

Research Article

Monotone and Concave Positive Solutions to Three-Point Boundary Value Problems of Higher-Order Fractional Differential Equations

Wenyong Zhong and Lanfang Wang

College of Mathematics and Statistics, Jishou University, 120 Renmin South Road, Jishou, Hunan 416000, China

Correspondence should be addressed to Wenyong Zhong; wyzhong@jsu.edu.cn

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We study the three-point boundary value problem of higher-order fractional differential equations of the form ${}^c D_{0+}^\rho u(t) + f(t, u(t)) = 0$, $0 < t < 1$, $2 \leq n-1 < \rho < n$, $u'(0) = u''(0) = \dots = u^{(n-1)}(0) = 0$, $u(1) + pu'(1) = qu'(\xi)$, where ${}^c D_{0+}^\rho$ is the Caputo fractional derivative of order ρ , and the function $f: [0, 1] \times [0, \infty) \mapsto [0, +\infty)$ is continuously differentiable. Here, $0 \leq q \leq p$, $0 < \xi < 1$, $2 \leq n-1 < \rho < n$. By virtue of some fixed point theorems, some sufficient criteria for the existence and multiplicity results of positive solutions are established and the obtained results also guarantee that the positive solutions discussed are monotone and concave.

1. Introduction

Applications of fractional differential equations can be found in various areas, including engineering, physics, and chemistry [1–4]. In recent years, the interest in the study of fractional differential equations has been growing rapidly.

As one of the focal topics in the research of fractional differential equations, the study of the boundary value problems (BVPs for short) recently has attracted a great deal of attention from many researchers. A series of works have been presented to discuss the existence of (positive) solutions in the BVPs for fractional differential equations [5–15].

However, there are few results in the literature to discuss the positive, monotone, and concave solutions to the BVPs of fractional differential equations; it is difficult to establish the relation between the monotonicity and concavity of a function and its fractional derivatives. It is worth pointing out that Wang et al. [7] obtained the existence and multiplicity results of the positive, monotone, and concave solutions to the following problem:

$$\begin{aligned} {}^c D_{0+}^\rho u(t) + f(t, u(t)) &= 0, & 0 < t < 1, & n-1 < \rho < n, \\ u(0) = u''(0) = \dots &= u^{(n-1)}(0) = 0, \\ u'(1) &= 0, \end{aligned} \quad (1)$$

where ${}^c D_{0+}^\rho$ is the Caputo fractional derivative of order ρ . The multiplicity results of solutions are obtained by using the Legget-Williams fixed point theorem. However, the question of how to establish the connection between the monotonicity and concavity of a function and its fractional derivatives is far from being solved; and the concavity of a function is also not used sufficiently.

Motivated by the aforementioned results, we then turn to investigating the existence of monotone and concave positive solutions for the following boundary value problem (BVP for short):

$$\begin{aligned} {}^c D_{0+}^\rho u(t) + f(t, u(t)) &= 0, & 0 < t < 1, & n-1 < \rho < n, \\ u'(0) = u''(0) = \dots &= u^{(n-1)}(0) = 0, \\ u(1) + pu'(1) &= qu'(\xi), \end{aligned} \quad (2)$$

where ${}^c D_{0+}^\rho$ is the Caputo fractional derivative of order ρ . The case $\rho = 2$ was discussed in [16] by virtue of the Avery-Henderson and Legget-Williams fixed point theorems. While in the setting of the fractional-order derivatives, as far as we

know, the existence of positive solutions for BVP (2) has not been discussed in the literature.

We now make the following assumptions to be used later:

- (A1) the function $f : [0, 1] \times [0, \infty) \mapsto [0, +\infty)$ is continuously differentiable;
- (A2) $0 \leq q \leq p, 0 < \xi < 1, 2 \leq n - 1 < \rho < n$.

