

## Research Article

# $C^2$ -Stably Limit Shadowing Diffeomorphisms

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Let  $f$  be a diffeomorphism on a  $C^\infty$  closed surface. In this paper, we show that if  $f$  has the  $C^2$ -stably limit shadowing property, then we have the following: (i)  $f$  satisfies the Kupka-Smale condition; (ii) if  $P(f)$  is dense in the nonwandering set  $\Omega(f)$  and if there is a dominated splitting on  $P_s(f)$ , then  $f$  satisfies both Axiom A and the strong transversality condition.

## 1. Introduction

The theory of shadowing was developed intensively in recent years and became a significant part of the qualitative theory of dynamical systems containing a lot of interesting and deep results (see [1]). Let  $M$  be a  $C^\infty$  closed manifold and let  $\text{Diff}(M)$  be the space of diffeomorphisms of  $M$  endowed with the  $C^1$ -topology. Denote by  $d$  the distance on  $M$  induced from a Riemannian metric  $\|\cdot\|$  on the tangent bundle  $TM$ . Let  $f \in \text{Diff}(M)$ . A sequence  $\{x_n\}_{n \in \mathbb{Z}}$  of points in  $M$  is called a  $\delta$ -pseudo orbit of  $f$  if  $d(f(x_n), x_{n+1}) < \delta$  for all  $n \in \mathbb{Z}$ . Let  $\Lambda$  be a closed  $f$ -invariant subset of  $M$ . We say that  $f$  satisfies the shadowing property on  $\Lambda$  if, for every  $\epsilon > 0$ , there is  $\delta > 0$  such that, for every  $\delta$ -pseudo orbit  $\{x_n\}_{n \in \mathbb{Z}} \subset \Lambda$  of  $f$ , there exists  $y \in M$  such that  $d(f^n(y), x_n) < \epsilon$  for all  $n \in \mathbb{Z}$ . If  $\Lambda = M$ , we say that  $f$  has the shadowing property.

The limit shadowing property was originally introduced by Eirola et al. [2], and it was slightly modified in [3]. In this paper we will adapt the definition of the limit shadowing property in [3] as follows. We say that  $f$  has the limit shadowing property on  $\Lambda$  if there is  $\delta > 0$  such that, for any  $\delta$ -pseudo orbit  $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$  with  $d(f(x_i), x_{i+1}) \rightarrow 0$  as  $i \rightarrow \pm\infty$ , which is called a  $\delta$ -limit pseudo orbit, there is a point  $y \in M$  such that  $d(f^i(y), x_{i+1}) \rightarrow 0$  as  $i \rightarrow \pm\infty$ . If  $\Lambda = M$ , we say that  $f$  has the limit shadowing property. From the numerical point of view this property of a dynamical system  $f$  means the following: if we apply a numerical method that approximates  $f$  with “improving accuracy” so that one-step errors tend to zero as time goes

to infinity, then the numerically obtained orbits tend to real ones. Such situations arise, for example, when we are not so interested in the initial behaviour of orbits but want to get areas where “interesting things” happen (e.g., attractors) and then improve accuracy.

Note that the limit shadowing property is different from the shadowing property. In fact, the limit shadowing property needs not have the shadowing property as we can see in [3, Example 4].

The following example shows that every irrational rotation map of the unit circle does not have the limit shadowing property which will be used in the proof of our main theorem.

*Example 1.* Let  $M = S^1$  be the unit circle and let  $g : S^1 \rightarrow S^1$  be defined by  $g(x) = x + \beta \pmod{1}$ , where  $x \in [0, 1)$  and  $\beta$  is irrational.

Assume, by contradiction, that  $g$  has the limit shadowing property. It is clear that, for any  $\delta > 0$ , there exists rational  $\gamma = p/m$  ( $p \in \mathbb{Z}, m \in \mathbb{Z} - \{0\}$ ) such that  $|\gamma - \beta| < \delta$ . Let  $g_1(x) = x + \gamma \pmod{1}$ . Then  $g_1^m(x) = x \pmod{1}$  for any  $x \in [0, 1)$ .

