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Research Article

On Distance (r, k)-Fibonacci Numbers and Their Combinatorial and Graph Interpretations

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We introduce three new two-parameter generalizations of Fibonacci numbers. These generalizations are closely related to k-distance Fibonacci numbers introduced recently. We give combinatorial and graph interpretations of distance (r, k)-Fibonacci numbers. We also study some properties of these numbers.

1. Introduction

In general we use the standard terminology of the combinatorics and graph theory; see [1]. The well-known Fibonacci sequence $\{F_n\}$ is defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = F_1 = 1$. The Fibonacci numbers have been generalized in many ways, some by preserving the initial conditions and others by preserving the recurrence relation. For example, in [2] k-Fibonacci numbers were introduced and defined recurrently for any integer $k \geq 1$ by $F(k, n) = kF(k, n-1) + F(k, n-2)$ for $n \geq 2$ with $F(k, 0) = 0$, $F(k, 1) = 1$. In [3] the following generalization of the Fibonacci numbers was defined: $x_n = 2^r x_{n-1} + x_{n-2}$ for an integer $r \geq 0$ such that $4^{r-1} + 1 \neq 0$ and $n \geq 2$ with $x_0 = 0$ and $x_1 = 1$. Other interesting generalizations of Fibonacci numbers are presented in [4, 5]. In the literature there are different kinds of distance generalizations of F_n. They have many graph interpretations closely related to the concept of k-independent sets. We recall some of such generalizations:

(1) Reference [6]. Consider $F(k, n) = F(k, n-1) + F(k, n-k)$ for $n \geq k+1$ with $F(k, n) = n+1$ for $n \leq k$.

(2) References [4, 7, 8]. Consider Fibonacci p-numbers $F_p(n) = F_p(n-1) + F_p(n-p-1)$ for any given p ($p = 1, 2, 3, \ldots$) and $n > p+1$ with $F_p(0) = 0$ and $F_p(n) = 1$ for $1 \leq n \leq p+1$.

(3) Reference [9]. Consider $Fd^{(1)}(k, n) = Fd^{(1)}(k, n-k+1) + Fd^{(1)}(k, n-k)$ for $n \geq k$ with $Fd^{(1)}(k, n) = 1$ for $n \leq k-1$.

(4) Reference [9]. Consider $Fd^{(2)}(k, n) = Fd^{(2)}(k, n-k+1) + Fd^{(2)}(k, n-k)$ for $n \geq k$ with $Fd^{(2)}(k, n) = 0$ for $n = 0, \ldots, k-2$, $Fd^{(2)}(k, k-1) = 1$, $Fd^{(2)}(1, 1) = 1$, $Fd^{(2)}(2, 2) = 2$, for $k \geq 3$ $Fd^{(2)}(k, k) = 1$.

(5) Reference [9]. Consider $Fd^{(3)}(k, n) = Fd^{(3)}(k, n-k+1) + Fd^{(3)}(k, n-k)$ for $n \geq 2k-1$ with $Fd^{(3)}(k, n) = 1$ for $n = 0, \ldots, k-1$, $Fd^{(3)}(2, 2) = 2$, for $k \geq 3$ $Fd^{(3)}(k, k) = Fd^{(3)}(k, 2k-2) = 3$, for $k+1 \leq n \leq 2k-1$ $Fd^{(3)}(k, n) = 4$.

(6) Reference [10]. Consider $F_2^{(1)}(k, n) = F_2^{(1)}(k, n-2) + F_2^{(1)}(k, n-k)$ for $n \geq k+1$ with

$$F_2^{(1)}(k, n) = \begin{cases} 1 & \text{if } n \leq k-1 \text{ or } n = k = 1, \\ 2 & \text{if } n = k \geq 2. \end{cases} \quad (1)$$

Table 1: Distance (r,k)-Fibonacci numbers $F_r^{(\mathrm{I})}(k,n)$ of the first kind.

