Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2016, Article ID 1796316, 9 pages
http://dx.doi.org/10.1155/2016/1796316

Research Article

Generation and Identification of Ordinary Differential Equations of Maximal Symmetry Algebra

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Received 13 June 2016; Revised 25 October 2016; Accepted 7 November 2016

Academic Editor: Jaume Giné

An effective method for generating linear ordinary differential equations of maximal symmetry in their most general form is found, and an explicit expression for the point transformation reducing the equation to its canonical form is obtained. New expressions for the general solution are also found, as well as several identification and other results and a direct proof of the fact that a linear ordinary differential equation is iterative if and only if it is reducible to the canonical form by a point transformation. New classes of solvable equations parameterized by an arbitrary function are also found, together with simple algebraic expressions for the corresponding general solution.

1. Introduction

Linear ordinary differential equations (LODEs) are quite probably the most common type of differential equations that occur in physics and in many other mathematically based fields of science. However, their most important properties such as their transformation properties, their general solutions, and even their symmetry properties remain largely unknown.

In a short paper published by Krause and Michel [1] in 1988 certain specific properties of LODEs of maximal symmetry were established. In particular, the said paper shows that such equations are precisely the iterative ones, and equivalently those which can be reduced by an invertible point transformation to the trivial equation $y^{(n)} = 0$, which we shall refer to as the canonical form. However that short paper left a number of important questions unanswered. It does not provide, for instance, any expression for the point transformation mapping a given equation of maximal symmetry to the canonical form.

Almost at the same time the problem of generating LODEs of maximal symmetry was considered by Mahomed and Leach [2] who found an algorithm for obtaining expressions for the most general normal form of these equations based on the direct computation of the symmetry algebra. Computations with this algorithm remain however quite tedious and the authors managed to provide a general expression for the LODEs of maximal symmetry only up to the order eight. In fact the expression of the corresponding eight-order equation of maximal symmetry found in that paper is incorrect.

A more direct algorithm for generating this class of equations based on the simple fact that they are iterative was proposed recently in [3]. Nevertheless, some of the main results of the latter paper concerning in particular the generation and the point transformations of the class of LODEs of maximal symmetry still have room for improvements.

It is worthwhile to mention that in recent years the study of ordinary differential equations (ODEs) of maximal symmetry algebra has given rise to a considerable number of research papers in the scientific literature. Examples of such studies include the determination of various types of symmetry subalgebras and some of their applications for systems of second-order ODEs [4, 5]. Momoniat and collaborators also studied the algebraic properties of first integrals of scalar second-order and third-order ODEs as well as for systems of second-order ODEs of maximal symmetry [6, 7]. On the other hand a solution algorithm for second-order and third-order ODEs of maximal symmetry based on Janet bases and Loewy decompositions was obtained in [8, 9]. A similar study of second-order LODEs of maximal Lie point symmetry and third-order LODEs of maximal contact symmetry was carried out in [10] for equations in canonical form ($y''=0$ and $y'''=0$) to investigate amongst others the properties of their first integrals and exceptional symmetries.

In this paper, we provide a much simpler differential operator than that found in [3] for generating linear iterative equations of a general order. This gives rise to a simple algorithm for testing LODEs for maximal symmetry based solely on their coefficients. The operator thus found corrects the wrong one obtained in [2, Equations (3.20) and (3.21)] as one of the main conclusions of that paper. On the other hand, we give a more direct proof than that of Krause and Michel [1] to the fact that LODEs reducible by an invertible point transformation to the canonical form $y^{(n)}=0$ are precisely those which are iterative.

We also establish several results concerning the solutions of this class of equations and in particular their transformation to canonical form. In contrast to the very well-known paper by Ermakov [11] who managed to find only some very specific cases from a restricted class for which the second-order source equation is solvable, we provide large families of second-order equations for which the general solution is given by simple algebraic formulas. All such families are parameterized by an entirely arbitrary nonzero function and the general solutions thus found for the second-order source equation yield through a very simple quadratic formula that for the whole corresponding class of equations of maximal symmetry of a general order.

2. Iterations of Linear Equations

Let $r\neq 0$ and s be two smooth functions of x, and consider the differential operator $\Psi = r(d/dx)+s$. We shall often denote by $F[x_1,\dots,x_m]$ a differential function of the variables $x_1,\dots,x_m$. Linear iterative equations are the iterations $\Psi^n[y]=0$ of the first-order LODE $\Psi[y]\equiv ry'+sy=0$, given by

$$\Psi^n[y]=\Psi^{n-1}(\Psi[y]),\quad \text{for } n\geq 1 \text{ with } \Psi^0=I, \tag{1}$$

where I is the identity operator. A linear iterative equation of a general order n thus has the form

$$\begin{aligned}\Psi^n[y]\equiv K_n^0 y^{(n)}+K_n^1 y^{(n-1)}+K_n^2 y^{(n-2)}+\cdots+K_n^{n-1}y' \\ +K_n^n y=0.\end{aligned}\tag{2}$$

