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Research Article

Certain Subclasses of Bistarlike and Biconvex Functions Based on Quasi-Subordination

Nanjundan Magesh,[1] Vitalrao Kupparao Balaji,[2] and Jagadesan Yamini[3]

[1]*Post-Graduate and Research Department of Mathematics, Government Arts College for Men, Krishnagiri, Tamil Nadu 635001, India*
[2]*Department of Mathematics, L. N. Government College, Ponneri, Tiruvalur, Tamil Nadu 601204, India*
[3]*Department of Mathematics, Government First Grade College, Vijayanagar, Bangalore, Karnataka 560104, India*

Correspondence should be addressed to Nanjundan Magesh; nmagi_2000@yahoo.co.in

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We introduce the unified biunivalent function class $\mathscr{M}_{q,\sigma}^{\delta,\lambda}(\gamma,\varphi)$ defined based on quasi-subordination and obtained the coefficient estimates for Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Several related classes of functions are also considered and connections to earlier known and new results are established.

1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$. Further, by $\mathscr{S}$ we denote the family of all functions in $\mathscr{A}$ which are univalent in $\mathbb{U}$. Let $h(z)$ be an analytic function in $\mathbb{U}$ and $|h(z)| \le 1$, such that

$$h(z) = h_0 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots, \quad (2)$$

where all coefficients are real. Also, let φ be an analytic and univalent function with positive real part in $\mathbb{U}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Taylor's series expansion of such function is of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (3)$$

where all coefficients are real and $B_1 > 0$. Throughout this paper we assume that the functions h and φ satisfy the above conditions one or otherwise stated.

For two functions f and g are analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \quad (4)$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$, with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathbb{U}), \quad (5)$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \quad (6)$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \quad (7)$$

For two analytic functions f and g, the function f is quasi-subordinate to g in the open unit disc $\mathbb{U}$ if there exist analytic functions h and w, with $|h(z)| \le 1$, $w(0) = 0$, and $|w(z)| < 1$, such that $f(z)/h(z)$ is analytic in $\mathbb{U}$ and written as

$$\frac{f(z)}{h(z)} \prec g(z) \quad (z \in \mathbb{U}). \quad (8)$$

We also denote the above expression by

$$f(z) \prec_q g(z) \quad (z \in \mathbb{U}) \tag{9}$$

and this is equivalent to

$$f(z) = h(z)\, g(w(z)) \quad (z \in \mathbb{U}). \tag{10}$$

Observe that if $h(z) \equiv 1$, then $f(z) = g(w(z))$, so that $f(z) \prec g(z)$ in $\mathbb{U}$. Also notice that if $w(z) = z$, then $f(z) = h(z)g(z)$ and it is said that f is majorized by g and written by $f(z) \ll g(z)$ in $\mathbb{U}$. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization (see [1]).

In [2] Ma and Minda introduced the unified classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ given below:

$$\mathcal{S}^*(\varphi) := \left\{ f : f \in \mathcal{A},\ \frac{zf'(z)}{f(z)} \prec \varphi(z);\ z \in \mathbb{U} \right\},$$
$$\mathcal{K}(\varphi) := \left\{ f : f \in \mathcal{A},\ 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z);\ z \in \mathbb{U} \right\}. \tag{11}$$

For the choice

$$\varphi(z) = \frac{1 + (1 - 2\alpha) z}{1 - z} \quad (0 \le \alpha < 1) \tag{12}$$

or

$$\varphi(z) = \left(\frac{1 + z}{1 - z} \right)^{\beta} \quad (0 < \beta \le 1) \tag{13}$$

the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ consist of functions known as the starlike (resp., convex) functions of order α or strongly starlike (resp., convex) functions of order β, respectively.

