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Research Article

A Note on the Existence of a Smale Horseshoe in the Planar Circular Restricted Three-Body Problem

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It has been proved that, in the classical planar circular restricted three-body problem, the degenerate saddle point processes transverse homoclinic orbits. Since the standard Smale-Birkhoff theorem cannot be directly applied to indicate the chaotic dynamics of the Smale horseshoe type, we in this note alternatively apply the Conley-Moser conditions to analytically prove the existence of a Smale horseshoe in this classical restricted three-body problem.

1. Introduction and Preliminaries

Few bodies problems [1–7] have been studied for long time in celestial mechanics, either as simplified models of more complex planetary systems or as benchmark models where new mathematical theories can be tested. The three-body problem has been the source of inspiration and study in celestial mechanics since Newton and Euler [8–14]. Especially, the following classical planar circular restricted three-body model has been extensively studied in the literature. Let two particles P_1 and P_2, of mass $1-\mu$ and μ, move uniformly in a circular orbit about their common center of mass with angular velocity ω. The orbit is located in the Oxy plane of the inertial frame of reference and the common center of mass is in the origin. The particle P_3 of infinitesimal mass m_3 moves in the gravitational field generated by P_1 and P_2. Note that since the mass of P_3 is so small, its effects on other three particles can be ignored. Without loss of generality, assume that, in the $O\overline{xy}$ plane of the rotating frame of reference, the particles P_1 and P_2 rest at the points $(\mu, 0)$ and $(\mu-1, 0)$, respectively. By denoting their polar coordinates by ρ and φ and using the polar angle $\tau = \omega t$ as a new independent variable, the equation of motion of the infinitesimal particle P_3 can be written as follows:

$$\frac{d\rho}{d\tau} = p_\rho,$$

$$\frac{dp_\rho}{d\tau} = \frac{p_\varphi^2}{\rho^3} - \frac{(1-\mu)(\rho-\mu\cos\varphi)}{(\rho^2+\mu^2-2\rho\mu\cos\varphi)^{3/2}} - \frac{\mu[\rho+(1-\mu)\cos\varphi]}{[\rho^2+(1-\mu)^2+2\rho(1-\mu)\cos\varphi]^{3/2}},$$

$$\frac{d\varphi}{d\tau} = \frac{p_\varphi}{\rho^2} - 1,$$

$$\frac{dp_\varphi}{d\tau} = -\mu(1-\mu) \times \rho\sin\varphi\left[\frac{1}{[\rho^2+\mu^2-2\rho\mu\cos\varphi]^{3/2}} - \frac{1}{[\rho^2+(1-\mu)^2+2\rho(1-\mu)\cos\varphi]^{3/2}}\right], \quad (1)$$

where p_ρ and p_φ are momenta canonically conjugate to the coordinates ρ and φ, respectively.

The Hamiltonian of the system (1) is

$$H = \frac{1}{2}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} - 2p_\varphi\right) - \frac{1-\mu}{\left(\rho^2+\mu^2-2\rho\mu\cos\varphi\right)^{1/2}} - \frac{\mu}{\left(\rho^2+(1-\mu)^2+2\rho(1-\mu)\cos\varphi\right)^{1/2}}. \tag{2}$$

For the above classical model, Xia [4] has showed, by proper coordinate change for transforming the points at infinity to the origin (i.e, the McGehee transformation [2]), that there is a periodic solution at infinity. Moreover, from [2, 4], we know that this periodic solution is a degenerate saddle in the sense [2] that, for the Poincaré map of the periodic orbit introduced at infinity, its derivative (i.e., the Jacobian) at the origin is the identity.

Further, Xia [4] and Zhu and Xiang [12] both proved the existence of transversal homoclinic orbits by the Melnikov method to the periodic solution at infinity, which corresponds to the origin under the coordinate change. However, since the origin is a degenerate fixed point, the standard Smale-Birkhoff theorem [15] cannot be directly applied to indicate the existence of a Smale horseshoe. This problem has also been pointed out by Dankowicz and Holmes [6] and Llibre and Perez-Chavela [8]. Thus, in this present note, we try to alternatively apply the Conley-Moser conditions to analytically prove the existence of a Smale horseshoe in the above classical model. For this, we introduce the Conley-Moser conditions [16] as follows.

Let $f : D \mapsto \mathbb{R}^2$ be an invertible map, where $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le 1\}$, and f is at least C^1. For two given $\mu_v > 0$ and $\mu_h > 0$, let $K = \{1, 2, \dots, N\}$ $(N \ge 2)$ be an index set, let $H_1, \dots, H_N$ be the N disjoint μ_h-horizontal strips, and $V_1, \dots, V_N$ be the N disjoint μ_v-vertical strips. For each $i, j \in K$, denote $f(H_i) \bigcap H_j$ as V_{ji} and $H_i \bigcap f^{-1}(H_j)$ as H_{ij}. Clearly, $H_{ij} = f^{-1}(V_{ji})$. Define $\mathscr{H} = \bigcup_{i,j\in K} H_{ij}$ and $\mathscr{V} = \bigcup_{i,j\in K} V_{ji}$. It is also obvious that $f(\mathscr{H}) = \mathscr{V}$.