Now, we can construct a  $\delta$ -limit pseudo orbit of  $g$  by

$$\begin{aligned} \xi &= \{\dots, x, x = x_0, g_1(x), \dots, g_1^m(x) = x, x, \dots\} \\ &= \{\dots, x_{-1} = x, x_0 = x, \dots, x_{m-1}, x_m = x, x_{m+1} = x, \dots\}. \end{aligned} \quad (1)$$

Then, since  $g$  has the limit shadowing property, there is a point  $y \in S^1$  such that  $d(g^n(y), x_n) \rightarrow 0$  as  $n \rightarrow \pm\infty$ . Then we consider the following cases.

Firstly, if  $x = y$  then  $g^n(y) = y + n\beta = x + n\beta$  for all  $n \in \mathbb{N}$ . And  $d(g^m(x), g_1^m(x)) = |x + m\beta - x| = m\beta$ . For any  $n \geq m$ ,

$$d(g^n(y), x_n) = d(g^n(x), x) = |x + n\beta - x| = n\beta \rightarrow 0. \quad (2)$$

Finally, if  $y \in S^1$  with  $x \neq y$  then it is clear that, for  $i \geq m$ ,

$$d(g^i(y), x_i) = d(g^i(y), x) = |y + i\beta - x| > |y - x| \rightarrow 0. \quad (3)$$

This is a contradiction. Therefore, the irrational rotation map does not have the limit shadowing property.

It is a well-known fact that  $C^\infty$  maps are dense in  $\text{Diff}(M)$  ( $r \geq 1$ ), and so we can consider the following. We say that  $f$  satisfies the  $C^2$ -stably shadowing property if there exists a  $C^2$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that, for every  $g \in \mathcal{U}(f)$ ,  $g$  satisfies the shadowing property. When  $M$  is a  $C^\infty$  closed surface, Sakai [4] proved that if  $f$  has the  $C^2$ -stably shadowing property then  $f$  is Kupka-Smale; that is, every periodic point of  $f$  is hyperbolic and all their invariant manifolds are transverse. If, in addition, the periodic points of  $f$  are dense in the nonwandering set and there is a dominated splitting on the closure of periodic points of saddle type, then  $f$  satisfies both Axiom A and the strong transversality condition; that is,  $f$  is structurally stable.

*Definition 2.* One says that  $f$  has the  $C^2$ -stably limit shadowing property if  $f$  is in the  $C^2$ -interior of the set of all diffeomorphisms having the limit shadowing property.

Let  $\Lambda$  be an invariant set for  $f \in \text{Diff}(M)$ . We say that a compact  $f$ -invariant set  $\Lambda$  admits a *dominated splitting* if the tangent bundle  $T_\Lambda M$  has a continuous  $Df$ -invariant splitting  $E \oplus F$  and there exist  $C > 0$ ,  $0 < \lambda < 1$  such that

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n \quad (4)$$

for all  $x \in \Lambda$ ,  $n \geq 0$ .

We say that  $\Lambda$  is *hyperbolic* for  $f$  if there is a tangent bundle  $T_\Lambda M$  which has a  $Df$ -invariant continuous splitting  $E^s \oplus E^u$  and constants  $C > 0$  and  $0 < \lambda < 1$  such that

$$\|D_x f^n|_{E^s}\| \leq C\lambda^n, \quad \|D_x f^{-n}|_{E^u}\| \leq C\lambda^n \quad (5)$$

for all  $x \in \Lambda$  and  $n \geq 0$ .

A set  $\Lambda$  is a *basic set* if it is compact and locally maximal, and  $f$  is transitive on  $\Lambda$ . A basic set  $\Lambda$  is called of *saddle type* if  $0 < \dim W^s(x) < \dim M$  for  $x \in \Lambda$ . As usual, we denote  $P(f)$  by the set of periodic points of  $f$  and let  $P_s(f)$  be the set of periodic points of saddle type. In this paper, we prove the following theorem.