$k\setminus n$	0	1	2	3	4	5	6	7	8
1	1	1	$r+1$	$2r+1$	r^2+3r+1	r^2+5r+2	r^3+4r^2+6r+2	$2r^3+9r^2+8r+2$	$r^4+6r^3+15r^2+10r+2$
2	1	1	$2r$	$2r$	$4r^2$	$4r^2$	$8r^3$	$8r^3$	$16r^4$
3	1	1	r	r^2+r	$2r^2$	$2r^3+r^2$	r^4+3r^3	$4r^4+r^3$	$3r^5+4r^4$
4	1	1	r	r	r^3+r^2	r^3+r^2	$2r^4+r^3$	$2r^4+r^3$	$r^6+3r^5+r^4$
5	1	1	r	r	r^2	r^4+r^2	r^4+r^3	$2r^5+r^3$	$2r^5+r^4$
6	1	1	r	r	r^2	r^2	r^5+r^3	r^5+r^3	$2r^6+r^4$
7	1	1	r	r	r^2	r^2	r^3	r^6+r^3	r^6+r^4

(7) Reference [11]. Consider $F_2^{(2)}(k,n) = F_2^{(2)}(k,n-2) + F_2^{(2)}(k,n-k)$ for $n \geq k+1$ with

$$F_2^{(2)}(k,n) = \begin{cases} 0 & \text{if } n \text{ is odd and } n \leq k-1, \\ 1 & \text{if } n \text{ is even and } n \leq k-1, \end{cases}$$
$$F_2^{(2)}(k,k) = \begin{cases} 0 & \text{if } k = 1, \\ 1 & \text{if } k \text{ is odd and } k \geq 3, \\ 2 & \text{if } k \text{ is even.} \end{cases} \quad (2)$$

(8) Reference [11]. Consider $F_2^{(3)}(k,n) = F_2^{(3)}(k,n-2) + F_2^{(3)}(k,n-k)$ for $n \geq k+1$ with

$$F_2^{(3)}(k,n) = \begin{cases} 1 & \text{if } n \text{ is even and } n \leq k-1, \\ 2 & \text{if } n \text{ is odd and } n \leq k-1, \end{cases}$$
$$F_2^{(3)}(k,k) = \begin{cases} 3 & \text{if } k \text{ is odd and } k \geq 3, \\ 2 & \text{if } k \text{ is even or } k = 1. \end{cases} \quad (3)$$

In this paper we introduce three new two-parameter generalizations of distance Fibonacci numbers. They are closely related with the numbers $F_2^{(j)}(k,n)$, $j = 1,2,3$, presented in [10, 11]. We show their combinatorial and graph interpretations and we present some identities for them.

2. Distance (r,k)-Fibonacci Numbers

Let $k \geq 1$, $n \geq 0$, and $r \geq 1$ be integers. We define distance (r,k)-Fibonacci numbers of the first kind $F_r^{(\mathrm{I})}(k,n)$ by the recurrence relation

$$F_r^{(\mathrm{I})}(k,n) = rF_r^{(\mathrm{I})}(k,n-2) + r^{k-1}F_r^{(\mathrm{I})}(k,n-k) \quad \text{for } n \geq k+1 \quad (4)$$

with the following initial conditions:

$$\begin{aligned} F_r^{(\mathrm{I})}(k,0) &= F_r^{(\mathrm{I})}(k,1) = 1, \\ F_r^{(\mathrm{I})}(k,n) &= r^{[n/2]} \quad \text{for } n = 2,3,\dots,k-2, \\ F_r^{(\mathrm{I})}(k,k-1) &= r^{[(k-1)/2]} \quad \text{for } k \geq 3, \\ F_r^{(\mathrm{I})}(k,k) &= r^{k-1} + r^{[k/2]} \quad \text{for } k \geq 2. \end{aligned} \quad (5)$$

For $r = 1$ we get $F_1^{(\mathrm{I})}(k,n) = F_2^{(1)}(k,n)$. These numbers were introduced in [10].