For an arbitrary point $z_0 = (x_0, y_0) \in \mathscr{H} \bigcup \mathscr{V}$, let (ξ_{z_0}, η_{z_0}) be a vector emanating from the point z_0 in the tangent space of z_0. The stable sector at z_0 is then defined as $\mathcal{S}^s_{z_0} = \{(\xi_{z_0}, \eta_{z_0}) \in \mathbb{R}^2 \mid |\eta_{z_0}| \le \mu_h |\xi_{z_0}|\}$. Similarly, the unstable sector at z_0 is defined as $\mathcal{S}^u_{z_0} = \{(\xi_{z_0}, \eta_{z_0}) \in \mathbb{R}^2 \mid |\xi_{z_0}| \le \mu_v |\eta_{z_0}|\}$. By taking the union of the stable and unstable sectors over all points in $\mathscr{H}$ and $\mathscr{V}$, we can define sector bundles as follows:

$$\mathcal{S}^s_{\mathscr{H}} = \bigcup_{z_0\in\mathscr{H}} \mathcal{S}^s_{z_0}, \qquad \mathcal{S}^s_{\mathscr{V}} = \bigcup_{z_0\in\mathscr{V}} \mathcal{S}^s_{z_0}; \qquad \mathcal{S}^u_{\mathscr{H}} = \bigcup_{z_0\in\mathscr{H}} \mathcal{S}^u_{z_0}, \qquad \mathcal{S}^u_{\mathscr{V}} = \bigcup_{z_0\in\mathscr{V}} \mathcal{S}^u_{z_0}. \tag{3}$$

Then, the Conley-Moser conditions for the map f are described by the following two assumptions.

Assumption 1. $0 \le \mu_v\mu_h < 1$ and, for each $i \in \{1, 2, \dots, N\}$, f maps H_i homeomorphically onto V_i; that is, $f(H_i) = V_i$. Moreover, the horizontal boundaries of H_i are mapped to the horizontal boundaries of V_i and the vertical boundaries of H_i are mapped to the vertical boundaries of V_i.

Assumption 2. $Df(\mathcal{S}^u_{\mathscr{H}}) \subset \mathcal{S}^u_{\mathscr{V}}$ and $Df^{-1}(\mathcal{S}^s_{\mathscr{V}}) \subset \mathcal{S}^s_{\mathscr{H}}$. Moreover, there exists a positive number λ satisfying $0 < \lambda < 1 - \mu_v\mu_h$ such that

(1) if $(\xi_{z_0}, \eta_{z_0}) \in \mathcal{S}^u_{z_0}$ and $(\xi_{f(z_0)}, \eta_{f(z_0)}) \doteq Df(z_0)(\xi_{z_0}, \eta_{z_0}) \in \mathcal{S}^u_{f(z_0)}$, then $|\eta_{f(z_0)}| \ge (1/\lambda)|\eta_{z_0}|$;

(2) if $(\xi_{z_0}, \eta_{z_0}) \in \mathcal{S}^s_{z_0}$ and $(\xi_{f^{-1}(z_0)}, \eta_{f^{-1}(z_0)}) \doteq Df^{-1}(z_0)(\xi_{z_0}, \eta_{z_0}) \in \mathcal{S}^s_{f^{-1}(z_0)}$, then $|\xi_{f^{-1}(z_0)}| \ge (1/\lambda)|\xi_{z_0}|$.

Based on Assumptions 1 and 2, we directly have the following.

Lemma 3 (see [16]). *If the map f satisfies Assumptions 1 and 2, then f has an invariant Cantor set, on which it is topologically conjugate to a full shift on N symbols and has*

(i) *a countable infinity of periodic orbits of arbitrarily high period;*

(ii) *an uncountable infinity of nonperiodic orbits;*

(iii) *a dense orbit.*

Remark 4 (see [16–18]). If f satisfies Assumption 2, we call that f satisfies the (μ_h, μ_v)-cone condition.

2. Main Result

In this section, we will analytically prove the existence of a Smale horseshoe in the classical planar circular restricted three-body problem introduced in Section 1, arriving at the following theorem.

Theorem 5. *For the classical planar circular restricted three-body problem introduced in Section 1, when the mass ratio μ is sufficiently small, there exists a Smale horseshoe and thus the system* (1) *processes chaotic dynamics of the Smale horseshoe type.*

In order to prove Theorem 5, we will construct an invertible map f and then verify that this f satisfies the Conley-Moser conditions.

2.1. Construction of an Invertible Map f. According to the McGehee transformation $\rho = 1/x^2$, $p_\rho = y$ [2], the Hamiltonian of the system (1) can be reformulated as follows:

$$H = \frac{1}{2}\left(y^2 + x^4 p_\varphi^2 - 2p_\varphi\right) - \frac{(1-\mu)x^2}{\left(1 + x^4\mu^2 - 2x^2\mu\cos\varphi\right)^{1/2}} - \frac{\mu x^2}{\left[1 + x^4(1-\mu)^2 + 2x^2(1-\mu)\cos\varphi\right]^{1/2}}. \tag{4}$$

Thus, the system (1) can be reformulated as

$$\begin{aligned} \frac{dx}{d\tau} &= -\frac{1}{2}x^3 y, \\ \frac{dy}{d\tau} &= p_\varphi^2 x^6 - \frac{(1-\mu)\left(1-\mu x^2\cos\varphi\right)x^4}{\left(1+\mu^2x^4 - 2\mu x^2\cos\varphi\right)^{3/2}} - \frac{\mu\left[1+(1-\mu)x^2\cos\varphi\right]x^4}{\left[1+(1-\mu)^2x^4 + 2(1-\mu)x^2\cos\varphi\right]^{3/2}}, \\ \frac{d\varphi}{d\tau} &= p_\varphi x^4 - 1, \\ \frac{dp_\varphi}{d\tau} &= \mu(1-\mu)x^4 \times \sin\varphi\left[\frac{1}{\left[1+(1-u)^2x^4 + 2(1-\mu)x^2\cos\varphi\right]^{3/2}} - \frac{1}{\left(1+\mu^2x^4 - 2\mu x^2\cos\varphi\right)^{3/2}}\right]. \end{aligned} \tag{5}$$

For the energy surface $H = h$, where h is a constant, there exists a 2π-periodic solution with respect to φ; that is, $(x, y, p_\varphi) = (0, 0, -h)$. Further, near this periodic solution, by solving the Jacobi integral for p_φ, we have $p_\varphi = -h + \nu_1(x, y, \varphi)$, where $\nu_1(x, y, \varphi)$ is second order in x and y and 2π-periodic with respect to φ.