**Theorem 3.** *Let  $M$  be a  $C^\infty$  closed surface. If  $f$  has the  $C^2$ -stably limit shadowing property, then one has the following:*

- (i)  $f$  satisfies the Kupka-Smale condition,
- (ii) if  $P(f)$  is dense in the nonwandering set  $\Omega(f)$  and if there is a dominated splitting on  $P_s(f)$ , then  $f$  satisfies both Axiom A and the strong transversality condition.

## 2. Proof of Theorem 3

First, we show that if  $f$  has the  $C^2$ -stably limit shadowing property, then every periodic point of  $f$  is hyperbolic. For the proof, we need to perturb some maps but, unfortunately, we cannot use a perturbation lemma, so-called ‘‘Franks’ Lemma’’ which only works for the  $C^1$ -topology. Therefore, by a technical reason in the proof, we restrict the manifold to a surface. The proof is motivated by [4].

**Proposition 4.** *If  $f$  has the  $C^2$ -stably limit shadowing property, then every  $p \in P(f)$  is hyperbolic.*

*Proof.* Let  $f$  have the  $C^2$ -stably limit shadowing property and fix  $p \in P(f)$  with period  $n > 0$ . Assume that  $p$  is not hyperbolic. To simplify, suppose  $n = 1$ . With a  $C^2$ -small perturbation, we can find  $g$   $C^2$ -nearby  $f$  such that  $g(p) = p$  and

$$D_p g = \begin{pmatrix} A & O \\ O & B \end{pmatrix}, \quad (6)$$

where  $A$  is a constant  $\mu \in \mathbb{R}$  or  $2 \times 2$  matrix and  $B$  is a hyperbolic matrix (with respect to some coordinates), satisfying one of the following three possible cases:

- (a)  $\mu = 1$ ,
- (b)  $\mu = -1$ ,
- (c) the eigenvalues of  $A$  are of the form  $\mu_1 = e^{i\theta}$ ,  $\mu_2 = e^{-i\theta}$  for some real  $\theta \neq k\pi$  and  $k \in \mathbb{Z}$ .

Since the dimension of  $M$  is 2, we can put

$$D_p g = \begin{pmatrix} \mu & O \\ O & \rho \end{pmatrix}. \quad (7)$$

In cases (a) and (b), we approximate  $g$  by  $C^3$ -diffeomorphism  $g_1$  (with respect to the  $C^2$ -topology) such that

- (i)  $g_1(p) = p$ ,
- (ii) there is a  $C^3$ - $g_1$ -invariant curve  $W^c(p, g_1)$ , so-called the center manifold of  $p$ , which is tangent to the eigenspace associated with  $\mu = 1$  (case (a)) or  $\mu = -1$  (case (b)),
- (iii) if we consider  $W^c(p, g_1) \subset \mathbb{R}$  and  $p$  is the origin 0 (with respect to corresponding coordinates), then the restriction  $g_1|_{W^c(p, g_1)}$  has the following expressions (see [5, page 38]):

$$\begin{aligned} g_1|_{W^c(p, g_1)}(t) &= t + at^2 + o(t^3), \\ g_1|_{W^c(p, g_1)}(t) &= -t + at^3 + o(t^4), \end{aligned} \quad (8)$$

for  $t \in W^c(p, g_1) \subset \mathbb{R}$  if  $|t|$  is small enough. We may assume that  $a > 0$  (since the condition is satisfied generically).