Recently, El-Ashwah and Kanas [3] introduced and studied the following two subclasses:

$$\mathcal{S}_q^*(\gamma, \varphi) := \left\{ f : f \in \mathcal{A},\ \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec_q \varphi(z) - 1;\ z \in \mathbb{U},\ 0 \ne \gamma \in \mathbb{C} \right\},$$
$$\mathcal{K}_q(\gamma, \varphi) := \left\{ f : f \in \mathcal{A},\ \frac{1}{\gamma} \frac{zf''(z)}{f'(z)} \prec_q \varphi(z) - 1;\ z \in \mathbb{U},\ 0 \ne \gamma \in \mathbb{C} \right\}. \tag{14}$$

We note that when $h(z) \equiv 1$, the classes $\mathcal{S}_q^*(\gamma, \varphi)$ and $\mathcal{K}_q(\gamma, \varphi)$ reduce, respectively, to the familiar classes $\mathcal{S}^*(\gamma, \varphi)$ and $\mathcal{K}(\gamma, \varphi)$ of Ma-Minda starlike and convex functions of complex order γ ($\gamma \in \mathbb{C} \setminus \{0\}$) in $\mathbb{U}$ (see [4]). For $\gamma = 1$, the classes $\mathcal{S}_q^*(\gamma, \varphi)$ and $\mathcal{K}_q(\gamma, \varphi)$ reduce to the classes $\mathcal{S}_q^*(\varphi)$ and $\mathcal{K}_q(\varphi)$, respectively, that are analogous to Ma-Minda starlike and convex functions, introduced by Mohd and Darus [5].

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$
$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \ge \frac{1}{4} \right), \tag{15}$$

where

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n \quad (|w| < r_0(f)), \tag{16}$$

where

$$b_n = \frac{(-1)^{n+1}}{n!} \left| A_{ij} \right| \tag{17}$$

and $|A_{ij}|$ is the $(n-1)$th order determinant whose entries are defined in terms of the coefficients of $f(z)$ by the following:

$$\left| A_{ij} \right| = \begin{cases} [(i - j + 1)n + j - 1]\, a_{i-j+2}, & i + 1 \ge j; \\ 0, & i + 1 < j. \end{cases} \tag{18}$$

For initial values of n, we have

$$b_2 = -a_2,$$
$$b_3 = 2a_2^2 - a_3, \tag{19}$$
$$b_4 = 5a_2a_3 - 5a_2^3 - a_4,$$

and so on. A function $f \in \mathcal{A}$ is said to be biunivalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let σ denote the class of biunivalent functions in $\mathbb{U}$ given by (1). For a brief history and interesting examples of functions which are in (or which are not in) the class σ, together with various other properties of the biunivalent function class σ, one can refer to the work of Srivastava et al. [6] and references therein. Recently, various subclasses of the biunivalent function class σ were introduced and nonsharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor–Maclaurin series expansion (1) were found in several recent investigations (see, e.g., [7–17]). But the problem of finding the coefficient bounds on $|a_n|$ ($n = 3, 4, \ldots$) for functions $f \in \sigma$ is still an open problem.

Motivated by the above mentioned works, we define the following subclass of function class σ.

A function $f \in \sigma$ given by (1) is said to be in the class $\mathcal{M}_{q,\sigma}^{\delta,\lambda}(\gamma, \varphi)$, $0 \ne \gamma \in \mathbb{C}$, $\delta \ge 0$, if the following quasi-subordination conditions are satisfied:

$$\frac{1}{\gamma} \left((1 - \delta) \frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)} + \delta \left(1 + \frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)} \right) - 1 \right) \prec_q \varphi(z) - 1 \quad (z \in \mathbb{U}),$$
$$\frac{1}{\gamma} \left((1 - \delta) \frac{w\mathcal{G}_\lambda'(w)}{\mathcal{G}_\lambda(w)} + \delta \left(1 + \frac{w\mathcal{G}_\lambda''(w)}{\mathcal{G}_\lambda'(w)} \right) - 1 \right) \prec_q \varphi(w) - 1 \quad (w \in \mathbb{U}), \tag{20}$$

where

$$\mathcal{F}_\lambda(z) = (1-\lambda) f(z) + \lambda z f'(z),$$
$$\mathcal{G}_\lambda(w) = (1-\lambda) g(w) + \lambda w g'(w) \quad (0 \le \lambda \le 1), \tag{21}$$

and the function g is the extension of f^{-1} to $\mathbb{U}$.

It is interesting to note that the special values of δ, γ, λ, and φ and the class $\mathcal{M}_{q,\sigma}^{\delta,\lambda}(\gamma,\varphi)$ unify the following known and new classes.