Setting

$$\begin{aligned}K_m^j&=0,\quad \text{for } j<0 \text{ or } j>m,\\ K_m^j&=1,\quad \text{for } m=j=0\end{aligned}\tag{3}$$

and applying (1) show that the coefficients K_n^j in the general expression (2) of an iterative equation satisfy the recurrence relation

$$K_n^j=rK_{n-1}^j+\Psi K_{n-1}^{j-1},\quad \text{for } 0\leq j\leq n,\ n\geq 1. \tag{4}$$

Moreover, setting $j=0$ or $j=n$ in (4) shows by induction on n that

$$\begin{aligned}K_n^0&=r^n,\\ K_n^n&=\Psi^{n-1}[s],\\ &\forall n\geq 1,\end{aligned}\tag{5}$$

and applying (4) recursively and using the conventions set in (3) give the new recurrence relation

$$K_n^j=\sum_{k=j}^{n} r^{n-k}\Psi K_{k-1}^{j-1},\quad \text{for } j=0,\dots,n,\ n\geq 1. \tag{6}$$

We note that (6) provides an algorithm for the computation of the coefficients K_n^j in terms of the parameters r and s of the source equation and the operator Ψ, and the resulting formula has effectively been obtained in [3, Theorem 2.2]. Moreover, it is of course also possible to compute K_n^j directly in terms of n and the parameters r and s, and for $j=1,2$ one finds that

$$K_n^1=r^{n-1}\left[ns+\binom{n}{2}r'\right], \tag{7a}$$

$$\begin{aligned}K_n^2=r^{n-2}\Bigg[\binom{n}{2}\Psi[s] \\ +\binom{n}{3}\left(3sr'+rr''+\frac{3n-5}{4}r'^2\right)\Bigg].\end{aligned}\tag{7b}$$

If we divide through the general nth order linear iterative equation $\Psi^n[y]=0$ in (2) by $K_n^0=r^n$, it takes the form

$$0=y^{(n)}+B_n^1y^{(n-1)}+\cdots+B_n^jy^{(n-j)}+\cdots+B_n^ny, \tag{8a}$$

$$B_n^j=\frac{K_n^j}{r^n}. \tag{8b}$$

It is clear that this equation represents the standard form of the general linear iterative equation with leading coefficient one. Moreover, the well-known change of the dependent variable $y\mapsto y\exp((1/n)\int_{x_0}^x B_n^1(t)dt)$ maps (8a) into its normal form in which the coefficient of the term of second highest order has vanished. This transformation however simply amounts to choosing r and s such that $B_n^1=0$; that is, $K_n^1=0$. Therefore, for given parameters r and s of the operator Ψ, an nth order LODE in normal form

$$y^{(n)}+A_n^2y^{(n-2)}+\cdots+A_n^jy^{(n-j)}+\cdots+A_n^ny=0 \tag{9a}$$

is iterative if and only if

$$A_n^j=\left.\frac{K_n^j}{r^n}\right|_{K_n^1=0},\quad (2\leq j\leq n), \tag{9b}$$

where K_n^j is given by (6). It follows from (7a) that the requirement that $K_n^1=0$ holds is equivalent to having

$$s=-\frac{1}{2}(n-1)r', \tag{10}$$

and this shows in particular that any iterative equation in normal form can be expressed in terms of the parameter r alone; that is, it depends on a single arbitrary function. Clearly, the coefficients A_n^j can also be expressed solely in terms of n, r, and the derivatives of r. For instance, by setting for any given function ξ,

$$\mathscr{A}(\xi)(x) = \frac{\left[\xi'(x)\right]^2 - 2\xi(x)\xi''(x)}{4\left[\xi(x)\right]^2}, \tag{11}$$

it follows from (7b) that in (9a)-(9b) we have

$$A_2^2 = \mathscr{A}(r), \qquad \text{and more generally } A_n^2 = \binom{n+1}{3}\mathscr{A}(r). \tag{12}$$

In fact, as already noted in [2, 3], the coefficients A_n^j depend only on the function $\mathscr{A}(r) = A_2^2$ and its derivatives. For simplicity, it will often be convenient to denote the coefficient A_2^2 of the term of third highest order in (9a) simply by $\mathfrak{q}$.

Having noted that the coefficients of every LODE of maximal symmetry in normal form depend solely on $\mathfrak{q}$ and its derivatives, an important problem considered in [2] was to find a linear ordinary differential operator $\Gamma_n[y]$ *depending solely on* $\mathfrak{q}$ and its derivatives, and which generates the most general form of the linear nth order equation of maximal symmetry in the dependent variable $y = y(x)$. In a recent paper [3], it was established that, for an arbitrary parameter r of the source equation, the operator

$$\Phi_n = \frac{1}{r^n}\Psi^n\Big|_{K_n^1=0} \tag{13}$$

generates the linear iterative equation of an arbitrary order n in normal form and in its most general form (9a)-(9b). Therefore, although the operator (13) and the equation it generates depend explicitly on r (and not on $\mathfrak{q} = A_2^2$) and its derivatives, on the basis of the result of [1] identifying linear iterative equations with LODEs of maximal symmetry, the two equations $\Gamma_n[y] = 0$ and $\Phi_n[y] = 0$ should always be the same for all $n \geq 3$, although such an operator $\Gamma_n[y]$ *has not yet been found*. This is due in part to the fact that a general expression for $A_n^j = A_n^j[\mathfrak{q}]$ is not available for all n and j. Nonetheless, we show here that it is naturally possible to make use of the differential operator Φ_n to generate directly a LODE of maximal symmetry of the general form (9a) in which the coefficients A_n^j depend only on $\mathfrak{q}$ and its derivatives.