If $r = 1$ and $k = 1$, then $F_1^{(\mathrm{I})}(1,n)$ gives the Fibonacci numbers F_n. For $r = 1$ and $k = 3$ the numbers $F_1^{(\mathrm{I})}(3,n)$ are the well-known Padovan numbers.

Table 1 includes the values of $F_r^{(\mathrm{I})}(k,n)$ for special values of k and n.

Let $k \geq 1$, $n \geq 0$, and $r \geq 1$ be integers. We define the distance (r,k)-Fibonacci numbers of the second kind $F_r^{(\mathrm{II})}(k,n)$ by the following recurrence relation:

$$F_r^{(\mathrm{II})}(k,n) = rF_r^{(\mathrm{II})}(k,n-2) + r^{k-1}F_r^{(\mathrm{II})}(k,n-k) \quad \text{for } n \geq k+1 \quad (6)$$

with initial conditions

$$F_r^{(\mathrm{II})}(k,n) = \begin{cases} r^{[n/2]} & \text{for even } n, \\ 0 & \text{for odd } n, \end{cases} \quad \text{for } n = 0,1,\dots,k-1$$
$$F_r^{(\mathrm{II})}(k,k) = \begin{cases} 0 & \text{for } k = 1, \\ r^{k-1} & \text{for odd } k,\ k \geq 3, \\ r^{k-1} + r^{k/2} & \text{for even } k. \end{cases} \quad (7)$$

For $r = 1$ we have then $F_1^{(\mathrm{II})}(k,n) = F_2^{(2)}(k,n)$; see [10]. Moreover, for $r = 1$ and $k = 1$, $n \geq 2$ $F_1^{(\mathrm{II})}(k,n) = F_{n-2}$.

In Table 2 a few first words of the distance (r,k)-Fibonacci numbers of the second kind $F_r^{(\mathrm{II})}(k,n)$ for special values of k and n are presented.

Let $k \geq 1$, $n \geq 0$, and $r \geq 1$ be integers. We define distance (r,k)-Fibonacci numbers of the third kind $F_r^{(\mathrm{III})}(k,n)$ by the following recurrence relation:

$$F_r^{(\mathrm{III})}(k,n) = rF_r^{(\mathrm{III})}(k,n-2) + r^{k-1}F_r^{(\mathrm{III})}(k,n-k) \quad \text{for } n \geq k+1 \quad (8)$$

Table 2: Distance (r,k)-Fibonacci numbers $F_r^{(\mathrm{II})}(k,n)$ of the second kind.

$k \setminus n$	0	1	2	3	4	5	6	7	8
1	1	0	r	r	r^2+r	$2r^2+r$	r^3+3r^2+r	$3r^3+4r^2+r$	$r^4+6r^3+5r^2+r$
2	1	0	$2r$	0	$4r^2$	0	$8r^3$	0	$16r^4$
3	1	0	r	r^2	r^2	$2r^3$	r^4+r^3	$3r^4$	$3r^5+r^4$
4	1	0	r	0	r^3+r^2	0	$2r^4+r^3$	0	$r^6+3r^5+r^4$
5	1	0	r	0	r^2	r^4	r^3	$2r^5$	r^4
6	1	0	r	0	r^2	0	r^5+r^3	0	$2r^6+r^4$
7	1	0	r	0	r^2	0	r^3	r^6	r^4

with initial conditions

$$F_r^{(\mathrm{III})}(1,1)=2,$$

$$F_r^{(\mathrm{III})}(k,n)=\begin{cases} r^{n/2} & \text{for even } n,\\ 2r^{\lfloor n/2\rfloor} & \text{for odd } n,\end{cases} \quad \text{for } n=0,1,\ldots,k-1 \tag{9}$$

$$F_r^{(\mathrm{III})}(k,k)=\begin{cases} r^{k-1}+r^{k/2} & \text{for even } k,\\ r^{k-1}+2r^{\lfloor k/2\rfloor} & \text{for odd } k\ge 3.\end{cases}$$

For $r=1$ we get $F_1^{(\mathrm{III})}(k,n)=F_2^{(3)}(k,n)$. These numbers were introduced in [11]. For $r=1$, $k=1$, and $n\ge 0$ we have $F_1^{(\mathrm{III})}(1,n)=F_{n+1}$. Moreover, for $r=1$, $k=4$, and $n\ge 1$ $F_1^{(\mathrm{III})}(4,2n)=F_n$.