In case (a), denote the two disjoint components of  $W^c(p, g_1) \setminus \{p\}$  by  $C^-(p)$  and  $C^+(p)$ . Take  $x \in C^-(p)$  and  $z \in C^+(p)$  and consider the limit pseudo orbit

$$\{\dots, g_1^{-2}(x), g_1^{-1}(x), x, g_1(x), \dots, g_1^{l-1}(x), z, g_1(z), \dots\} \tag{9}$$

in  $W^c(p, g_1) \setminus \{p\}$ . We can make  $d(g_1^l(x), z)$  as small as we want by letting  $l \rightarrow \infty$  and  $d(p, z) \rightarrow 0$ . If there exists the limit shadowing point  $y \in C^-(p)$ , then its forward orbit is contained in  $C^-(p)$ . If  $y \in C^+(p)$  then its backward orbit is contained in  $C^+(p)$ . If  $y \notin C^-(p) \cup C^+(p)$  then there is  $n$  such that  $g_1^n(y) \notin V$ , where  $V$  is a small fixed neighborhood of  $p$ . Therefore, there does not exist the limit shadowing point of  $g_1$ . This is a contradiction since  $g_1$  has the limit shadowing property.

In case (b), since

$$\begin{aligned} \frac{d}{dt}g_1 \Big|_{W^c(p, g_1)}(t) &= \frac{d}{dt}(-t + at^3 + o(t^4)) \\ &= -1 + 3at^2 + o(t^3), \end{aligned} \tag{10}$$

$$\frac{d^2}{dt^2}g_1 \Big|_{W^c(p, g_1)}(t) = 6at + o(t^2)$$

with respect to the corresponding coordinates, we see that

$$\frac{d}{dt}g_1 \Big|_{W^c(p, g_1)}(0) = -1, \quad \frac{d^2}{dt^2}g_1 \Big|_{W^c(p, g_1)}(0) = 0. \tag{11}$$

Thus perturbing  $g_1$  in a neighborhood of  $p$  with respect to the  $C^2$ -topology, there exists  $g_2$  ( $C^3$ -nearby  $g_1$ ) which has the limit shadowing property and  $\epsilon_0 > 0$  such that

- (i) there exists the center manifold  $W^c(p, g_2)$  of  $p$  such that  $W^c(p, g_2) \cap B_{\epsilon_0}(p) = W^c(p, g_1) \cap B_{\epsilon_0}(p)$ ,
- (ii)  $g_2|_{W^c(p, g_2)}(t) = -t$  for  $t \in W^c(p, g_2)$  if  $|t|$  is small enough.

Clearly,  $g_2^2|_{W^c(p, g_2) \cap B_{\epsilon_0}(p)}$  is the identity map.

On the other hand, since  $g_2$  has the limit shadowing property,  $g_2^2$  has to have the limit shadowing property. However, we can see that the identity map does not satisfy the limit shadowing property (see [3, Example 3]). This is a contradiction.

In case (c), by the proof of [6, page 23, Theorem 5.2 and Remark 5.3], we will derive a contradiction.

First, we may suppose that there exists a smooth arc  $\{\varphi_t\}_{t \in \mathbb{R}}$  of diffeomorphisms on  $M$  (the corresponding map  $\phi : M \times \mathbb{R} \rightarrow M \times \mathbb{R}$  defined by  $\phi(x, t) = (\varphi_t(x), t)$  for  $(x, t) \in M \times \mathbb{R}$  is  $C^\infty$ ) such that  $\varphi_0 = g$  and  $(p, 0) \in M \times \mathbb{R}$  is a Hopf point unfolding generically (see [6, page 22]). Then, we approximate the arc by an arc  $\{\varphi'_t\}_{t \in \mathbb{R}}$  (with respect to the  $C^\infty$ -topology) such that the eigenvalues of  $D_p\varphi'_0$  have the form  $e^{2\pi i\alpha}$  with  $\alpha$  irrational and such that the center manifold  $W^c(p, \varphi'_0)$  is  $C^\infty$  (see [6, page 15]).