Thus, the system (5) can be further reformulated as

$$\begin{aligned} \frac{dx}{d\tau} &= -\frac{1}{2}x^3 y, \\ \frac{dy}{d\tau} &= -(1-2\mu)\left(x^4 + g_1(x, y, \varphi, \mu)\right), \\ \frac{d\varphi}{d\tau} &= -1 + g_2(x, y, \varphi, \mu), \end{aligned} \tag{6}$$

where g_1 and g_2 are 2π-periodic with respect to φ, g_1 is the third order in (x, y), and g_2 is fourth order in (x, y).

From [4, 12], the origin $(0, 0)$ can be regarded as a periodic orbit γ_μ with period 2π with respect to φ in the system (6). Moreover, the Poincaré map of the periodic orbit $(x, y) = (0, 0)$ has the form

$$\begin{aligned} P_0 : x &\longrightarrow x + k_1 x^3\left(y + r_1(x, y)\right) \\ y &\longrightarrow y + k_2 x^3\left(x + r_2(x, y)\right), \end{aligned} \tag{7}$$

where $k_1 = \pi$, $k_2 = 2\pi(1-2\mu)$, and r_1, r_2 are real analytic and contain terms of at least second order in (x, y).

Using polar coordinates (ρ, θ), the Poincaré map P_0 can be reformulated as

$$\begin{aligned} P_0 : r &\longrightarrow r - k_1 r^4\cos^4\theta\left((4\mu - 3)\sin\theta + o(r)\right) \\ \theta &\longrightarrow \theta - k_2 r^3\cos^3\theta \times \left(\frac{1}{2(1-2\mu)}\sin^2\theta - \cos^2\theta + o(r)\right). \end{aligned} \tag{8}$$

According to formula (8), by making the following linear transformation:

$$\begin{aligned} x &= u + v, \\ y &= -\sqrt{2(1-2\mu)}(u - v), \end{aligned} \tag{9}$$

the system (6) can be reformulated as follows:

$$\begin{aligned} \frac{du}{d\tau} &= (u+v)^3 k_3 u, \\ \frac{dv}{d\tau} &= -(u+v)^3\left(k_3 v + h_1(u, v, \varphi, \mu)\right), \\ \frac{d\varphi}{d\tau} &= -1 + h_2(u, v, \varphi, \mu), \end{aligned} \tag{10}$$

where $k_3 = \sqrt{2(1-2\mu)}/2$. Due to the symmetry of the problem, we subsequently restrict our discussion to the positive quadrant.

We neglect the higher order terms of (10) and then obtain that $du/dv = -u/v$. It is clear that its solution remains on the hyperbolae $uv = c_0 > 0$, where c_0 is a constant. We substitute $v = c_0/u$ into the first expression of (10) and neglect the higher order terms, arriving at $du/d\varphi = -k_3((u^2 + c_0)^3/u^2)$.

Let Σ be a plane transversal to the periodic orbit γ_μ at the origin $(0, 0)$ and let U_0 be a sufficiently small neighborhood of the origin $(0, 0)$ in the plane Σ. For an arbitrary but fixed point $(u_0, v_0) \in U_0 \setminus \{(0, 0)\}$, we define $T_\varphi(u_0, v_0) = (u_\varphi, v_\varphi)$ with $T_0(u_0, v_0) = (u_0, v_0)$.

Assume that $u_\varphi = \sqrt{c_0}\tan(\phi_\varphi/4)$; then $v_\varphi = \sqrt{c_0}\cot(\phi_\varphi/4)$, where ϕ_φ is an auxiliary variable. Substituting $u_\varphi = \sqrt{c_0}\tan(\phi_\varphi/4)$ into $du/d\varphi = -k_3((u^2 + c_0)^3/u^2)$, we can obtain

$$\phi_\varphi - \sin\phi_\varphi = k_0 - 32kc_0^{3/2}\varphi, \quad k_0 = \phi_0 - \sin\phi_0, \tag{11}$$

where $c_0 = u_0 v_0$ and $\phi_0 = 4\arctan\sqrt{u_0/v_0}$.

Moreover, we can calculate

$$DT_{\varphi} = \begin{bmatrix} \dfrac{\partial u_{\varphi}}{\partial u_0} & \dfrac{\partial u_{\varphi}}{\partial v_0} \\ \dfrac{\partial v_{\varphi}}{\partial u_0} & \dfrac{\partial v_{\varphi}}{\partial v_0} \end{bmatrix} = \begin{bmatrix} \dfrac{u_{\varphi}}{2u_0}\left(1+\Delta-3k_3\left(u_{\varphi}+v_{\varphi}\right)^3\varphi\right) & \dfrac{u_0 u_{\varphi}}{2c}\left(1-\Delta-3k_3\left(u_{\varphi}+v_{\varphi}\right)^3\varphi\right) \\ \dfrac{c}{2u_0 u_{\varphi}}\left(1-\Delta+3k_3\left(u_{\varphi}+v_{\varphi}\right)^3\varphi\right) & \dfrac{u_0}{2u_{\varphi}}\left(1+\Delta+3k_3\left(u_{\varphi}+v_{\varphi}\right)^3\varphi\right) \end{bmatrix}, \quad (12)$$

where $\Delta = ((u_{\varphi}+v_{\varphi})/(u_0+v_0))^3$. Clearly, $\det DT_{\varphi} = \Delta \neq 0$.