Indeed, for any value of $n \geq 2$ it follows from (6) and (9a), (9b), (11), and (12) and a simple induction on $j \geq 2$ that each coefficient A_n^j in (9a)-(9b) depends linearly on $r^{(j)}$. Moreover, it follows from (11)-(12) that

$$r^{(j)} = D_x^{j-2}\left(\frac{r'^2 - 4\mathfrak{q}r^2}{2r}\right) := F_j[r, \mathfrak{q}], \quad \text{for } j \geq 2, \tag{14}$$

where $D_x = d/dx$. Therefore, applying the substitution (14) to $\Phi_n[y]$ yields the desired equation, that is, the LODE of the form (9a)-(9b) in which $A_n^j = A_n^j[\mathfrak{q}]$ depends only on $\mathfrak{q}$ and its derivatives. More formally, the resulting differential operator can be represented as

$$\Phi_n^r = \frac{1}{r^n}\Psi^n\Big|_{K_n^1=0,\ r^{(j)}=F_j[r,\mathfrak{q}],\ j\geq 2} = \Phi_n\Big|_{r^{(j)}=F_j[r,\mathfrak{q}],\ j\geq 2}. \tag{15}$$

For example, for $n = 3$ or 4, evaluating $\Phi_n[y]$ yields the following expressions directly in terms of r and its derivatives alone.

$$\begin{aligned}
\Phi_3[y] &= -\frac{y\left(r'^3 - 2rr'r'' + r^2r^{(3)}\right)}{r^3} + \frac{y'\left(r'^2 - 2rr''\right)}{r^2} + y^{(3)}, \\
\Phi_4[y] &= \frac{3y\left(27r'^4 - 68rr'^2r'' + 24r^2r'r^{(3)} + 4r^2\left(7r''^2 - 2rr^{(4)}\right)\right)}{16r^4} \\
&\quad - \frac{5y'\left(r'^3 - 2rr'r'' + r^2r^{(3)}\right)}{r^3} + \frac{5\left(r'^2 - 2rr''\right)y''}{2r^2} \\
&\quad + y^{(4)}.
\end{aligned} \tag{16}$$

However, if in addition we also apply to these expressions for $\Phi_n[y]$ the substitution (14), which amounts to applying directly the operator Φ_n^r to y we obtain

$$\Phi_3^r[y] = 2\mathfrak{q}'y + 4\mathfrak{q}y' + y''', \tag{17a}$$

$$\Phi_4^r[y] = 3y\left(3\mathfrak{q}^2 + \mathfrak{q}''\right) + 10y'\mathfrak{q}' + 10\mathfrak{q}y'' + y^{(4)}. \tag{17b}$$

Comparing this with the known expressions for LODEs of maximal symmetry expressed solely in terms of $\mathfrak{q}$ and its derivatives [2, 3] shows that Φ_3^r and Φ_4^r yield indeed the indicated expressions.

Another important observation made in this paper is that if we set $r = u^2$ for a certain nonzero function u, the expression for $\mathscr{A}(r) = \mathscr{A}(u^2)$ in (11) is much simpler and reduces to $-u''/u$. Setting $\mathfrak{q} = \mathscr{A}(r)$ is thus equivalent to letting u be a solution of the equation

$$y'' + \mathfrak{q}y = 0, \tag{18}$$

which is referred to as the second-order source equation for (9a)-(9b). Thus, if we express $\Phi_n[y]$ with r replaced by u^2, the substitution rule for $u^{(j)}$ similar to that given for $r^{(j)}$ in (14) takes the much simpler form

$$u^{(j)} = D_x^{j-2}(-u\mathfrak{q}) := H_j[u, \mathfrak{q}], \quad \text{for } j \geq 2. \tag{19}$$

Denote by Θ_n^u the operator Φ_n in which r is replaced with u^2 and to which the substitution (19) is applied. That is,

$$\Theta_n^u = \Phi_n\Big|_{r=u^2,\ u^{(j)}=H_j[u,\mathfrak{q}],\ j\geq 2}. \tag{20}$$

In other words Θ_n^u is just Φ_n as in (13), in which the source parameter r has been replaced by u^2 and to which the substitution (19) is then applied. Then for the same reasons that