Table 3 includes a few initial words of distance $F_r^{(\mathrm{III})}(k,n)$ for special values of k and n.

By the definition of distance (r,k)-Fibonacci numbers of three kinds we get for $k\ge 1$ and $n\ge 0$ the following relations:

$$F_r^{(\mathrm{III})}(k,n)=2F_r^{(\mathrm{I})}(k,n) \quad \text{for even } k \text{ and odd } n,$$

$$F_r^{(\mathrm{I})}(k,n)=F_r^{(\mathrm{II})}(k,n)=F_r^{(\mathrm{III})}(k,n) \quad \text{for even } k \text{ and even } n, \tag{10}$$

$$F_r^{(\mathrm{II})}(k,n)=0 \quad \text{for even } k \text{ and odd } n.$$

3. Combinatorial and Graph Interpretations of Distance (r,k)-Fibonacci Numbers

In this section we present some combinatorial and graph interpretations of distance (r,k)-Fibonacci numbers. The classical Fibonacci numbers have many combinatorial interpretations. One of them is the interpretation related to set decomposition. We recall it. Let $X=\{1,2,\ldots,n\}$, $n\ge 1$, and $\mathcal{Y}^\star=\{Y_t^\star : t\in T\}$ be a family of disjoint subsets of X such that

(1) $|Y_t^\star|\in\{1,2\}$,

(2) if $|Y_t^\star|=2$ then $Y_t^\star$ contains two consecutive integers,

(3) $X\setminus\bigcup_{t\in T}Y_t^\star=\emptyset$.

It is well known that the number of all families $\mathcal{Y}^\star$ is equal to the classical Fibonacci numbers F_n. We introduce analogous interpretation of distance (r,k)-Fibonacci numbers.

Let $r\ge 1$ and $X=\{1,2,\ldots,n\}$, $n\ge 2$, be the set of n integers. Let $k\ge 3$. Assume that $\mathcal{R}_n$ is a multifamily of two-element subsets of X such that

$$\mathcal{R}_n=\Big\{\underbrace{\{1,2\},\{1,2\},\ldots,\{1,2\}}_{r\text{-times}},\underbrace{\{2,3\},\ldots,\{2,3\}}_{r\text{-times}},\ldots,\underbrace{\{n-1,n\},\ldots,\{n-1,n\}}_{r\text{-times}}\Big\}. \tag{11}$$

For fixed t, $1\le t\le n-k$ by $\mathcal{R}(k,t)$ we denote a subfamily of $\mathcal{R}_n$ such that $\mathcal{R}(k,t)=\{\{t+j,t+j+1\} : j=0,1,\ldots,k-2,\ t=1,2,\ldots,n-k+1\}$. Analogously for fixed t' we define $\mathcal{R}(k,t')=\{\{t,t+1\} : t=1,2,\ldots,n-1\}$.

Let $\mathcal{R}_{t,t'}^{(j)}$, $j=1,2,3$, be a subfamily of $\mathcal{R}_n$ such that $\mathcal{R}_{t,t'}^{(j)}=\mathcal{R}(k,t)\cup\mathcal{R}(k,t')$ and

(a) for each $R(k,t_1'),R(k,t_2')\in\mathcal{R}(k,t')$, $t_1'\ne t_2'$ holds $|t_1'-t_2'|\ge 2$, for each $R(k,t_1),R(k,t_2)\in\mathcal{R}(k,t)$, $t_1\ne t_2$, holds $|t_1-t_2|\ge k$,