For the approximate system obtained by neglecting the higher order terms in the system (10), we can describe the Poincaré map P defined over the plane Σ by using the truncated flow near the degenerate saddle as follows:

$$P : (u_0, v_0) \longmapsto (u_{2\pi}, v_{2\pi}), \quad \text{where } (u_0, v_0) \in U_0. \quad (13)$$

Since the terms neglected in (10) are both $o(u^4, v^4)$ and $O(\mu)$, we can use this Poincaré map P to approximate P_0.

Letting $u_0 = \sqrt{c_0}\tan(\phi_0/4)$ and $v_0 = \sqrt{c_0}\cot(\phi_0/4)$, then we can obtain

$$P^k(u_0, v_0) \triangleq (u_k, v_k) = \left(\sqrt{c_0}\tan\frac{\phi_{2k\pi}}{4}, \sqrt{c_0}\cot\frac{\phi_{2k\pi}}{4}\right). \quad (14)$$

For the system (10), the coordinate axis $v = 0$ corresponds to the local stable manifold $W^s_{\text{loc}}(\gamma_\mu)$ and $u = 0$ corresponds to the unstable manifold $W^u_{\text{loc}}(\gamma_\mu)$, respectively. Moreover, from [4, 12], when the mass ratio μ is sufficiently small, there exists a transversal homoclinic orbit, denoted as γ, of the periodic orbit γ_μ. Thus, there exist two points p and q such that $p \in W^s_{\text{loc}}(\gamma_\mu)$, $q \in W^u_{\text{loc}}(\gamma_\mu)$, and $p, q \in \Sigma \bigcap \gamma$. For convenience, by introducing a scale transformation, we can further assume that $p = (1, 0)$ and $q = (0, 1)$.

We define $B = \{(u, v) \mid |u - 1| \leq \delta_1, |v| \leq \delta_2\}$ and $\widetilde{B} = \{(u, v) \mid |u| \leq \delta_2, |v - 1| \leq \delta_1\}$ as the corresponding neighborhoods of p and q, respectively. For sufficiently small positive numbers δ_1 and δ_2, B and $\widetilde{B}$ satisfy $PB \bigcap B = \emptyset$, $P^{-1}\widetilde{B} \bigcap \widetilde{B} = \emptyset$. Let $D_k = P^{-k}\widetilde{B} \bigcap B$. When k is sufficiently large, $D_k \neq \emptyset$. Moreover, we also can obtain $D_k \bigcap D_m = \emptyset$ for $k \neq m$. Again let $\widetilde{D}_k = P^k D_k$. When k is sufficiently large, $\widetilde{D}_k \neq \emptyset$. The relation between D_k and $\widetilde{D}_k$ can be seen from Figure 1.

Since $p, q \in \Sigma \bigcap \gamma$, when δ_1 and δ_2 are sufficiently small, every positive half-orbit of the system (10) that starts from $\widetilde{B}$ intersects a neighborhood U_p of the point p at a point, where $U_p \subset \Sigma$. This can be depicted by the map $F : \widetilde{B} \to U_p$. It is clear that F is a C^1 diffeomorphism. Let $F(u, v) = (F_U, F_V)$. Since the stable manifold and the unstable manifold of the periodic orbit γ_μ transversally intersect along γ, we can obtain $(\partial F_V/\partial v)|_q \neq 0$.

Let $B_h = B \bigcap \{v = 0\}$, $\widetilde{B}_v = \widetilde{B} \bigcap \{u = 0\}$, $\partial B_h = \{u = 1 \pm \delta_1, v = 0\}$, and $\partial \widetilde{B}_v = \{u = 0, v = 1 \pm \delta_1\}$. Then, there exists a sufficiently small δ_1 such that $F\widetilde{B}_v \bigcap B_h = \{p\}$, $(\partial F_V/\partial v)|_{\widetilde{B}_v} \neq 0$, $F\widetilde{B}_v \bigcap \partial B_h = \emptyset$, $F\partial\widetilde{B}_v \bigcap B_h = \emptyset$. Moreover, let $\partial_v B = \{(u, v) \in B \mid u = 1 \pm \delta_1\}$ and $\partial_h \widetilde{B} = \{(u, v) \in \widetilde{B} \mid v = 1 \pm \delta_1\}$. We can further obtain that there exists a sufficiently small δ_2 such that $(\partial F_V/\partial v)|_{\widetilde{B}} \neq 0$, $F\widetilde{B} \bigcap \partial_v B = \emptyset$, $F(\partial_h \widetilde{B}) \bigcap B = \emptyset$.

Based on P and F, we construct a successor map $\Delta_k = F \circ P^k : D_k \to U_p$. Further, we define another map f over the set $\bigcup_k D_k$ such that $f|_{D_k} = \Delta_k$. Clearly, f is also a homeomorphism.

2.2. Proofs of Some Propositions for f. In order to prove that f satisfies the Conley-Moser conditions, we need to introduce one lemma and then prove four propositions in this subsection.

Lemma 6 (see [17, 18]). *Consider two invertible linear operators of $R^1 \times R^1$ into itself:*

$$I = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad J = \begin{bmatrix} \Lambda & E \\ G & M \end{bmatrix}, \quad (15)$$

where $dM \neq 0$. Let $L > 0$ be a constant such that the following conditions hold:

$$\|I\| < L, \qquad \left\|I^{-1}\right\| < L, \qquad \left|d^{-1}\right| < L, \qquad \left|EM^{-1}\right| < L. \quad (16)$$

Then, for arbitrary $0 < \mu_h < \mu_v^{-1} \ll 1$, there exists a positive constant δ_0, which is dependent on L, μ_h and μ_v, such that if the following conditions hold:

$$\left|M^{-1}\right| < \delta_0, \quad \left|\Lambda - EM^{-1}G\right| < \delta_0, \quad \left|\Lambda M^{-1}\right| < \delta_0, \quad \left|GM^{-1}\right| < \delta_0, \quad \left|cEM^{-1}\right| < \delta_0, \quad (17)$$

the linear map $A = IJ$ satisfies the (μ_h, μ_v)-cone condition.