$\Phi_n^r[y, \mathbf{q}]$ generates the most general form of linear iterative equations, $\Theta_n^u[y, \mathbf{q}]$ also does the same. However, in view of the much simpler substitution rule (19), the computational cost for generating these equations is much lower using Θ_n^u rather than Φ_n^r. Indeed for arbitrary values of $\mathbf{q}$, generating equations of maximal symmetry of order greater than ten has been up to now a very tedious task using the original algorithm of [2], but it is now easy to generate such equations at much higher orders in most standard computers using computing systems such as MATHEMATICA. For instance, the coefficient A_{15}^{15} of $y = y^{(0)}$ which is the largest expression in size in the equation $\Theta_{15}^u[y] = 0$ is given by

$$\begin{aligned}
A_{15}^{15} = 14\Big(&2\Big(52022476800\mathbf{q}^6\mathbf{q}' + 2132810240\mathbf{q}'^5 \\
&+ 6656237568\mathbf{q}^5\mathbf{q}^{(3)} + 341232100\mathbf{q}'^3\mathbf{q}^{(4)} \\
&+ 1024\mathbf{q}^4\left(73853676\mathbf{q}'\mathbf{q}'' + 286397\mathbf{q}^{(5)}\right) \\
&+ 32\mathbf{q}^3\left(2646561024\mathbf{q}'^3 + 207959056\mathbf{q}''\mathbf{q}^{(3)}\right. \\
&\left.+ 123346720\mathbf{q}'\mathbf{q}^{(4)} + 184297\mathbf{q}^{(7)}\right) + 2\mathbf{q}'^2\left(863911980\mathbf{q}''\mathbf{q}^{(3)}\right. \\
&\left.+ 784597\mathbf{q}^{(7)}\right) + 8\mathbf{q}^2\left(2094143648\mathbf{q}'^2\mathbf{q}^{(3)}\right. \\
&+ 33615550\mathbf{q}^{(3)}\mathbf{q}^{(4)} + 22652990\mathbf{q}''\mathbf{q}^{(5)} \\
&\left.+ \mathbf{q}'\left(2835404512\mathbf{q}''^2 + 9750858\mathbf{q}^{(6)}\right) + 7323\mathbf{q}^{(9)}\right) \\
&+ 13\left(457296\left[\mathbf{q}^{(3)}\right]^3 + \mathbf{q}^{(5)}\left(740970\mathbf{q}''^2 + 2717\mathbf{q}^{(6)}\right)\right. \\
&+ 2068\mathbf{q}^{(4)}\mathbf{q}^{(7)} + 2\mathbf{q}^{(3)}\left(1100540\mathbf{q}''\mathbf{q}^{(4)} + 591\mathbf{q}^{(8)}\right) \\
&\left.+ 491\mathbf{q}''\mathbf{q}^{(9)}\right) + \mathbf{q}'\left(780095196\mathbf{q}''^3 + 8478360\left[\mathbf{q}^{(4)}\right]^2\right. \\
&\left.+ 14231306\mathbf{q}^{(3)}\mathbf{q}^{(5)} + 8279146\mathbf{q}''\mathbf{q}^{(6)} + 1793\mathbf{q}^{(10)}\right) \\
&+ 2\mathbf{q}\left(12696730880\mathbf{q}'^3\mathbf{q}'' + 151883460\mathbf{q}'^2\mathbf{q}^{(5)}\right. \\
&+ 2519682\mathbf{q}^{(4)}\mathbf{q}^{(5)} + 2\mathbf{q}^{(3)}\left(257461378\mathbf{q}''^2 + 913529\mathbf{q}^{(6)}\right) \\
&+ 937436\mathbf{q}''\mathbf{q}^{(7)} + 139\mathbf{q}^{(11)} \\
&\left.\left.+ 4\mathbf{q}'\left(95070193\left[\mathbf{q}^{(3)}\right]^2 + 152580285\mathbf{q}''\mathbf{q}^{(4)} + 79587\mathbf{q}^{(8)}\right)\right)\Big) \\
&+ \mathbf{q}^{(13)}\Big).
\end{aligned} \tag{21}$$

Of course, if one can generate a LODE of maximal symmetry of a general order and in its most general form, then the identification of any given LODE as a member or not in the class of linear equations of maximal symmetry is also achieved. This is due to the fact that, in their normal form, nth order LODES (9a) of maximal symmetry are completely and uniquely determined by the coefficients A_n^2. Indeed, suppose that

$$\begin{aligned}
\mathscr{L}[y] \equiv y^{(n)} + F_n^2 y^{(n-2)} + \cdots + F_n^j y^{(n-j)} + \cdots + F_n^n y \\
= 0
\end{aligned} \tag{22}$$

is a given LODE in normal form that we wish to test for the maximality of its symmetry algebra, where F_n^j are given functions of x. If this equation is indeed of maximal symmetry, then by (12) the coefficient $\mathbf{q}$ of the corresponding second-order source equation should satisfy $\mathbf{q} = F_n^2/\binom{n+1}{3}$. Consequently, $\mathscr{L}[y] = 0$ is of maximal symmetry if and only if it coincides with the corresponding generated equation $\Theta_n^u[y, \mathbf{q}] = 0$.

Using the operator Θ_n^u, we shall give in the next section a much simpler and direct proof than that of [1] to the fact that a LODE is iterative if and only if it is reducible by a point transformation to the canonical form. To close this section, we note that by Abel's Identity the Wronskian of any two linearly independent solutions u and v of the source equation (18) is a nonzero constant and will be normalized to one. It should be noted that linearly independent solutions of the second-order equation (18) are not known for arbitrary values of the coefficient $\mathbf{q}$.

