0$ as $t\to\infty$ by Theorem 2. In fact, it is easy to see that one oscillatory solution of (60) is $x(t)=\cos 4t$.

Example 7. The neutral differential equation

$$\left(t^3\left(z'(t)\right)^5\right)'+t^2\left(z'(t)\right)^5+\left(\sqrt[5]{t}-1\right)^3x^3\left(\frac{t}{2}\right)=0, \quad t\geq 1, \tag{64}$$

is a special case of (1) with $z(t)=x(t)-(1/3)x(t-\pi/4)$, $r(t)=t^3$, $h(t)=t^2$, $q(t)=(\sqrt[5]{t}-1)^3$, $\tau(t)=t-\pi/4$, $\delta(t)=t/2$, $f(u)=u^3$, $\alpha=5$, and $\beta=3$. Clearly,

$$\gamma(t)=\exp\left(\int_1^t\frac{1}{s}ds\right)=e^{\ln t-\ln 1}=t,$$
$$\int_1^\infty\frac{ds}{(\gamma(s)r(s))^{1/\alpha}}=\int_1^\infty\frac{ds}{s^{4/5}}=\infty, \tag{65}$$

so (2) holds. Since

$$\phi(t,T_*)=\phi(t,1)=\left(\frac{\int_1^{t/2}\left(ds/s^{4/5}\right)}{\int_1^t\left(ds/s^{4/5}\right)}\right)^3 = \frac{1}{2^{3/5}}\frac{\left(\sqrt[5]{t}-\sqrt[5]{2}\right)^3}{\left(\sqrt[5]{t}-1\right)^3}, \tag{66}$$

condition (42) with $\rho(t)=1$ and $T>T_*=1$ becomes

$$\limsup_{t\to\infty}\int_T^t s\frac{1}{2^{3/5}}\left(\sqrt[5]{s}-\sqrt[5]{2}\right)^3ds = \frac{1}{2^{3/5}}\limsup_{t\to\infty}\int_T^t s\left(\sqrt[5]{s}-\sqrt[5]{2}\right)^3ds=\infty; \tag{67}$$

that is, (42) holds. Therefore, by Theorem 3, a solution of (64) either is oscillatory or converges to zero.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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