The rest of paper is organized as follows. Section 2 preliminarily provides some definitions and lemmas which are crucial to the following discussion, and the connection between the monotonicity and concavity of a function and its Caputo derivatives is established in this section. Section 3 gives some sufficient conditions for the existence of at least two positive solutions of BVP (2) by means of the Avery-Henderson fixed point theorem. Section 4 gives some sufficient conditions for the existence of at least three positive solutions by virtue of the five-functional fixed point theorem. In addition, the sufficient conditions also guarantee that the positive solutions obtained are monotone and concave. Finally, Section 5 provides an example to illustrate a possible application of the obtained results.

2. Preliminaries

In this section, we preliminarily provide some definitions and lemmas to be used in the following discussion.

Definition 1 (see [3]). The fractional integral of order $\rho > 0$ of a function $y : (0, \infty) \mapsto \mathbb{R}$ is given by

$$I_{0+}^{\rho} y(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} y(s) ds, \quad (3)$$

provided the right side is pointwise defined on $(0, \infty)$.

Definition 2 (see [3]). The Riemann-Liouville fractional derivative of order $\rho > 0$ of a continuous function $y : (0, \infty) \mapsto \mathbb{R}$ is given by

$$D_{0+}^{\rho} y(t) = \frac{1}{\Gamma(n-\rho)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\rho-1} y(s) ds, \quad (4)$$

where $n = [\rho] + 1$, provided the right side is pointwise defined on $(0, \infty)$.

Remark 3. Consider $D_{0+}^m y(t) = D^m y(t)$, where $D^m y(t) \triangleq d^m y(t)/dt^m, m \in N_0 \triangleq \{0, 1, 2, \dots\}$.

Definition 4 (see [4]). For a function y given on the interval $[0, \infty)$, the Caputo fractional derivative of order $\rho > 0$ of y is defined by

$${}^c D_{0+}^{\rho} y(t) = \frac{1}{\Gamma(n-\rho)} \int_0^t (t-s)^{n-\rho-1} y^{(n)}(s) ds, \quad (5)$$

where $n = [\rho] + 1, [\rho]$ denotes the integer part of ρ .

Lemma 5 (see [3]). *Let ρ and μ be positive numbers. If $y \in C[0, 1]$, then $I_{0+}^{\rho} y(t) \in C[0, 1]$, and the equations $D_{0+}^{\rho} I_{0+}^{\rho} y(t) = y(t)$ and $I_{0+}^{\rho} I_{0+}^{\mu} y(t) = I_{0+}^{\rho+\mu} y(t)$ are satisfied for each t in $[0, 1]$.*

Lemma 6 (see [3]). *Let $\rho > 0$. If $y \in C^n[0, 1]$ or $y \in AC^n[0, 1]$, then*

$$I_{0+}^{\rho} {}^c D_{0+}^{\rho} y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad (6)$$

for some c_i in $\mathbb{R}, i = 0, 1, \dots, n-1, n = [\rho] + 1$.

The following two lemmas are fundamental in finding an integral representation of solutions of BVP (2).

Lemma 7. *Let $g : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ be a continuously differentiable function. If a function y in $C[0, 1]$ is a solution of the equation $u(t) = I_{0+}^{\rho} g(t, u(t))$, then $y \in C^{(n-1)}[0, 1]$ and $y^{(n)} \in C(0, 1] \cap L(0, 1)$, and the relation ${}^c D_{0+}^{\rho} y(t) = g(t, y(t))$ holds for each t in $[0, 1]$.*

Proof. Let $y \in C[0, 1]$ be a solution of the equation $u(t) = I_{0+}^{\rho} g(t, u(t))$.

Since g is continuous on $[0, 1] \times \mathbb{R}$, Lemma 5 implies that

$$I_{0+}^{\rho-1} g(t, y(t)) \in C[0, 1] \quad (7)$$

and that

$$I_{0+}^1 I_{0+}^{\rho-1} g(t, y(t)) = I_{0+}^{\rho} g(t, y(t)). \quad (8)$$

The above equation, together with Remark 3 and Lemma 5, yields

$$D_{0+}^1 I_{0+}^1 I_{0+}^{\rho-1} g(t, y(t)) = I_{0+}^{\rho-1} g(t, y(t)). \quad (9)$$

Hence $y(t)$ is continuously differentiable on $[0, 1]$ and $y'(t) = I_{0+}^{\rho-1} g(t, y(t))$.