Finally, apply the arguments in [6, page 23, Theorem 5.2 and Remark 5.3]. Then slightly perturbing the arc  $\{\varphi'_t\}_{t \in \mathbb{R}}$  if necessary (with respect to the  $C^\infty$ -topology), we may have the following assertions:

- (i) there is a  $\varphi'_t$ -invariant attracting (or repelling) circle  $\mathcal{C}$  (in the manifold) near  $p$  for  $t > 0$  small enough,
- (ii) the restriction  $\varphi'_t|_{\mathcal{C}}$  is conjugated to a rotation map.

Recall that  $\varphi'_t$  has the limit shadowing property and  $B_t$  is hyperbolic, where  $B_t$  is the matrix (for  $\varphi_t$ ) corresponding to  $B$ . Thus we can see that  $\varphi'_t|_{\mathcal{C}}$  satisfies limit shadowing, but this is a contradiction because any rotation map does not have the limit shadowing property (cf. Example 1) and so complete the proof.  $\square$

The notion of  $C^0$  transversality between the stable and unstable manifolds of basic sets  $\Lambda_i$  and  $\Lambda_j$  was introduced in [7] as follows. If there exists  $x \in W^s(\Lambda_i) \cap W^u(\Lambda_j) \setminus \Lambda_i \cup \Lambda_j$ , then for  $\epsilon > 0$  we denote by  $C_\epsilon^\sigma(x)$  the connected component of  $x$  in  $W^\sigma(x) \cap B_\epsilon(x)$  ( $\sigma = s, u$ ). Let  $B_\epsilon^+(x)$  and  $B_\epsilon^-(x)$  be the connected components of  $B_\epsilon(x) \setminus C_\epsilon^s(x)$ . Here  $B_\epsilon(x) = \{y \in M : d(x, y) \leq \epsilon\}$ . We say that  $W^s(x)$  and  $W^u(x)$  meet  $C^0$  transversely at  $x$  if  $\dim W^\sigma(x) = 1$  ( $\sigma = s, u$ ),  $B_\epsilon^+(x) \cap C_\epsilon^u(x) \neq \emptyset$ , and  $B_\epsilon^-(x) \cap C_\epsilon^u(x) \neq \emptyset$ , for every  $\epsilon > 0$ . Let  $\Lambda \subset M$  be an invariant submanifold of  $f$ . We say that  $\Lambda$  is normally hyperbolic if there is a splitting  $T_\Lambda M = T\Lambda \oplus N^\sigma$ ,  $\sigma = s, u$ , such that

- (a) the splitting depends continuously on  $x \in \Lambda$ ,
- (b)  $D_x f(N_x^\sigma) = N_{f(x)}^\sigma$  ( $\sigma = s, u$ ) for all  $x \in \Lambda$ ,
- (c) there are constants  $C > 0$ , and  $\lambda \in (0, 1)$  such that, for every triple of unit vectors  $v \in T_x \Lambda$ ,  $v^s \in N_x^s$ , and  $v^u \in N_x^u$  ( $x \in \Lambda$ ), we have

$$\frac{\|D_x f^n(v^s)\|}{\|D_x f^n(v)\|} \leq C\lambda^n, \quad \frac{\|D_x f^n(v^u)\|}{\|D_x f^n(v)\|} \geq C^{-1}\lambda^{-n}, \tag{12}$$

for all  $n \geq 0$ .

Let  $\Lambda$  be a closed  $f$ -invariant set. We say that  $f$  has the limit shadowing property in  $\Lambda$  if there is  $\delta > 0$  such that, for any  $\delta$ -limit pseudo orbit, there is a point  $y \in \Lambda$  such that  $d(f^i(y), x_i) \rightarrow 0$  as  $i \rightarrow \pm\infty$ . Note that the definition is different from that we defined before. In fact, if  $f$  has the limit shadowing property on  $\Lambda$ , the limit shadowing point needs not be in  $\Lambda$ .