There are of course many other obvious criteria that can be used to identify expressions $\mathscr{L}[y]$ which do not correspond to LODES of maximal symmetry. For example, the coefficients $A_n^j = A_n^j[\mathbf{q}]$ are in fact differential polynomials in $\mathbf{q}$. This implies in particular that if, for instance, $\mathbf{q}$ is polynomial (in x), then all A_n^j must be polynomials, and if it is a sine function, then the A_n^j must all be polynomials of sine and cosine functions.

3. Reduction to Canonical Form and General Solution

The equivalence group of linear nth order equations in normal form (9a) is well known to be given by invertible point transformations of the form

$$\begin{aligned}
x &= f(z), \\
y &= \lambda\left[f'(x)\right]^{(n-1)/2} w,
\end{aligned} \tag{23}$$

where f is an arbitrary locally invertible function and λ an arbitrary nonzero constant [9, 12, 13]. In other words, an equation of the form (9a) is reducible by a point transformation to the canonical form if and only if there exists a transformation of the form (23) that maps such an equation to the canonical form.

Let us denote by

$$S(\xi)(z) = \frac{\left(-3\xi''^2 + 2\xi'\xi^{(3)}\right)}{2\xi'^2} \tag{24}$$

the Schwarzian derivative of the function $\xi = \xi(z)$. By studying the expression of the source parameter of the transformed equation under equivalence transformations, a simple characterization of the point transformation that maps an iterative equation to the canonical form was found in [3, Theorem 4.3]. This result states that a point transformation reduces a given iterative equation, which without loss of generality may be assumed to be of the form (9a), to the canonical form $w^{(n)}(z) = 0$ if and only if it is of the form (23), where f is the inverse of the function $z = h(x)$ satisfying

$$\begin{aligned} A_2^2(x) &= \frac{1}{2} S(h)(x), \\ \text{or equivalently, } \mathscr{A}(r)(x) &= \frac{1}{2} S(h)(x), \end{aligned} \tag{25}$$

on account of (12), assuming that r is the source parameter of the equation. By making use of this result of [3], we can now provide a more direct proof than that given in [1] for the following result.

Theorem 1. *A linear ordinary differential equation is iterative if and only if it can be reduced to the canonical form by an invertible point transformation.*

Proof. As usual one may assume that the iterative equation is in its normal form and has source parameter r. On the basis of the above stated result from Theorem 4.3 of [3], it follows that the transformation (23) where f is the inverse of the function $h = \int(dx/r)$ maps the iterative equation to its canonical form, as the latter expression for h solves (25). More directly, one can prove that the class of iterative equations and that of equations reducible by an invertible point transformation to the canonical form are the same. Indeed, let $\Omega_n[f, \lambda, x, y] \equiv \Omega_n[f, \lambda]$ represent the element of the equivalence group of linear nth order ODEs $\Delta \equiv \Delta_n[y] = 0$ in normal form, acting on the space of independent variable x and dependent variable y, and given by (23). Denoting by $\Omega_n[f, \lambda] \cdot \Delta = 0$ the transformed equation, it follows that we have exactly

$$\Phi_n[y] = K_n(\lambda, r)\, \Omega_n\left[\int \frac{dx}{r}, \lambda, z, w\right] \cdot w^{(n)}(z), \tag{26a}$$

where

$$K_n(\lambda, r) = \frac{1}{\left(\lambda [r(x)]^{(n+1)/2}\right)}. \tag{26b}$$

In other words, generating a linear iterative equation in $y = y(x)$ for any given value of r using the differential operator Φ_n and transforming the canonical equation $w^{(n)}(z) = 0$ in the new variables x and y using the point transformation operator $K_n(\lambda, r)\Omega_n[\int(dx/r), \lambda, z, w]$ yield exactly the same equations, and this proves the result. □

In [1] Krause and Michel proved this theorem indirectly as a consequence of equivalence relations, by proving the equivalence between equations of maximal symmetry and iterative equations on one hand and between equations reducible by invertible point transformations and equations of maximal symmetry on the other hand. The part in that proof stating that an ODE (linear or nonlinear) is reducible to the canonical form if and only if it has maximal symmetry was however already proved by Lie [14–16]. The direct proof given here will allow us to construct various point transformations of practical importance for iterative equations as well as various forms of their general solutions.

It should first be noted that the right hand side of (26a) gives other methods for generating linear iterative equations with source parameter r. In practice, however, linear iterative equations arise generally with no reference to any source parameter but are expressed solely in terms of the coefficient $\mathfrak{q} = A_2^2(x)$ and its derivatives. In this case, on the basis of a remark made in the previous section, a solution h to the equation $\mathfrak{q} = (1/2)S(h)(x)$ in (25) is simply given by $h = \int(dx/u^2)$, where u is a solution of the second-order source equation (18). Consequently, if we denote by Ω_n^u the transformation operator given by

$$\begin{aligned} &\Omega_n^u\left[\int \frac{dx}{u^2}, \lambda, z, w\right] \\ &\quad = K_n\left(\lambda, u^2\right) \Omega_n\left[\int \frac{dx}{u^2}, \lambda, z, w\right]\Bigg|_{u^{(j)} = H_j[u, \mathfrak{q}],\ j \geq 2}, \end{aligned} \tag{27}$$

where $H_j[u, \mathfrak{q}] = D_x^{j-2}(-u\mathfrak{q})$ as in (19), then $\Theta_n^u[y]$ and $\Omega_n^u \cdot w^{(n)}(z)$ generate exactly the same linear iterative equation expressed solely in terms of $\mathfrak{q}$ and its derivatives.