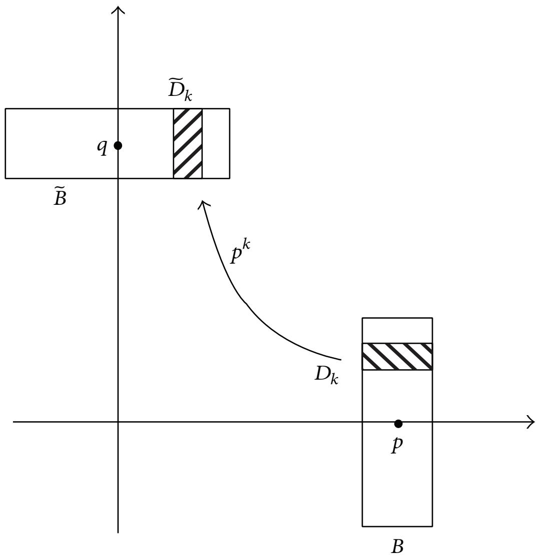

FIGURE 1: The relation between D_k and $\widetilde{D}_k$.

By Lemma 6, we have the following proposition.

Proposition 7. *For two arbitrary constants μ_h and μ_v with $0 < \mu_h < \mu_v^{-1} \ll 1$, when k is sufficiently large, $f|_{D_k}$ satisfies the (μ_h, μ_v)-cone condition.*

Proof. Based on the chain rule on the derivative of a composite function, we can obtain

$$Df|_{D_k} = DF \cdot DP^k = DF \cdot DT_\varphi = \begin{bmatrix} \dfrac{\partial F_U}{\partial u} & \dfrac{\partial F_U}{\partial v} \\ \dfrac{\partial F_V}{\partial u} & \dfrac{\partial F_V}{\partial v} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial u_\varphi}{\partial u_0} & \dfrac{\partial u_\varphi}{\partial v_0} \\ \dfrac{\partial v_\varphi}{\partial u_0} & \dfrac{\partial v_\varphi}{\partial v_0} \end{bmatrix}, \tag{18}$$

where $\varphi = 2k\pi$. Let $L = \sup_{\beta\in\widetilde{B}}\{\|DF(\beta)\|, \|DF(\beta)^{-1}\|, |\partial F_V/\partial v|^{-1}\}$. Since $\partial F_V/\partial v|_{\widetilde{B}} \neq 0$ and F is C^1, we have $L < +\infty$. Let $u_0 = \delta \approx 1$ and $v_0 = c_0/\delta \ll 1$. Then $(u_k, v_k) \approx (c_0/\delta, \delta)$.

Let $E = \partial u_\varphi/\partial v_0$, $\Lambda = \partial u_\varphi/\partial u_0$, $G = \partial v_\varphi/\partial u_0$, $M = \partial v_\varphi/\partial v_0$. Similar to the proof of Condition 1 in [19], after some simple calculations, we can obtain that $\lim_{k\to+\infty}|EM^{-1}| \approx c_0/\delta^2$, $\lim_{k\to+\infty}|M^{-1}| = 0$, $\lim_{k\to+\infty}|\Lambda - EM^{-1}G| = 0$, $\lim_{k\to+\infty}|\Lambda M^{-1}| \approx c_0/\delta^4$, $\lim_{k\to+\infty}|GM^{-1}| \approx c_0/\delta^2$, and $\lim_{k\to+\infty}|EM^{-1}| \approx c_0/\delta^2$. Further, when $c_0 \to 0$, we can obtain that $c_0/\delta^2 \to 0$, $c_0/\delta^4 \to 0$ and $(\partial F_V/\partial u)(c_0/\delta^2) \to 0$.

Thus, there exists a $\delta > 0$ such that, for sufficiently large k, inequalities (16) and (17) in Lemma 6 hold. Thus, according to Lemma 6, we obtain that when k is large enough, $f|_{D_k}$ satisfies the (μ_h, μ_v)-cone condition. □

In fact, we can further prove the following.

Proposition 8. *When k is sufficiently large, P^k satisfies the (μ_h, μ_v)-cone condition.*

Proof. Let $N \geq 2$ be an arbitrary but fixed integer. For sufficiently large k, let $H_l = D_{l+k-1}$, $V_l = f(D_{l+k-1})$, $V_{ji} = P^k H_i \bigcap H_j$, and $H_{ij} = H_i \bigcap P^{-k} H_j$, where $1 \leq l, i, j \leq N$. Moreover, let $\mathscr{H} = \bigcup_{i,j} H_{ij}$ and $\mathscr{V} = \bigcup_{i,j} V_{ij}$.

For an arbitrary point $z_0 \in \mathscr{H} \bigcup \mathscr{V}$, let (ξ_{z_0}, η_{z_0}) be a vector emanating from the point z_0 in the tangent space of z_0. In addition, for given μ_h and μ_v, let $\mathscr{S}^u_{z_0} = \{(\xi_{z_0}, \eta_{z_0}) \mid |\xi_{z_0}| < \mu_v|\eta_{z_0}|\}$ be the unstable sector at z_0 and let $\mathscr{S}^s_{z_0} = \{(\xi_{z_0}, \eta_{z_0}) \mid |\eta_{z_0}| < \mu_h|\xi_{z_0}|\}$ be the stable sector at z_0. Similar to Section 1, we also have $\mathscr{S}^u_{\mathscr{H}}$, $\mathscr{S}^u_{\mathscr{V}}$, $\mathscr{S}^s_{\mathscr{H}}$, and $\mathscr{S}^s_{\mathscr{V}}$.