**Lemma 5.** *Let  $\Lambda \subset M$  be a normally hyperbolic set for  $f$ . If  $f$  has the limit shadowing property on  $\Lambda$ , then the limit shadowing point is in  $\Lambda$ .*

*Proof.* Since  $\Lambda$  is compact and normally hyperbolic, for any  $\epsilon > 0$  and for every  $y \in M \setminus \Lambda$ , there is  $n \in \mathbb{Z}$  such that  $d(f^n(y), \Lambda) > \epsilon$ . Let  $0 < \delta < \epsilon$  be as in the limit shadowing property of  $f$ . Since  $f$  has the limit shadowing property on  $\Lambda$ , for any  $\delta$ -limit pseudo orbit  $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$  of  $f$ , there exists  $y \in M$  such that  $d(f^n(y), x_n) \rightarrow 0$  as  $n \rightarrow \pm\infty$ . Hence if  $y \notin \Lambda$ , then  $d(f^n(y), x_n) \not\rightarrow 0$  as  $n \rightarrow \pm\infty$ . Thus the limit shadowing point is in  $\Lambda$ .  $\square$

**Proposition 6.** Let  $\Lambda_1$  and  $\Lambda_2$  be two basic sets of  $f \in \text{Diff}(M)$  and let  $x \in W^s(\Lambda_1) \cap W^u(\Lambda_2) \setminus \Lambda_1 \cup \Lambda_2$ . If  $f$  has the  $C^2$ -stably limit shadowing property, then  $W^s(x)$  and  $W^u(x)$  meet transversely at  $x$ .

To prove Proposition 6, we need the following two lemmas.

**Lemma 7** (see [8, Lemma 3.2]). Let  $\Lambda_i$  ( $i = 1, 2$ ) be basic sets of  $f \in \text{Diff}^2(M)$  and suppose that  $x \in W^s(p) \cap W^u(q) \setminus \{\Lambda_1 \cup \Lambda_2\}$  ( $p \in \Lambda_1, q \in \Lambda_2$ ). If  $f$  has the limit shadowing property, then  $W^s(p)$  and  $W^u(q)$  meet  $C^0$  transversely at  $x$ .

**Lemma 8** (see [9, Lemma 2]). Let  $\Lambda_i$  ( $i = 1, 2$ ) be basic sets of  $f \in \text{Diff}^r(M)$  ( $r \geq 1$ ) and suppose that  $x \in W^s(\Lambda_1) \cap W^u(\Lambda_2) \setminus \Lambda_1 \cup \Lambda_2$ . Then there are  $\epsilon > 0$  and a  $C^r$ -diffeomorphism  $\psi_x : B_\epsilon(x) \rightarrow \mathbb{R}^2 = \{(y, z) \mid y, z \in \mathbb{R}\}$  such that  $\psi_x(x) = (0, 0)$  and  $\psi(C_\epsilon^s(x)) \subset y$ -axis. Here  $C_\epsilon^s(x)$  is defined as before.

*Proof of Proposition 6.* Let  $\Lambda_i$  ( $i = 1, 2$ ) be basic sets of  $f \in \text{Diff}^2(M)$  and suppose  $x \in W^s(\Lambda_1) \cap W^u(\Lambda_2) \setminus \{\Lambda_1 \cup \Lambda_2\}$ . We will prove that if there is a  $C^2$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that every  $g \in \mathcal{U}(f)$  has the limit shadowing property, then  $T_x M = T_x W^s(x) + T_x W^u(x)$ . By Lemma 8, there are  $\delta > 0$  and a  $C^2$ -diffeomorphism  $\psi_x : B_\delta(x) \rightarrow \mathbb{R}^2$  such that

$$\psi_x(C_\delta^s(x)) \subset y\text{-axis}, \quad \psi_x(x) = (0, 0). \quad (13)$$