(b) for each $R_t\in\mathcal{R}_{t,t'}^{(j)}$ holds $|R_t|\in\{2,k\}$ for $t\in T$

and exactly one of the following conditions for $R_t\in\mathcal{R}_{t,t'}^{(j)}$ and $j=\mathrm{I},\mathrm{II},\mathrm{III}$, respectively, is satisfied:

(c1) $X\setminus\bigcup_{t\in T}R_t=\emptyset$ or $X\setminus\bigcup_{t\in T}R_t=\{n\}$,

(c2) $X\setminus\bigcup_{t\in T}R_t=\emptyset$,

(c3) $|X\setminus\bigcup_{t\in T}R_t|\in\{0,1\}$ and if $p\in X\setminus\bigcup_{t\in T}R_t$ then either $p=1$ or $p=n$.

Assume that the condition (c1) is satisfied. Then the subfamily $\mathcal{R}_{t,t'}^{(1)}$ we will call a decomposition with repetitions of the set X with the rest at the end.

Assume that the condition (c2) is satisfied. Then the subfamily $\mathcal{R}_{t,t'}^{(2)}$ we will call a perfect decomposition with repetitions of the set X.

Assume that the condition (c3) is satisfied. Then the subfamily $\mathcal{R}_{t,t'}^{(3)}$ we will call a decomposition with repetitions of the set X with the rest at the end or at the beginning.

Theorem 1. *Let $k\ge 3$, $n\ge 2$, and $r\ge 1$ be integers. Then the number of all decompositions with repetitions of the set X with the rest at the end is equal to the number $F_r^{(I)}(k,n)$.*

Table 3: Distance (r,k)-Fibonacci numbers $F_r^{(III)}(k,n)$ of the third kind.

$k\setminus n$	0	1	2	3	4	5	6	7	8
1	1	2	$r+2$	$3r+2$	r^2+5r+2	$4r^2+7r+2$	r^3+9r^2+9r+2	$5r^3+9r^2+16r+4$	$r^4+14r^3+18r^2+18r+4$
2	1	2	$2r$	$4r$	$4r^2$	$8r^2$	$8r^3$	$16r^3$	$16r^4$
3	1	2	r	r^2+2r	$3r^2$	$2r^3+2r^2$	r^4+5r^3	$5r^4+2r^3$	$3r^5+7r^4$
4	1	2	r	$2r$	r^3+r^2	$2r^3+2r^2$	$2r^4+r^3$	$4r^4+2r^3$	$r^6+3r^5+r^4$
5	1	2	r	$2r$	r^2	r^4+2r^2	$2r^4+r^3$	$2r^5+2r^3$	$4r^5+r^4$
6	1	2	r	$2r$	r^2	$2r^2$	r^5+r^3	$2r^5+2r^3$	$2r^6+r^4$
7	1	2	r	$2r$	r^2	$2r^2$	r^3	r^6+2r^3	$2r^6+2r^4$

Proof (induction on n). Let $k \geq 3$, $n \geq 2$, and $r \geq 1$ be integers. Let $X = \{1,2,\ldots,n\}$. Denote by $d(n)$ the number of all decompositions with repetitions of X with the rest at the end. Let $n = 2$. Then it is easily seen that there are exactly r decompositions of X. Thus we get $d(2) = r = F_r^{(I)}(k,2)$. Let $n \geq 3$. Assume that equality $d(n) = F_r^{(I)}(k,n)$ holds for an arbitrary n. We will show that $d(n+1) = F_r^{(I)}(k,n+1)$.