Remark 1. Setting $\lambda = 0$ in the above class, we have

$$\mathcal{M}_{q,\sigma}^{\delta,0}(\gamma,\varphi) := \mathcal{M}_{q,\sigma}^{\delta}(\gamma,\varphi). \tag{22}$$

In particular, for $\gamma = 1$, we have

$$\mathcal{M}_{q,\sigma}^{\delta}(1,\varphi) := \mathcal{M}_{q,\sigma}^{\delta}(\varphi) \tag{23}$$

which was introduced and studied by Goyal and Kumar [18, Definition 2.3, p. 541]. Also, we note that for $h(z) \equiv 1$ the class $\mathcal{M}_{q,\sigma}^{\delta}(\varphi) := \mathcal{M}_{\sigma}^{\delta}(\varphi)$ was introduced and studied by Ali et al. [7] (see also [19]). If we take $\varphi(z)$ by (12) in the class $\mathcal{M}_{\sigma}^{\delta}(\varphi)$, we are led to the class which we denote, for convenience, by $\mathcal{M}_{\sigma}^{\delta}(\alpha)$, introduced and studied by Li and Wang [12, Definition 3.1., p. 1500], and upon replacing φ by (13) in the class $\mathcal{M}_{\sigma}^{\delta}(\varphi)$, we have $\mathcal{M}_{\sigma}^{\delta}(\beta)$; this class was introduced and studied by Li and Wang [12, Definition 2.1., p. 1497].

Remark 2. Taking $\lambda = 0$ and $\delta = 0$ in the class $\mathcal{M}_{q,\sigma}^{\delta,\lambda}(\gamma,\varphi)$, we have

$$\mathcal{M}_{q,\sigma}^{0,0}(\gamma,\varphi) := \mathcal{S}_{q,\sigma}^{*}(\gamma,\varphi). \tag{24}$$

In particular, for $\gamma = 1$, we have

$$\mathcal{S}_{q,\sigma}^{*}(1,\varphi) := \mathcal{S}_{q,\sigma}^{*}(\varphi). \tag{25}$$

The class $\mathcal{S}_{q,\sigma}^{*}(\varphi)$ is particular case of the class $\mathcal{M}_{q,\sigma}^{\delta}(\varphi)$, when $\delta = 0$ and it was introduced and studied by Goyal and Kumar [18, Definition 2.3, p. 541]. We note that, for $h(z) \equiv 1$, the class $\mathcal{S}_{q,\sigma}^{*}(\gamma,\varphi) := \mathcal{S}_{\sigma}^{*}(\gamma,\varphi)$ was introduced and studied by Deniz [10]. Further, for $h(z) \equiv 1$, the class $\mathcal{S}_{q,\sigma}^{*}(\varphi) := \mathcal{S}_{\sigma}^{*}(\varphi)$ was introduced by Ali et al. [7] and Srivastava et al. [16]. For $\varphi(z)$ given by (12), the class $\mathcal{S}_{\sigma}^{*}(\alpha)$ was introduced by Brannan and Taha [20] and studied by Bulut [8], Çaglar et al. [9], Li and Wang [12], and others.

Remark 3. Setting $\lambda = 0$ and $\delta = 1$ in the class $\mathcal{M}_{q,\sigma}^{\delta,\lambda}(\gamma,\varphi)$, we have

$$\mathcal{M}_{q,\sigma}^{1,0}(\gamma,\varphi) := \mathcal{K}_{q,\sigma}(\gamma,\varphi). \tag{26}$$

In particular, for $\gamma = 1$, we get

$$\mathcal{K}_{q,\sigma}(1,\varphi) := \mathcal{K}_{q,\sigma}(\varphi). \tag{27}$$

The class $\mathcal{K}_{q,\sigma}(\varphi)$ is particular case of the class $\mathcal{M}_{q,\sigma}^{\delta}(\varphi)$, when $\delta = 1$ and it was introduced and studied by Goyal and Kumar [18, Definition 2.3, p. 541]. We note that, for $h(z) \equiv 1$, the class $\mathcal{K}_{q,\sigma}(\gamma,\varphi) := \mathcal{K}_{\sigma}(\gamma,\varphi)$ was introduced and studied by Deniz [10]. Further, for $h(z) \equiv 1$, the class $\mathcal{K}_{q,\sigma}(\varphi) := \mathcal{K}_{\sigma}(\varphi)$ was considered by Ali et al. [7]. For $\varphi(z)$ given by (12), we get the class $\mathcal{K}_{\sigma}(\alpha)$, introduced by Brannan and Taha [20] and studied by Li and Wang [12] and others.