On the other hand, since the transformation operator $\Omega_n^u[\int(dx/u^2), \lambda, z, w]$ maps the trivial equation to the most general form (9a) of the iterative equation, its explicit expression can be used to derive the general solution of (9a). Indeed, under this operator we have

$$\begin{aligned} y &= \frac{1}{\lambda} u^{n-1} w, \\ z &= \int \frac{dx}{u^2}. \end{aligned} \tag{28}$$

Consequently, n linearly independent solutions to the general nth order iterative equation (9a) are given by

$$y_k = u^{n-1}\left(\int \frac{dx}{u^2}\right)^k, \quad k = 0, \ldots, n-1. \tag{29}$$

In particular, if u and v are two linearly independent solutions of the source equation (18), then $\int(dx/u^2) = v/u$. Consequently, in terms of u and v, the n linearly independent solutions to (9a) in (29) above can be rewritten as

$$y_k = u^{n-1-k} v^k, \quad k = 0, \ldots, n-1. \tag{30}$$

Formula (30) is well known and was cited without proof in [1], and it is an important result for which we have not been able to find the proof in the recent literature. We naturally also found formula (28) only in this paper. Moreover, formula (29) established above provides a much simpler result by showing that linearly independent solutions of (9a) can be expressed solely in terms of a single nonzero solution of the second-order source equation. Indeed, the solutions y_k in (29) are clearly linearly independent as their Wronskian equals the nonzero constant $\prod_{j=1}^{n-1} j!$.

Theorem 2. *A linear ordinary differential equation is iterative if and only if it has n linearly independent solutions y_k of the form (30), where u and v are two linearly independent solutions of the corresponding second-order source equation (18).*

Proof. The fact that a linear iterative equation has linearly independent solutions of the stated form is established in (30). Conversely, if a LODE has n linearly independent solutions of the form (30), then since the second-order source equation is reducible to the canonical form $w'' = 0$ by a point transformation, without loss of generality we may assume that such a transformation reduces u to 1 and v to z. Consequently, the corresponding linearly independent solutions of the LODE are polynomials of degree at most $n-1$, and thus the transformed equation is in canonical form. It then clearly follows from Theorem 1 that the equation is iterative. □

Recall that the mth symmetric power of a LODE $\Delta[y] = 0$ is the LODE $\Omega[y] = 0$ of minimal order such that for every set of m solutions $y_1, \ldots, y_m$ of $\Delta[y] = 0$ the product $y_1 \ldots y_m$ is a solution of $\Omega[y] = 0$. Moreover, it is well known that the mth symmetric power of a second-order LODE has order $m+1$ (see [17] and the references therein). It thus follows from Theorem 2 that for $m \geq 1$ an $(m+1)$th order LODE is of maximal symmetry if and only if it is the symmetric power of a second-order LODE. It has in fact been established in [17, Theorems 1 and 2] that the symmetric power of a LODE is an iterative LODE.

4. Solvability of Equations of Maximal Symmetry

The results obtained thus far in this paper will be used in this section to show amongst others that LODEs of maximal symmetry are highly solvable and that it is indeed the case for the second-order source equation (18) whose solutions completely determine those of the corresponding LODEs of maximal symmetry. In a very popular paper published in Russian in 1880 and recently translated into English [11], Ermakov stated that the majority of second-order linear homogeneous ODEs for which it is possible to find conditions for their solvability are of the form

$$\left(\alpha_1 x^2 + \alpha_2 x + \alpha_3\right) y'' + \left(\alpha_4 x + \alpha_5\right) y' + \alpha_6 y = 0, \tag{31}$$

where the α_j are some arbitrary constants. He then moved on in the paper to obtain some very specific cases of equations which are solvable from this class, together with their general solutions.

A couple of years later in the same decade Hill [18] considered in the study of lunar stability the most general form of the second-order LODE in normal form, that is, in the form (18), but in which the coefficient $\mathfrak{q}$ is a periodic function. Hill's equation and its important variants such as Meissner Equation and Mathieu Equation have been amply studied [19], and it is well known that its solutions and their related properties can be described by means of Floquet theory. These solutions can also be expressed in terms of Hill's determinant [20]. It should be mentioned that, in the context of the present paper, a differential equation is called solvable if the closed form expression of its general solution is known, regardless of the type of functions in terms of which this solution is expressed.