Let $d^2(n+1)$ and $d^k(n+1)$ denote the number of all decompositions R with repetitions of the set $X = \{1,2,\ldots,n+1\}$ with the rest at the end such that $\{1,2\} \in R$ and $\{1,2,\ldots,k\} \in R$, respectively. It is easily seen that

$$d(n+1) = d^2(n+1) + d^k(n+1). \tag{12}$$

Moreover, we get

$$\begin{gathered} d^2(n+1) = d^k(n-1), \\ d^k(n+1) = d^k(n+1-k). \end{gathered} \tag{13}$$

By the induction hypothesis and by recurrence (4) we obtain

$$\begin{aligned} d(n+1) &= d(n-1) + d(n+1-k) \\ &= F_r^{(I)}(k,n-1) + F_r^{(I)}(k,n+1-k) \\ &= F_r^{(I)}(k,n+1), \end{aligned} \tag{14}$$

which ends the proof. □

Analogously as Theorem 1 we can prove the following.

Theorem 2. *Let $k \geq 3$, $n \geq 2$, and $r \geq 1$ be integers. Then the number of all perfect decompositions with repetitions of the set X is equal to the number $F_r^{(II)}(k,n)$.*

Theorem 3. *Let $k \geq 3$, $n \geq 2$, and $r \geq 1$ be integers. Then the number of all decompositions with repetitions of the set X with the rest at the end or at the beginning is equal to the number $F_r^{(III)}(k,n)$.*

Distance (r,k)-Fibonacci numbers of three kinds have a graph interpretation, too. It is connected with k-distance H-matchings in graphs. We recall the definition of a k-distance H-matching. Let G and H be any two graphs, let $k \geq 1$ be an integer, and a k-distance H-matching M of G is a subgraph of G such that all connected components of M are isomorphic to H and for each two components H_1 and H_2 from M for each $x \in V(H_1)$ and $y \in V(H_2)$ holds $d_G(x,y) \geq k$. In case of $k = 1$ and $H = K_2$ we obtain the definition of matching in classical sense. If M covers the set $V(G)$ (i.e., $V(M) = V(G)$), then we say that M is a perfect matching of G. For $k = 2$ and $H = K_1$ the definition of k-distance H-matchings reduces to the definition of an independent set of a graph G. In the literature the generalization of H-matching of a graph G is considered, too. For a given collection $\mathscr{H} = H_1, H_2, \ldots, H_n$ of graphs a $\mathscr{H}$-matching $\mathscr{M}$ of G is a family of subgraphs of G such that each connected component of $\mathscr{M}$ is isomorphic to some H_i, $1 \leq i \leq n$. Moreover, the empty set is a $\mathscr{H}$-matching of G, too. If $H_i = H$ for all $i = 1,2,\ldots,n$, then we obtain the definition of H-matching.

Among $\mathscr{H}$-matchings we consider such $\mathscr{H}$-matchings, where H_i, $i = 1,2,\ldots,n$, belong to the same class of graphs, namely, 2-vertex or k-vertex paths (P_2 and P_k, resp.), $k \geq 3$.

Consider a multipath P_n^r, where $n \geq 2$, $r \geq 1$, $V(P_n^r) = \{x_1, x_2, \ldots, x_n\}$, and

$$\begin{aligned} E(P_n^r) = \Bigl\{ &\underbrace{\{x_1,x_2\},\ldots,\{x_1,x_2\}}_{r\text{-times}}, \\ &\underbrace{\{x_2,x_3\},\ldots,\{x_2,x_3\}}_{r\text{-times}}, \ldots, \\ &\underbrace{\{x_{n-1},x_n\},\ldots,\{x_{n-1},x_n\}}_{r\text{-times}} \Bigr\}. \end{aligned} \tag{15}$$

Let $n \geq 2$, $k \geq 3$, and $r \geq 1$ be integers. In the graph terminology the number $F_r^{(I)}(k,n)$ is equal to the number of special $\{P_2, P_k\}$-matchings M of the multipath P_n^r such that at most one vertex, namely, x_n, does not belong to a $\{P_2, P_k\}$-matching of the graph P_n^r. We will call such matchings M a quasi-perfect matching of P_n^r. The number $F_r^{(II)}(k,n)$ is equal to the number of such $\{P_2, P_k\}$-matchings of P_n^r that both vertex x_1 and vertex x_n belong to some $\{P_2, P_k\}$-matchings M and M', respectively, of the graph P_n^r. In other words the number $F_r^{(II)}(k,n)$ is equal to all perfect $\{P_2, P_k\}$-matchings M of the graph P_n^r.