In order to prove that P^k satisfies the (μ_h, μ_v)-cone condition, by Remark 4, we need to prove that P^k satisfies Assumption 2. That is, we need to prove the following:

(1) $DP^k(\mathscr{S}^u_{\mathscr{H}}) \subset \mathscr{S}^u_{\mathscr{V}}$ and $DP^{-k}(\mathscr{S}^s_{\mathscr{V}}) \subset \mathscr{S}^s_{\mathscr{H}}$;

(2) there exists a constant λ satisfying $0 < \lambda < 1 - \mu_h\mu_v$ such that $|\eta_{P^k(z_0)}| \geq \lambda^{-1}|\eta_{z_0}|$ if $(\xi_{z_0}, \eta_{z_0}) \in \mathscr{S}^u_{z_0}$ and $(\xi_{P^k(z_0)}, \eta_{P^k(z_0)}) \doteq DP^k(z_0)(\xi_{z_0}, \eta_{z_0}) \in \mathscr{S}^u_{P^k(z_0)}$, where $(\xi_{P^k(z_0)}, \eta_{P^k(z_0)})$ is a vector emanating from the point $P^k(z_0)$ in the tangent space of $P^k(z_0)$; $|\xi_{P^{-k}(z_0)}| \geq \lambda^{-1}|\xi_{z_0}|$ if $(\xi_{z_0}, \eta_{z_0}) \in \mathscr{S}^s_{z_0}$ and $(\xi_{P^{-k}(z_0)}, \eta_{P^{-k}(z_0)}) \doteq DP^{-k}(z_0)(\xi_{z_0}, \eta_{z_0}) \in \mathscr{S}^s_{P^{-k}(z_0)}$, where $(\xi_{P^{-k}(z_0)}, \eta_{P^{-k}(z_0)})$ is a vector emanating from the point $P^{-k}(z_0)$ in the tangent space of $P^{-k}(z_0)$.

First, we want to prove that $DP^k(\mathscr{S}^u_{\mathscr{H}}) \subset \mathscr{S}^u_{\mathscr{V}}$. For this, it is sufficient to prove that, for an arbitrary $z_0 = (u_0, v_0) \in \mathscr{H}$ with $(\xi_{z_0}, \eta_{z_0}) = (1, \vartheta) \in \mathscr{S}^u_{z_0}$, $(\xi_{P^k(z_0)}, \eta_{P^k(z_0)}) = DP^k(z_0)(\xi_{z_0}, \eta_{z_0}) \in \mathscr{S}^u_{\mathscr{V}}$.

Clearly, $P^k(z_0) \in \mathscr{V}$ and ϑ is bounded. According to the definitions of T_φ and P, $DP^k(z_0)(\xi_{z_0}, \eta_{z_0}) = DT_{2k\pi}(z_0)(\xi_{z_0}, \eta_{z_0}) = ((\partial u_{2k\pi}/\partial u_0) + (\partial u_{2k\pi}/\partial v_0)\vartheta, (\partial v_{2k\pi}/\partial u_0) + (\partial v_{2k\pi}/\partial v_0)\vartheta)$.

Since δ_1 and δ_2 for defining B and $\widetilde{B}$ are chosen to be sufficiently small, $u_0 \approx 1$ and $v_0 \leq \delta_2 \ll 1$. Letting $u_0 = \delta$ and $c = u_0 v_0$, then $(u_{2k\pi}, v_{2k\pi}) \approx (c/\delta, \delta)$. According to DT_φ in Section 2.1, when k is sufficiently large, we have

$$\begin{aligned} &\left|\frac{(\partial v_{2k\pi}/\partial u_0) + (\partial v_{2k\pi}/\partial v_0)\,\vartheta}{(\partial u_{2k\pi}/\partial u_0) + (\partial u_{2k\pi}/\partial v_0)\,\vartheta}\right| \\ &\quad = \frac{c}{u_{2k\pi}^2} \\ &\qquad \times \Big| \left(\Delta\left(u_0\vartheta - v_0\right) + \left(1 + 6k_3\left(u_{2k\pi} + v_{2k\pi}\right)^3 k\pi\right) \right. \\ &\qquad\qquad \left. \times \left(v_0 + u_0\vartheta\right)\right) \\ &\qquad\quad \times \left(\Delta\left(u_0\vartheta - v_0\right) - \left(1 - 6k_3\left(u_{2k\pi} + v_{2k\pi}\right)^3 k\pi\right) \right. \\ &\qquad\qquad\quad \left. \times \left(v_0 + u_0\vartheta\right)\right)^{-1} \Big| \end{aligned}$$

$$\approx \frac{\delta^2}{c}\left|\left(\Delta\left(\delta\vartheta-\frac{c}{\delta}\right)+\left(1+6k_3\left(\frac{\delta^2+c}{\delta}\right)^3 k\pi\right)\right.\right.$$
$$\left.\times\left(\delta\vartheta+\frac{c}{\delta}\right)\right)$$
$$\times\left(\Delta\left(\delta\vartheta-\frac{c}{\delta}\right)-\left(1-6k_3\left(\frac{\delta^2+c}{\delta}\right)^3 k\pi\right)\right.$$
$$\left.\left.\times\left(\delta\vartheta+\frac{c}{\delta}\right)\right)^{-1}\right|$$
$$\approx \frac{\delta^2}{c}\left|1+\frac{1}{3k_3\delta^3 k\pi}\right|. \tag{19}$$

Clearly, $\lim_{k\to+\infty}(\delta^2/c)|1+1/3k_3\delta^3k\pi| = \delta^2/c$. Moreover, when $c \to 0$, $\delta^2/c \to +\infty$. So, for sufficiently large k and sufficiently small δ_1 and δ_2, $|((\partial v_{2k\pi}/\partial u_0)+(\partial v_{2k\pi}/\partial v_0)\vartheta)/((\partial u_{2k\pi}/\partial u_0)+(\partial u_{2k\pi}/\partial v_0)\vartheta)| > 1/\mu_v$. Thus, $DP^k(z_0)(\xi_{z_0},\eta_{z_0}) \in \mathcal{S}^u_{\mathcal{V}}$. This directly implies that $DP^k(\mathcal{S}^u_{\mathcal{H}}) \subset \mathcal{S}^u_{\mathcal{V}}$.