Remark 4. Taking $\delta = 0$, we have the class $\mathcal{M}_{q,\sigma}^{0,\lambda}(\gamma,\varphi) \equiv \mathcal{P}_{q,\sigma}(\gamma,\lambda,\varphi)$ as defined below.

A function $f \in \sigma$ is said to be in the class $\mathcal{P}_{q,\sigma}(\gamma,\lambda,\varphi)$, $0 \ne \gamma \in \mathbb{C}$, $0 \le \lambda \le 1$, if the following quasi-subordinations hold:

$$\frac{1}{\gamma}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda) f(z) + \lambda z f'(z)} - 1\right) \prec_q \varphi(z) - 1,$$
$$\frac{1}{\gamma}\left(\frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda) g(w) + \lambda w g'(w)} - 1\right) \prec_q \varphi(w) - 1, \tag{28}$$

where $g(w) = f^{-1}(w)$. A function in the class $\mathcal{P}_{q,\sigma}(\gamma,\lambda,\varphi)$ is called both bi-λ-convex functions and bi-λ-starlike functions of complex order γ of Ma-Minda type. For $h(z) \equiv 1$, the class $\mathcal{P}_{q,\sigma}(\gamma,\lambda,\varphi) := \mathcal{P}_{\sigma}(\gamma,\lambda,\varphi)$ was introduced and studied by Deniz [10].

Remark 5. Putting $\delta = 1$, we have the class $\mathcal{M}_{q,\sigma}^{1,\lambda}(\gamma,\varphi) \equiv \mathcal{K}_{q,\sigma}(\gamma,\lambda,\varphi)$ as defined below.

A function $f \in \sigma$ is said to be in the class $\mathcal{K}_{q,\sigma}(\gamma,\lambda,\varphi)$, $0 \ne \gamma \in \mathbb{C}$, $0 \le \lambda \le 1$, if the following quasi-subordinations hold:

$$\frac{1}{\gamma}\left(\frac{zf'(z) + (1+2\lambda) z^2 f''(z) + \lambda z^3 f'''(z)}{zf'(z) + \lambda z^2 f''(z)} - 1\right) \prec_q \varphi(z) - 1,$$
$$\frac{1}{\gamma}\left(\frac{wg'(w) + (1+2\lambda) w^2 g''(w) + \lambda w^3 g'''(w)}{wg'(w) + \lambda w^2 g''(w)} - 1\right) \prec_q \varphi(w) - 1, \tag{29}$$

where $g(w) = f^{-1}(w)$.

Remark 6. For $h(z) \equiv 1$, the class $\mathcal{M}_{q,\sigma}^{\delta,\lambda}(\gamma,\varphi) := \mathcal{M}_{\sigma}(\delta,\lambda,\gamma,\varphi)$ was introduced in [21].

In this paper we introduce the unified biunivalent function class $\mathcal{M}_{q,\sigma}^{\delta,\lambda}(\gamma,\varphi)$ as defined above and obtain the coefficient estimates for Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to $\mathcal{M}_{q,\sigma}^{\delta,\lambda}(\gamma,\varphi)$. Some interesting applications of the results presented here are also discussed.

In order to derive our results, we need the following lemma.

Lemma 7 (see [22]). *If $p \in \mathscr{P}$, then $|p_i| \le 2$ for each i, where $\mathscr{P}$ is the family of all functions p, analytic in $\mathbb{U}$, for which*

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}), \tag{30}$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}). \tag{31}$$

2. Coefficient Estimates for the Class $\mathscr{M}_{q,\sigma}^{\delta,\lambda}(\gamma,\varphi)$

Theorem 8. *Let $f(z)$ given by (1) be in the class $\mathscr{M}_{q,\sigma}^{\delta,\lambda}(\gamma,\varphi)$, $0 \le \lambda < 1$, $0 \ne \gamma \in \mathbb{C}$, and $\delta \ge 0$. Then*

$$|a_2| \le \frac{|\gamma||h_0| B_1 \sqrt{B_1}}{\sqrt{\left|\gamma\left[2(1+2\delta)(1+2\lambda) - (1+3\delta)(1+\lambda)^2\right] h_0 B_1^2 - (1+\delta)^2(1+\lambda)^2(B_2 - B_1)\right|}}, \tag{32}$$

$$\begin{aligned} |a_3| \le{}& \frac{|\gamma||h_1| B_1}{2(1+2\delta)(1+2\lambda)} + \frac{|\gamma||h_0||B_2 - B_1|}{|(1+\delta)(1+2\lambda) - \lambda^2(1+3\delta)|} \\ &+ \frac{|\gamma||h_0| B_1\left[(1+3\delta)(1+\lambda)^2 + |(3+5\delta)(1+2\lambda) - \lambda^2(1+3\delta)|\right]}{4(1+2\delta)(1+2\lambda)|(1+\delta)(1+2\lambda) - \lambda^2(1+3\delta)|}. \end{aligned} \tag{33}$$