The number $F_r^{(III)}(k,n)$ is equal to the number of special $\{P_k, P_2\}$-matchings of the multipath P_n^r such that at most one vertex either vertex x_1 or x_n does not belong to a $\{P_2, P_k\}$-matching of the graph P_n^r.

Let $\sigma(P_n^r)$ be the number of all perfect $\{P_2, P_k\}$-matchings M of the graph P_n^r.

Theorem 4. *Let $r \geq 1$, $k \geq 3$, and $n \geq 2$ be integers. Then $\sigma(P_n^r) = F_r^{(II)}(k,n)$.*

Proof. Consider a multipath P_n^r where vertices from $V(P_n^r) = \{x_1, x_2, \ldots, x_n\}$ are numbered in the natural fashion. Let $\sigma_k(n)$ and $\sigma_2(n)$ be the number of perfect $\{P_2, P_k\}$-matchings M of P_n^r such that $x_n, x_{n-1} \in V(M)$ and $x_n, x_{n-1}, \ldots, x_{n-k} \in V(M)$, respectively. It is easily seen that $\sigma_k(n) + \sigma_2(n) = \sigma(P_n^r)$.

Let M be an arbitrary perfect $\{P_2, P_k\}$-matching of P_n^r, $k \geq 3$. Consider two cases:

(1) $\{x_{n-1}, x_n\} \in E(P_k)$, where $P_k \in M$.

Then we can choose the edge $\{x_{n-1}, x_n\}$ on r ways. Moreover, $M = M' \cup \{P_k\}$, where M' is an arbitrary $\{P_2, P_k\}$-matching of the graph $P_n^r \setminus \{x_n, x_{n-1}, \ldots, x_{n-k+1}\}$ which is isomorphic to the multipath P_{n-k}^r. Hence $\sigma_k(n) = r^{k-1}\sigma(P_{n-k}^r)$.

(2) $\{x_{n-1}, x_n\} \in E(P_2)$, where $P_2 \in M$.

Proving analogously as in case (1) we obtain $\sigma_2(n) = r\sigma(P_{n-2}^r)$.

Consequently

$$\sigma(P_n^r) = \sigma_k(n) + \sigma_2(n) = r^{k-1}\sigma(P_{n-k}^r) + r\sigma(P_{n-2}^r). \quad (16)$$

Claim

$$\sigma(P_n^r) = r^{k-1}F_r^{(II)}(k, n-k) + rF_r^{(II)}(k, n-2). \quad (17)$$

Proof. Assume now that the set $X = \{1, 2, \ldots, n\}$ corresponds to $V(P_n^r)$ with the numbering in the natural fashion. Let $\mathcal{R}(t, t') = \{R_t : t \in T\} \cup \{R'_t : t' \in T\}$ be a multifamily of X which gives a perfect decomposition of the set X. Then every R_t and $R_{t'}$ correspond to subgraph $P_{|R_t|}$ and $P_{|R'_t|}$ for $t, t' \in T$, respectively, of P_n^r. By Theorem 2 we get

$$\begin{aligned}\sigma(P_n^r) &= \sigma_k(n) + \sigma_2(n) \\ &= r^{k-1}F_r^{(II)}(k, n-k) + rF_r^{(II)}(k, n-2).\end{aligned} \quad (18)$$

Moreover, by (6) we obtain $\sigma(P_n^r) = F_r^{(II)}(k,n)$, which ends the proof. □

Analogously we can prove combinatorial interpretations of numbers $F_r^{(I)}(k,n)$ and $F_r^{(III)}(k,n)$.

4. Identities for Distance (r,k)-Fibonacci Numbers

In this section we give some identities and some relations between distance (r,k)-Fibonacci numbers of three types.