If  $T_{(0,0)}\psi_x(C_\delta^u(x)) \neq y$ -axis, then we have  $T_x M = T_x W^s(x) + T_x W^u(x)$ ; that is,  $W^s(x)$  and  $W^u(x)$  meet transversely at  $x$ . Thus we assume that  $T_{(0,0)}\psi_x(C_\delta^u(x)) = y$ -axis. It is easy to see that there are  $\epsilon > 0$  and a  $C^2$ -function  $\alpha : [-\epsilon, \epsilon] \rightarrow \mathbb{R}$  such that

$$\text{graph}(\alpha) \subset \psi_x(C_\delta^u(x)), \quad (0, \alpha(0)) = \psi_x(x) = (0, 0). \quad (14)$$

If  $\alpha'(0) \neq 0$ , then, since  $\alpha'(0) = 0$ ,  $W^s(x)$  and  $W^u(x)$  do not meet  $C^0$  transversely at  $x$ . This is inconsistent with Lemma 7 and so  $\alpha''(0) = 0$ . If we denote a  $C^2$ -metric by  $d_2$ , then for every  $\delta'$ , there exists  $0 < \epsilon' < \epsilon$  such that

$$d_2(\psi_x^{-1}(\text{graph}(\alpha(-\epsilon', \epsilon'))), C_{\epsilon'}^s(x)) < \delta' \quad (15)$$

since  $\alpha'(0) = 0$  and  $\alpha''(0) = 0$ . Thus, by using a standard procedure, for every  $\eta > 0$  and every  $C^2$ -neighborhood  $\mathcal{U}(f)$  of  $f$  such that every  $g \in \mathcal{U}(f)$  has the limit shadowing property, we can construct a  $C^2$ -diffeomorphism  $g : M \rightarrow M$  such that (1)  $g(x) = x$ , (2)  $g|_{M \setminus B_\eta(x)} = id$ , (3)  $g(W^s(x)) \cap B_{\eta'}(x) \subset W^u(x)$ , and (4)  $h = g^{-1} \circ f \in \mathcal{U}(f)$ , where  $0 < \eta' < \eta$  is sufficiently small.

From this we have

$$W^s(x, h) \cap B_{\eta'}(x) = W^u(x, h) \cap B_{\eta'}(x). \quad (16)$$

Here  $W^\sigma(x, h)$  ( $\sigma = s, u$ ) are the stable and unstable manifolds of  $h$  at  $x$ . By Lemma 7, this is a contradiction since  $h$  has the limit shadowing property and so the proof is completed.  $\square$

Let  $f : M^n \rightarrow M^n$  ( $n \geq 2$ ) be a diffeomorphism. For given  $x, y \in M$ , we write  $x \rightsquigarrow y$  if, for any  $\delta > 0$ , there is a finite  $\delta$ -pseudo orbit  $\{x_i\}_{i=0}^n$  ( $n \geq 1$ ) of  $f$  such that  $x_0 = x$  and  $x_n = y$ . For a closed  $f$ -invariant set  $\Lambda \subset M$ , we say that  $f$  is chain transitive in  $\Lambda$  (or  $f|_\Lambda$  is chain transitive) if for any  $x, y \in \Lambda$ ,  $x \rightsquigarrow y \subset \Lambda$ .

To prove Theorem 3, we need some results as follows from [10–12]. Since [10] is preprint, we give the proof here.

**Lemma 9** (see [10, Lemma 2.1]). If  $f|_\Lambda$  is chain transitive, then  $f$  has neither sinks nor sources.

*Proof.* Let  $p$  be a sink. Then there exist  $\epsilon > 0$  and  $\lambda < 1$  such that if  $d(x, p) < \epsilon$  then  $d(f^i(x), p) < \lambda d(x, p)$  for all  $i \geq 1$ . Take  $y \in \Lambda$  such that  $d(y, p) \geq 2\epsilon$ . For any  $\delta > 0$ , let  $\xi = \{p = x_0, x_1, \dots, x_m = y\}$  ( $m \geq 1$ ) be a  $\delta$ -pseudo orbit of  $f$  such that  $x_i \in \Lambda$ . For simplicity, we may assume that  $f(p) = p$ . Then we have  $d(p, x_1) < \delta$  and  $d(p, x_2) \leq d(p, f(x_1)) + d(f(x_1), x_2) < \lambda d(p, x_1) + \delta < \delta(\lambda + 1)$ . Thus we obtain

$$\begin{aligned} d(p, x_i) &\leq d(p, f(x_{i-1})) + d(f(x_{i-1}), x_i) \\ &< d(p, f(x_{i-1})) + \delta \\ &< \lambda d(p, x_{i-1}) + \delta < \dots < \delta(1 + \lambda + \dots + \lambda^i) \\ &\leq \frac{\delta}{1 - \lambda}. \end{aligned} \quad (17)$$