Second, following the proof of $DP^k(\mathcal{S}^u_{\mathcal{H}}) \subset \mathcal{S}^u_{\mathcal{V}}$, we want to prove that there exists a constant λ satisfying $0<\lambda<1-\mu_h\mu_v$ such that $|\eta_{P^k(z_0)}| \geq \lambda^{-1}|\eta_{z_0}|$.

Similarly, for the above (u_0,v_0) and $(u_{2k\pi},v_{2k\pi})$, when k is sufficiently large,

$$\left|\frac{\partial v_{2k\pi}}{\partial u_0}+\frac{\partial v_{2k\pi}}{\partial v_0}\vartheta\right|$$
$$=\frac{1}{2u_{2k\pi}}\left|\Delta\left(u_0\vartheta-v_0\right)+\left(1+6k_3\left(u_{2k\pi}+v_{2k\pi}\right)^3k\pi\right)\right.$$
$$\left.\times\left(v_0+u_0\vartheta\right)\right|$$
$$\approx\frac{1}{2c}\left|\Delta\left(\delta^2\vartheta-c\right)\right.$$
$$+\left(1+6k_3\left(\frac{\delta^2+c}{\delta}\right)^3k\pi\right)$$
$$\left.\times\left(c+\delta^2\right)\right|$$
$$\approx\frac{\delta^2}{c}\left|\left(1+3k_3\delta^3k\pi\right)\vartheta\right|. \tag{20}$$

For given B and $\widetilde{B}$, $\lim_{k\to+\infty}(\delta^2/c)|1+3k_3\delta^3k\pi| = \infty$. So, for any constant λ satisfying $0<\lambda<1-\mu_h\mu_v$, when k is sufficiently large, $(\delta^2/c)|1+3k_3\delta^3k\pi| > \lambda^{-1}$. Thus, we have $|(\partial v_{2k\pi}/\partial u_0)+(\partial v_{2k\pi}/\partial v_0)\vartheta| > \lambda^{-1}|\vartheta|$, implying that $|\eta_{P^k(z_0)}| \geq \lambda^{-1}|\eta_{z_0}|$.

Third, we want to prove $DP^{-k}(\mathcal{S}^s_{\mathcal{V}}) \subset \mathcal{S}^s_{\mathcal{H}}$ and $|\xi_{P^{-k}(z_0)}| \geq \lambda^{-1}|\xi_{z_0}|$. For this, let $T^{-1}_{\varphi}(u_\varphi,v_\varphi)=(u_0,v_0)$ be the inverse map of T_φ. Then,

$$DT^{-1}_{\varphi}=\left(DT_\varphi\right)^{-1}=\begin{bmatrix}\dfrac{\partial u_\varphi}{\partial u_0} & \dfrac{\partial u_\varphi}{\partial v_0}\\ \dfrac{\partial v_\varphi}{\partial u_0} & \dfrac{\partial v_\varphi}{\partial v_0}\end{bmatrix}^{-1}$$
$$=\frac{1}{\triangle}\begin{bmatrix}\dfrac{\partial v_\varphi}{\partial v_0} & -\dfrac{\partial u_\varphi}{\partial v_0}\\ -\dfrac{\partial v_\varphi}{\partial u_0} & \dfrac{\partial u_\varphi}{\partial u_0}\end{bmatrix}. \tag{21}$$

For any $z_0=(u_0,v_0)\in\mathcal{V}$ with $(\xi_{z_0},\eta_{z_0})=(\varpi,1)\in\mathcal{S}^s_{z_0}$, $(\xi_{P^{-k}(z_0)},\eta_{P^{-k}(z_0)}) = DP^{-k}(\xi_{z_0},\eta_{z_0}) = DT^{-1}_{2k\pi}(\xi_{z_0},\eta_{z_0}) = (1/\triangle)((\partial v_{2k\pi}/\partial v_0)\varpi - (\partial u_{2k\pi}/\partial v_0), (-\partial v_{2k\pi}/\partial u_0)\varpi + (\partial u_{2k\pi}/\partial u_0))$. Similarly, we can prove that $|(\partial v_{2k\pi}/\partial v_0)\varpi - (\partial u_{2k\pi}/\partial v_0)/(-\partial v_{2k\pi}/\partial u_0)\varpi + (\partial u_{2k\pi}/\partial u_0)| > \mu_h$ and $|(\partial v_{2k\pi}/\partial v_0)\varpi - (\partial u_{2k\pi}/\partial v_0)| > \lambda^{-1}|\varpi|$. Thus, $DP^{-k}(\mathcal{S}^s_{\mathcal{V}}) \subset \mathcal{S}^s_{\mathcal{H}}$ and $|\xi_{P^{-k}(z_0)}| \geq \lambda^{-1}|\xi_{z_0}|$.

Based on all above analysis and Remark 4, we can then obtain that when k is sufficiently large, P^k satisfies the (μ_h,μ_v)-cone condition. □

Based on Proposition 8, we can prove that f satisfies the boundary condition.