Proof. Since $f \in \mathscr{M}_{q,\sigma}^{\delta,\lambda}(\gamma,\varphi)$, there exist two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$\begin{aligned} &\frac{1}{\gamma}\left((1-\delta)\frac{z\mathscr{F}_\lambda'(z)}{\mathscr{F}_\lambda(z)} + \delta\left(1 + \frac{z\mathscr{F}_\lambda''(z)}{\mathscr{F}_\lambda'(z)}\right) - 1\right) \\ &\quad = h(z)(\varphi(r(z)) - 1), \\ &\frac{1}{\gamma}\left((1-\delta)\frac{w\mathscr{G}_\lambda'(w)}{\mathscr{G}_\lambda(w)} + \delta\left(1 + \frac{w\mathscr{G}_\lambda''(w)}{\mathscr{G}_\lambda'(w)}\right) - 1\right) \\ &\quad = h(w)(\varphi(s(w)) - 1). \end{aligned} \tag{34}$$

Define the functions u and v by

$$\begin{aligned} u(z) &= \frac{1 + r(z)}{1 - r(z)} = 1 + u_1 z + u_2 z^2 + u_3 z^3 + \cdots, \\ v(z) &= \frac{1 + s(z)}{1 - s(z)} = 1 + v_1 z + v_2 z^2 + v_3 z^3 + \cdots \end{aligned} \tag{35}$$

or equivalently

$$\begin{aligned} r(z) &= \frac{u(z) - 1}{u(z) + 1} = \frac{1}{2}\left(u_1 z + \left(u_2 - \frac{u_1^2}{2}\right) z^2 \right. \\ &\quad \left. + \left(u_3 + \frac{u_1}{2}\left(\frac{u_1^2}{2} - u_2\right) - \frac{u_1 u_2}{2}\right) z^3 + \cdots\right), \\ s(z) &= \frac{v(z) - 1}{v(z) + 1} = \frac{1}{2}\left(v_1 z + \left(v_2 - \frac{v_1^2}{2}\right) z^2 \right. \\ &\quad \left. + \left(v_3 + \frac{v_1}{2}\left(\frac{v_1^2}{2} - v_2\right) - \frac{v_1 v_2}{2}\right) z^3 + \cdots\right). \end{aligned} \tag{36}$$

Using (36) in (34), we have

$$\begin{aligned} &\frac{1}{\gamma}\left((1-\delta)\frac{z\mathscr{F}_\lambda'(z)}{\mathscr{F}_\lambda(z)} + \delta\left(1 + \frac{z\mathscr{F}_\lambda''(z)}{\mathscr{F}_\lambda'(z)}\right) - 1\right) \\ &\quad = h(z)\left[\varphi\left(\frac{u(z) - 1}{u(z) + 1}\right) - 1\right], \\ &\frac{1}{\gamma}\left((1-\delta)\frac{w\mathscr{G}_\lambda'(w)}{\mathscr{G}_\lambda(w)} + \delta\left(1 + \frac{w\mathscr{G}_\lambda''(w)}{\mathscr{G}_\lambda'(w)}\right) - 1\right) \\ &\quad = h(w)\left[\varphi\left(\frac{q(w) - 1}{q(w) + 1}\right) - 1\right]. \end{aligned} \tag{37}$$