Put  $\eta = \delta/(1 - \lambda)$ . Then if  $\delta$  is sufficiently small, we can make  $\eta < \epsilon$ . This is a contradiction since  $d(y, p) \geq 2\epsilon$ .  $\square$

**Theorem 10** (see [11, Lemma 2.3]). Let  $\Lambda$  be a locally maximal set. If  $f$  has the limit shadowing property on  $\Lambda$  then  $f|_\Lambda$  is chain transitive.

**Proposition 11** (see [12]). Let  $f$  be a  $C^2$ -diffeomorphism on a  $C^\infty$  closed surface  $M$  and let  $\Lambda$  be a compact  $f$ -invariant set having a dominated splitting  $T_\Lambda M = E \oplus F$ . Assume that all the periodic points in  $\Lambda$  are hyperbolic of saddle type. Then,  $\Lambda = \Lambda_1 \cup \Lambda_2$ , where  $\Lambda_1$  is hyperbolic and  $\Lambda_2$  consist of a finite union of normally hyperbolic periodic simple closed curves  $\mathcal{C}_1 \cup \dots \cup \mathcal{C}_n$  such that  $f^{m_i} : \mathcal{C}_i \rightarrow \mathcal{C}_i$  is conjugated to an irrational rotation. Here  $m_i$  denotes the minimal number such that  $f^{m_i}(\mathcal{C}_i) = \mathcal{C}_i$ .

*Proof of Theorem 3.* Let  $M$  be a  $C^\infty$  closed manifold and let  $f$  have the  $C^2$ -stably limit shadowing property. Then (i) follows from Propositions 4 and 6 directly.

To prove (ii), we suppose  $\overline{P(f)} = \Omega(f)$  and there is a dominated splitting on  $\overline{P_s(f)}$ . By Proposition 4, every  $p \in P(f)$  is hyperbolic. Hence by Proposition 11, we have  $\overline{P_s(f)} = \Lambda_1 \cup \Lambda_2$ , where  $\Lambda_1$  is hyperbolic and  $\Lambda_2$  consists of a finite union of normally hyperbolic periodic simple closed curves  $\mathcal{C}_1, \dots, \mathcal{C}_n$  such that  $f^{m_i} : \mathcal{C}_i \rightarrow \mathcal{C}_i$  is conjugated to an irrational rotation. Here  $m_i$  denotes the smallest number such that  $f^{m_i}(\mathcal{C}_i) = \mathcal{C}_i$ . Since  $\mathcal{C}_i$  is normally hyperbolic, by Lemma 5, the limit shadowing point  $y$  is in  $\mathcal{C}_i$ . That is,  $f^{m_i}$  satisfies the limit shadowing property on  $\mathcal{C}_i$  such that the limit shadowing point is in  $\mathcal{C}_i$ .

On the other hand, by Example 1, the irrational rotation map does not have the limit shadowing property. Since the limit shadowing property is invariant under a topological conjugacy,  $\Lambda_2 = \emptyset$  is concluded. Thus  $\overline{P_s(f)} = \Lambda_1 \cup \Lambda_2$  is hyperbolic. Since  $f$  has the limit shadowing property, by Theorem 10, it is chain transitive and, by Lemma 9, it has neither sinks nor sources. Therefore  $f$  satisfies Axiom A. The strong transversality condition follows from Proposition 6, and so the proof is completed.  $\square$

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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