The results of this paper extend the scope of solvability of linear second-order equations obtained by Ermakov, Hill, and others such as Loewy [21] not only to more larger families of the coefficient function $\mathfrak{q}$ in (18), but moreover to all LODEs of maximal symmetry of a general order. This statement holds provided that the parameter r of the source equation (18) is known. Indeed, it follows from (29) and the condition $r = u^2$ relating the parameter of the source equation and the solution of the corresponding second-order equation, that for *arbitrary* values $r = r(x)$ of the source parameter, two linearly independent solutions of the second-order equation of the form

$$y'' + \mathscr{A}(r)\, y = 0 \tag{32a}$$

are given by

$$y_j = \sqrt{r}\left(\int \frac{dx}{r}\right)^j, \quad \text{for } j = 0, 1. \tag{32b}$$

Therefore, n linearly independent solutions of the corresponding $(n-2)$th iteration of (32a) can also be obtained from (29) through the substitution $r = u^2$. More directly in terms of r, we have the following result which follows immediately from (32a)-(32b) and (30).

Theorem 3. *Let r be a given nonzero function considered as the source parameter of the differential operator generating an nth order LODE $\Delta_n[y] = 0$ of maximal symmetry and in normal form. Then n linearly independent solutions s_k of $\Delta_n[y] = 0$ are given by*

$$s_k = \left(\sqrt{r}\right)^{n-1}\left(\int \frac{dx}{r}\right)^k, \quad k = 0, \ldots, n-1. \tag{33}$$

On one hand and on the basis of the results obtained in the previous section, it follows from Theorem 3 that not only can we generate a LODE of maximal symmetry of any order and for any nonzero function r, but also the general solution of any such equation is given in closed form in terms of the source parameter r by (33). On the other hand, for any given LODE of maximal symmetry, as long as its corresponding source parameter r is known, its general solution is given also by (33).

Now, to make a more direct comparison with the class of equations (31) considered by Ermakov, we note that linearly independent solutions can also be found for the standard form of (32a) which in terms of the arbitrary coefficient $B = B(x)$ of y' takes the form

$$y'' + By' + \frac{1}{4}\left(4\mathscr{A}(r) + B^2 + 2B'\right) y = 0. \tag{34}$$

The two linearly independent solutions of (34) are then given by

$$y_j = \sqrt{r}\left(\int \frac{dx}{r}\right)^j e^{-(1/2)\int B(x)dx}, \quad j = 0, 1. \tag{35}$$

We note that as opposed to the very specific cases of solvable equations found by Ermakov from the restricted class of equations (31), not only does (34) depend intrinsically on one *arbitrary function*, but also its general solution is given by the simple formula (35) in terms of the source parameter r.

We further clarify the application of Theorem 3 by an example.

Example 4. Let α be a nonzero real number and consider the particular case where the source equation (18) is given by

$$y'' + \mathfrak{q}y = 0, \quad \text{with } \mathfrak{q} = \frac{\alpha^2 + [\ln(x^\alpha)]^2}{4x^2[\ln(x^\alpha)]^2}. \tag{36}$$

Then it is not obvious how to solve (36) using standard methods. However, the latter coefficient $\mathfrak{q}$ is an expression of the form $\mathscr{A}(r)$ as defined in (11) and (12) and more precisely with $r = x\ln(x^\alpha)$. Therefore, since we know the source parameter r of the differential operator generating (36), it follows that an explicit expression of its general solution is given in closed form by (32b). Moreover, it also follows from Theorem 3 that the closed form expression of the general solution of any equation of arbitrary order n generated by the same differential operator with source parameter r is given by (33).

In practice, a LODE occurs however with no reference to a source parameter of the differential operator that generated it, but rather in terms of its coefficients. As already indicated any LODE can always be assumed to be of the normal form (9a)-(9b), and even in such a case the results of Theorem 3 can also be used to find the general solution of LODES of maximal symmetry, in virtually all known solvable cases, in addition to those cases which can be solved only with the proposed method. It follows from Theorem 3 that to find the general solution it suffices to find the parameter r of the source equation and by (12) this amounts to solving for r the nonlinear second-order equation.

$$A_n^2 = \binom{n+1}{3}\mathscr{A}(r), \tag{37}$$

where A_n^2 is the coefficient appearing in (9a)-(9b) while $\mathscr{A}(r)$ is given by (11).

This approach for solving linear differential equations has some similarities with the Loewy decomposition method [21, 22] in which the solution of a linear differential equation is known once a factorization of the differential operator generating the equation has been computed. In this approach, finding the coefficients of each factor in the factorization is also achieved for second- and third-order LODES with rational coefficients by solving certain Riccati equations. We illustrate the comparison of these two methods by considering a very simple example.