Theorem 5. *For $k \geq 1$, $n \geq 2k-2$, and $j = I, II, III$,*

$$\begin{aligned}F_r^{(j)}(k,n) &= rF_r^{(j)}(k, n-2) + r^{k-2}F_r^{(j)}(k, n-k+2) \\ &\quad - r^{2k-3}F_r^{(j)}(k, n-2k+2).\end{aligned} \quad (19)$$

Proof. We give the proof for distance (r,k)-Fibonacci numbers of the first kind. By the definition of numbers $F_r^{(I)}(k,n)$, we have

$$\begin{aligned}&rF_r^{(I)}(k, n-2) + r^{k-2}F_r^{(I)}(k, n-k+2) \\ &\quad - r^{2k-3}F_r^{(I)}(k, n-2k+2) = rF_r^{(I)}(k, n-2) \\ &\quad + r^{k-2}\left(rF_r^{(I)}(k, n-k) + r^{k-1}F_r^{(I)}(k, n-2k+2)\right) \\ &\quad - r^{2k-3}F_r^{(I)}(k, n-2k+2) = rF_r^{(I)}(k, n-2) \\ &\quad + r^{k-1}F_2^{(I)}(k, n-k) = F_r^{(I)}(k,n),\end{aligned} \quad (20)$$

which ends the proof. □

Corollary 6. *For $n \geq 2$ $F_n = (1/2)(F_{n-2} + F_{n+1})$.*

Proof. For $r = 1$, $j = \mathrm{I}$, and $k = 1$ by (19) we obtain

$$F_1^{(I)}(1, n) = F_n = F_{n-2} + F_{n+1} - F_n. \quad (21)$$

Hence

$$F_n = \frac{1}{2}\left(F_{n-2} + F_{n+1}\right). \quad (22)$$

□

Theorem 7. *For $r \geq 1$, $k \geq 2$, and $n \geq 1$,*

$$F_r^{(I)}(k,n) = F_r^{(II)}(k,n) + F_r^{(II)}(k, n-1). \quad (23)$$

Proof (induction on n). For $n = 1$ we have

$$F_r^{(I)}(k, 1) = 1 = F_r^{(II)}(k, 1) + F_r^{(II)}(k, 0). \quad (24)$$

Assume that equality (23) is true for an arbitrary n. We will prove it for $n+1$. By the recurrence (6) and by induction hypothesis we get

$$\begin{aligned}&F_r^{(I)}(k, n+1) \\ &\quad = rF_r^{(I)}(k, n-1) + r^{k-1}F_r^{(I)}(k, n+1-k) \\ &\quad = r\left(F_r^{(II)}(k, n-1) + F_r^{(II)}(k, n-2)\right) \\ &\qquad + r^{k-1}\left(F_r^{(II)}(k, n+1-k) + F_r^{(II)}(k, n-k)\right) \\ &\quad = rF_r^{(II)}(k, n-1) + r^{k-1}F_r^{(II)}(k, n+1-k) \\ &\qquad + rF_r^{(II)}(k, n-2) + r^{k-1}F_2^{(II)}(k, n-k) \\ &\quad = F_r^{(II)}(k, n+1) + F_r^{(II)}(k, n),\end{aligned} \quad (25)$$

which ends the proof. □

Analogously we can prove the following.

Theorem 8. *For $r \geq 1$, $k \geq 3$, and $n \geq 0$,*

$$2F_r^{(I)}(k,n) = F_r^{(II)}(k,n) + F_r^{(III)}(k,n). \quad (26)$$

Theorem 9. *For $r \geq 1$, $k \geq 2$, $n \geq 2k$, and $j = I, II, III$,*

$$\begin{aligned}F_r^{(j)}(k,n) &= r^2F_r^{(j)}(k, n-4) + 2r^kF_r^{(j)}(k, n-k-2) \\ &\quad + r^{2k-2}F_r^{(j)}(k, n-2k).\end{aligned} \quad (27)$$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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