Again using (36) along with (3), it is evident that

$$\begin{aligned} &h(z)\left[\varphi\left(\frac{u(z) - 1}{u(z) + 1}\right) - 1\right] = \frac{1}{2} h_0 B_1 u_1 z \\ &\quad + \left(\frac{1}{2} h_1 B_1 u_1 + \frac{1}{2} h_0 B_1\left(u_2 - \frac{1}{2} u_1^2\right) + \frac{1}{4} h_0 B_2 u_1^2\right) z^2 \\ &\quad + \cdots, \\ &h(w)\left[\varphi\left(\frac{q(w) - 1}{q(w) + 1}\right) - 1\right] = \frac{1}{2} h_0 B_1 v_1 w \\ &\quad + \left(\frac{1}{2} h_1 B_1 v_1 + \frac{1}{2} h_0 B_1\left(v_2 - \frac{1}{2} v_1^2\right) + \frac{1}{4} h_0 B_2 v_1^2\right) w^2 \\ &\quad + \cdots. \end{aligned} \tag{38}$$

It follows from (37) and (38) that

$$\frac{1}{\gamma}(1+\delta)(1+\lambda)a_2 = \frac{1}{2}h_0B_1u_1, \tag{39}$$

$$\frac{1}{\gamma}\left[2(1+2\delta)(1+2\lambda)a_3 - (1+3\delta)(1+\lambda)^2a_2^2\right] = \frac{1}{2}\cdot h_1B_1u_1 + \frac{1}{2}h_0B_1\left(u_2 - \frac{1}{2}u_1^2\right) + \frac{1}{4}h_0B_2u_1^2, \tag{40}$$

$$-\frac{1}{\gamma}(1+\delta)(1+\lambda)a_2 = \frac{1}{2}h_0B_1v_1, \tag{41}$$

$$\frac{1}{\gamma}\left[4\left((1+2\delta)(1+2\lambda)-(1+3\delta)(1+\lambda)^2\right)a_2^2 - 2(1+2\delta)(1+2\lambda)a_3\right] = \frac{1}{2}h_1B_1v_1 + \frac{1}{2}h_0B_1\left(v_2 - \frac{1}{2}v_1^2\right) + \frac{1}{4}h_0B_2v_1^2. \tag{42}$$

From (39) and (41), we find that

$$a_2 = \frac{\gamma h_0B_1u_1}{2(1+\delta)(1+\lambda)} = \frac{-\gamma h_0B_1v_1}{2(1+\delta)(1+\lambda)}; \tag{43}$$

it follows that

$$u_1 = -v_1, \tag{44}$$

$$8(1+\delta)^2(1+\lambda)^2a_2^2 = h_0^2B_1^2\gamma^2\left(u_1^2+v_1^2\right). \tag{45}$$

Adding (40) and (42), we have

$$a_2^2\frac{1}{\gamma}\left[4(1+2\delta)(1+2\lambda) - 2(1+3\delta)(1+\lambda)^2\right] = \frac{h_0B_1}{2}(u_2+v_2) + \frac{h_0(B_2-B_1)}{4}\left(u_1^2+v_1^2\right). \tag{46}$$

Substituting (43) and (44) into (46), we get

$$a_2^2 = \frac{\gamma^2h_0^2B_1^3(u_2+v_2)}{4\gamma\left[2(1+2\delta)(1+2\lambda)-(1+3\delta)(1+\lambda)^2\right]h_0B_1^2 - 4(1+\delta)^2(1+\lambda)^2(B_2-B_1)}. \tag{47}$$

Applying Lemma 7 in (47), we get desired inequality (32). Subtracting (40) from (42) and a computation using (44) finally lead to

$$a_3 = a_2^2 + \frac{\gamma h_1B_1u_1}{4(1+2\delta)(1+2\lambda)} + \frac{\gamma h_0B_1(u_2-v_2)}{8(1+2\delta)(1+2\lambda)}. \tag{48}$$

Again applying Lemma 7, (48) yields desired inequality (33). This completes the proof of Theorem 8. □

In light of Remarks 1–5, we have following corollaries.

Corollary 9. *If $f \in \mathcal{S}^*_{q,\sigma}(\gamma,\varphi)$, $0 \neq \gamma \in \mathbb{C}$, then*

$$|a_2| \le \frac{|\gamma||h_0|B_1\sqrt{B_1}}{\sqrt{\left|\gamma h_0B_1^2 - B_2 + B_1\right|}}, \qquad |a_3| \le \frac{|\gamma||h_1|B_1}{2} + |\gamma||h_0|\left[B_1 + |B_2-B_1|\right]. \tag{49}$$

Remark 10. Corollary 9 reduces to [23, Corollary 2.3, p. 82].