Example 5. Consider the second-order equation

$$0 = y'' + \left(2 + \frac{1}{x}\right)y' + y, \tag{38a}$$

whose solution by Loewy decomposition is discussed in [22]. This equation has Loewy decomposition

$$0 = \operatorname{Lclm}\left(\frac{d}{dx} + a_2, \frac{d}{dx} + a_1\right)[y] = 0, \tag{38b}$$

where Lclm stands for least common left multiple [22] and the coefficients a_1 and a_2 given by

$$\begin{aligned} a_1 &= 2 + \frac{2}{x} - \frac{1}{x + 3/2}, \\ a_2 &= \frac{2}{x} - \frac{2x - 2}{x^2 - 2x + 3/2} \end{aligned} \tag{38c}$$

are rational nonequivalent solutions of the Riccati equation

$$a' - a^2 + \left(2 + \frac{1}{x}\right)a + \frac{4}{x^2} = 0. \tag{39}$$

It then follows from a result of Loewy [22, Lemma 2.4] that a fundamental set of solutions of (38a) is given by

$$\begin{aligned} y_1 &= \frac{2}{3} - \frac{4}{3x} + \frac{1}{x^2}, \\ y_2 &= \frac{e^{-2x}(3 + 2x)}{x^2}. \end{aligned} \tag{40}$$

Given that every second-order LODE is of maximal symmetry, to solve (38a) with the method proposed in this paper, we first reduce it to its normal form (although from (34) this is not necessary), through the usual transformation given in Section 2. The reduced equation takes the simpler form

$$0 = y'' + \mathfrak{q}(x)y, \quad \text{with } \mathfrak{q}(x) = -\frac{[15 + 4x(1 + x)]}{4x^2}. \tag{41}$$

The nonlinear equation (37) for finding the source parameter r reduces to $\mathscr{A}(r) = \mathfrak{q}(x)$, where $\mathfrak{q}(x)$ is given here by (41). It has general solution

$$r = \frac{e^{-2x}\left(3e^{2x} - 4e^{2x}x + 2e^{2x}x^2 + 6k_1 + 4xk_1\right)^2 k_2}{x^3} \tag{42}$$

for some constants of integration k_1 and k_2. Therefore, by (32b) linearly independent solutions of (41) are given by

$$s_j = \frac{e^x(3 + 2x(-2 + x))}{\sqrt{x^3}}\left[-\frac{e^{-2x}(3 + 2x)}{8(3 + 2x(-2 + x))}\right]^j, \quad j = 0, 1. \tag{43}$$

The inverse of the transformation mapping (38a)–(38c) to (41) then shows that a fundamental system of solutions of (38a)–(38c) is given by

$$\begin{aligned} y_1 &= \frac{3 + 2x(-2 + x)}{x^2}, \\ y_2 &= \frac{e^{-2x}(3 + 2x)}{x^2}, \end{aligned} \tag{44}$$

which is the same as that obtained in (40).

The drawback with these two methods for solving LODES is that they require the solutions of nonlinear ODES which, in general, are more difficult to solve than the original LODE.

However, for the method proposed in this paper, equation (37) for finding r turns out to be linearizable and the linearized equation might be much easier to solve than the original equation.

As far as ODEs are concerned, the Loewy decomposition method is designed in principle for LODEs of all orders. However, that method does not seem to be accessible because its scope is quite limited, due in particular to the fact that the implementation of its algorithm involves a very high computational cost. The Loewy decomposition theory has also quite a large and very specific vocabulary of its own taken from differential algebra. Moreover, there are too many types of Loewy decompositions for a given linear differential operator, and there are in particular 12 such types for third-order linear operators alone. On the other hand the Loewy decomposition method is limited only to equations with rational coefficients for which the solutions of the corresponding (nonlinear) Riccati equations for finding the coefficients of the factorized operator are available.

Of course the method proposed in this paper for finding solutions of LODEs is limited to the case of equations of maximal symmetry. The determining nonlinear second-order equation (37) for finding the source parameter r is however the same for equations of all orders since A_n^2 is an arbitrary coefficient in (37), and it is also linearizable as already indicated. Moreover, as Theorem 3 shows, it is easily applicable to equations of all orders as long as (37) can be solved for r.

Nevertheless, it should be emphasized that the most important result obtained in this section is not a method for solving a given lode. It is rather the fact that to any arbitrary smooth function r we have associated a LODE of maximal symmetry and arbitrary order and for which the closed form solution is given by (33). This shows that such equations are highly solvable, because traditionally most solvable LODEs are limited to those having rational coefficients.

5. Concluding Remarks

In this paper, we found a cost effective algorithm for generating LODEs of maximal symmetry. This algorithm corrects the wrong one proposed in [2], as quoted in the paper. We have also shown how such an algorithm can be used to easily identify linear ODEs of maximal symmetry and proved some fundamental results concerning the reduction of ODEs of maximal symmetry to canonical form. We also found general and optimal expressions for their general solutions.

Clearly, the source equation (18) is hardly solvable for arbitrary values of the corresponding coefficient $\mathfrak{q}$. We have however shown that for every given nonzero smooth function r, where the function r may include in particular any kind of special function, the general solution of any nth order LODE of maximal symmetry in normal form generated by the differential operator having r as coefficient is given by the closed form expression (33).

On the other hand, the inverse problem of finding r for a given value of $\mathfrak{q}$ turns out to be equivalent to solving a different LODE of the form (18), because the nonlinear second-order ODE $\mathscr{A}(r) = \mathfrak{q}$ is linearizable. Nevertheless, it seems quite possible to describe all possible values of the arbitrary coefficient function $\mathfrak{q}$ by an appropriate choice of a source parameter r. This in turn would then mean that all linear ODEs of maximal symmetry are exactly solvable.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this article.

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