Proposition 9. *When i and j are sufficiently large, $f(\partial_h D_i)\bigcap D_j = \emptyset$ and $fD_i\bigcap\partial_v D_j = \emptyset$, where $\partial_h D_i$ is the horizontal boundary of D_i and $\partial_v D_j$ is the vertical boundary of D_j.*

Proof. Due to Proposition 8, when k is sufficiently large, P^k satisfies the (μ_h,μ_v)-cone condition. This implies that P^k contracts in the horizontal direction and expands in the vertical direction. Moreover, μ_v-vertical curves are mapped to μ_v-vertical curves under the map P^k and μ_h-horizontal curves are mapped to μ_h-horizontal curves under the map P^{-k}. Thus, for sufficiently large i and j, $D_j\subset B$, $P^iD_i\subset\widetilde{B}$, $\partial_vD_j\subset\partial_vB$ and $P^i(\partial_hD_i)\subset\partial_h\widetilde{B}$. In addition, $f(\partial_hD_i)\bigcap D_j = (F\circ P^i(\partial_hD_i))\bigcap D_j = (F\circ(P^i(\partial_hD_i)))\bigcap D_j$. Thus, according to the expression $F(\partial_h\widetilde{B})\bigcap B=\emptyset$ in Section 2.1, $f(\partial_hD_i)\bigcap D_j = \emptyset$. Similarly, since $F\widetilde{B}\bigcap\partial_vB = \emptyset$, we can obtain that $fD_i\bigcap\partial_vD_j=\emptyset$. □

Finally, we can prove that f satisfies the intersection condition as follows.

Proposition 10. *When i and j are sufficiently large, $fD_i\bigcap D_j\neq\emptyset$.*

Proof. Let $C^i_v(u)=\{u\}\times(D_i)_v$ be the family of vertical curves in D_i, where $u\in B_h$ and $(D_i)_v=\{v \mid (u,v)\in B, P^i(u,v)\in\widetilde{B}\}$. From Proposition 8, P^i with sufficiently large i satisfies the (μ_h,μ_v)-cone condition. Thus, $P^iC^i_v(u)$ infinitely approaches

$\widetilde{B}_v$ when $i \to +\infty$. Similarly, letting $C_h^i(v) = B_h \times \{v\}$ be the family of horizontal curves in D_i, where $v \in (D_i)_v$, we can obtain that $C_h^j(v)$ infinitely approaches B_h when $j \to +\infty$.

Since F is C^1, $F(P^i C_v^i(u))$ infinitely approaches $F(\widetilde{B}_v)$ when $i \to +\infty$. By the expression $F\widetilde{B}_v \bigcap B_h = \{p\}$ in Section 2.1, $F(P^i C_v^i(u)) \bigcap C_h^j(v) \neq \emptyset$. Thus, $fD_i \bigcap D_j \neq \emptyset$. □

Remark 11. In fact, we can prove that when $i, j \to +\infty$, $fC_v^i(u)$ and $C_h^j(v)$ intersect at a unique point near p.

2.3. Proof of Our Theorem 5. In order to prove our Theorem 5, similar to [19], we try to use Propositions 7, 9, and 10 to verify that f satisfies Assumptions 1 and 2. Then, from Lemma 3, we can obtain that f is a horseshoe map as follows.

Proof. From Proposition 7, when k is sufficiently large, $f|_{D_k}$ satisfies the (μ_h, μ_v)-cone condition. Thus, the map f contracts in the horizontal direction and expands in the vertical direction. Moreover, μ_v-vertical curves are mapped to μ_v-vertical curves under the map f and μ_h-horizontal curves are mapped to μ_h-horizontal curves under the map f^{-1}. Therefore, from Propositions 9 and 10, for sufficiently large i and j, $fD_i \bigcap D_j \neq \emptyset$ is a (μ_h, μ_v)-curved rectangle and satisfies $\partial_h(fD_i \bigcap D_j) \subset \partial_h D_j$ and $\partial_v(fD_i \bigcap D_j) \subset \partial_v(fD_i)$. Similarly, for sufficiently large i and j, $f^{-1}(fD_i \bigcap D_j) \neq \emptyset$ is also a (μ_h, μ_v)-curved rectangle and satisfies $\partial_v f^{-1}(fD_i \bigcap D_j) \subset \partial_v D_i$ and $\partial_h f^{-1}(fD_i \bigcap D_j) \subset \partial_h(f^{-1}D_j)$.

Let $N \geq 2$ be an arbitrary but fixed positive integer. For sufficiently large k, i, and j, by letting $H_l = D_{l+k-1}$, $V_l = f(D_{l+k-1})$, $fD_i \bigcap D_j = V_{(j-k+1)(i-k+1)}$, and $D_i \bigcap f^{-1}D_j = H_{(i-k+1)(j-k+1)}$, where $1 \leq l \leq N$, we can obtain that f satisfies Assumption 1. In addition, due to Remark 4, f obviously satisfies Assumption 2.

Thus, for an arbitrary but fixed $N \geq 2$, when k is sufficiently large, the map f over the set $\bigcup_{i=0}^{N-1} D_{k+i}$ satisfies Assumptions 1 and 2; that is, f satisfies the Conley-Moser conditions.

By Lemma 3, when k is sufficiently large, f has an invariant Cantor set, on which it is topologically conjugate to a full shift on N symbols. This directly implies that f is a horseshoe map. □

3. Conclusions

In this present note, we studied the existence of a Smale horseshoe in a planar circular restricted three-body problem by first defining an invertible map f and then proving that this f satisfies the Conley-Moser conditions. This implies that the planar circular restricted three-body problem processes chaotic dynamics of the Smale horseshoe type.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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