Corollary 11. *If $f \in \mathcal{K}_{q,\sigma}(\gamma,\varphi)$, $0 \neq \gamma \in \mathbb{C}$, then*

$$|a_2| \le \frac{|\gamma||h_0|B_1\sqrt{B_1}}{\sqrt{\left|2\gamma h_0B_1^2 - 4(B_2-B_1)\right|}}, \qquad |a_3| \le \frac{|\gamma||h_1|B_1}{6} + \frac{|\gamma||h_0|\left[B_1+|B_2-B_1|\right]}{2}. \tag{50}$$

Corollary 12. *If $f \in \mathcal{M}^{\delta}_{q,\sigma}(\gamma,\varphi)$, $0 \neq \gamma \in \mathbb{C}$, and $\delta \ge 0$, then*

$$|a_2| \le \frac{|\gamma||h_0|B_1\sqrt{B_1}}{\sqrt{\left|\gamma h_0B_1^2(1+\delta) - (B_2-B_1)(1+\delta)^2\right|}}, \qquad |a_3| \le \frac{|\gamma||h_1|B_1}{2+4\delta} + \frac{|\gamma||h_0|\left[B_1+|B_2-B_1|\right]}{1+\delta}. \tag{51}$$

Corollary 13. *If $f \in \mathcal{P}_{q,\sigma}(\gamma,\lambda,\varphi)$, $0 \neq \gamma \in \mathbb{C}$, and $0 \le \lambda \le 1$, then*

$$|a_2| \le \frac{|\gamma||h_0|B_1\sqrt{B_1}}{\sqrt{\left|\gamma(1+2\lambda-\lambda^2)h_0B_1^2 - (1+\lambda)^2(B_2-B_1)\right|}}, \qquad |a_3| \le \frac{|\gamma||h_1|B_1}{2+4\lambda} + \frac{|\gamma||h_0||B_2-B_1|}{|1+2\lambda-\lambda^2|} + \frac{|\gamma||h_0|B_1\left[(1+\lambda)^2 + |3+6\lambda-\lambda^2|\right]}{4(1+2\lambda)|1+2\lambda-\lambda^2|}. \tag{52}$$

Corollary 14. *If $f \in \mathcal{K}_{q,\sigma}(\gamma,\lambda,\varphi)$, $0 \neq \gamma \in \mathbb{C}$, and $0 \le \lambda \le 1$, then*

$$|a_2| \le \frac{|\gamma||h_0|B_1\sqrt{B_1}}{\sqrt{\left|\gamma(2+4\lambda-4\lambda^2)h_0B_1^2 - 4(1+\lambda)^2(B_2-B_1)\right|}}, \tag{53}$$

$$\begin{aligned}
|a_3| &\le \frac{|\gamma|\,|h_1|\,B_1}{6+12\lambda}+\frac{|\gamma|\,|h_0|\,|B_2-B_1|}{|2+4\lambda-4\lambda^2|} \\
&\quad+\frac{|\gamma|\,|h_0|\,B_1\left[(1+\lambda)^2+\left|2+4\lambda-\lambda^2\right|\right]}{3(1+2\lambda)\,|2+4\lambda-4\lambda^2|}.
\end{aligned} \tag{54}$$

Remark 15. Taking $h(z)\equiv 1$ in Corollary 9, we get estimates in [10, Corollary 2.3, p. 54] and setting $h(z)\equiv 1$ in Corollary 11 we have bounds in [10, Corollary 2.2, p. 53]. For $h(z)\equiv 1$ and $\gamma=1$, the inequalities obtained in Corollary 11 coincide with [7, Corollary 2.2, p. 349]. For $h(z)\equiv 1$ and $\gamma=1$, the estimates in Corollary 12 reduce to a known result in [7, Theorem 2.3, p. 348]. Further, for $h(z)\equiv 1$, $\gamma=1$, and φ given by (12) the inequalities in Corollary 12 reduce to a result proven earlier by [12, Theorem 3.2, p. 1500] and for $h(z)\equiv 1$, $\gamma=1$, and φ given by (13) the inequalities in Corollary 12 would reduce to known result in [12, Theorem 2.2, p. 1498]. Also, for $h(z)\equiv 1$, the estimates in Corollary 13 provide improvement over the estimates derived by Deniz [10, Theorem 2.1, p. 32]. For $h(z)\equiv 1$, the results obtained in this paper coincide with results in [21]. Furthermore, various other interesting corollaries and consequences of our results could be derived similarly by specializing φ.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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