Generally, noticing $n - 1 < \rho < n$, we have

$$y^{(k)}(t) = I_{0+}^{\rho-k} g(t, y(t)), \quad (10)$$

which implies $y^{(k)} \in C[0, 1]$ for $k = 1, 2, \dots, n - 1$.

Furthermore, using the assumptions imposed on the function $g(t, u)$ and integrating by parts, we obtain

$$\begin{aligned}
 & y^{(n-1)}(t) \\
 &= \frac{1}{\Gamma(\rho - n + 1)} \int_0^t (t - s)^{\rho - n} g(s, y(s)) ds \\
 &= \frac{g(0, y(0))}{\Gamma(\rho - n + 2)} t^{\rho - n + 1} \\
 &\quad + \frac{1}{\Gamma(\rho - n + 2)} \int_0^t (t - s)^{\rho - n + 1} \left(\frac{\partial g}{\partial s} + y'(s) \frac{\partial g}{\partial y} \right) ds \\
 &= \frac{g(0, y(0))}{\Gamma(\rho - n + 2)} t^{\rho - n + 1} + I_{0+}^{\rho - n + 2} \left(\frac{\partial g}{\partial t} + y'(t) \frac{\partial g}{\partial y} \right). \tag{11}
 \end{aligned}$$

This yields that, for every t in $(0, 1]$,

$$y^{(n)}(t) = \frac{g(0, y(0))}{\Gamma(\rho - n + 1)} t^{\rho - n} + I_{0+}^{\rho - n + 1} \left(\frac{\partial g}{\partial t} + y'(t) \frac{\partial g}{\partial y} \right). \tag{12}$$

Since the second term of the right-hand side of the above equality is continuous on the interval $[0, 1]$, $y^{(n)} \in C(0, 1] \cap L(0, 1]$. Consequently, direct computations produce

$$\begin{aligned}
 {}^c D_{0+}^\rho y(t) &= I_{0+}^{n-\rho} y^{(n)}(t) \\
 &= \frac{g(0, y(0))}{\Gamma(\rho - n + 1)} I_{0+}^{n-\rho} t^{\rho - n} \\
 &\quad + I_{0+}^{n-\rho} I_{0+}^{\rho - n + 1} \left(\frac{\partial g}{\partial t} + y'(t) \frac{\partial g}{\partial y} \right) \tag{13} \\
 &= g(0, y(0)) + I_{0+}^1 Dg(t, y(t)) \\
 &= g(t, y(t)).
 \end{aligned}$$

The proof is completed. □

By Lemma 6, we next present an integral representation of the solution of the linearized problem corresponding to BVP (2).

Lemma 8. *Let $h \in C^1[0, 1]$; if (A1)-(A2) hold, then BVP*

$$\begin{aligned}
 & {}^c D_{0+}^\rho u(t) + h(t) = 0, \quad 0 < t < 1, \quad 2 \leq n - 1 < \rho < n, \\
 & u'(0) = u''(0) = \dots = u^{(n-1)}(0) = 0, \tag{14} \\
 & u(1) + pu'(1) = qu'(\xi)
 \end{aligned}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s) h(s) ds, \tag{15}$$

where

$$G(t, s) = G_1(t, s) + G_2(t, s). \tag{16}$$

$$G_1(t, s) = \begin{cases} \frac{(1-s)^{\rho-1} - (t-s)^{\rho-1}}{\Gamma(\rho)}, & s \leq t, \\ \frac{(1-s)^{\rho-1}}{\Gamma(\rho)}, & s \geq t, \end{cases} \tag{17}$$

$$G_2(t, s) = \frac{1}{\Gamma(\rho - 1)} [p(1-s)^{\rho-2} - q(\xi - s)^{\rho-2} \chi_{E_\xi}(s)]. \tag{18}$$

Here $E_\xi = \{s : s \leq \xi\}$, and χ_{E_ξ} denotes the characteristic function of the set E_ξ .

Proof. Lemma 6 implies

$$u(t) = -I_{0+}^\rho h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}. \tag{19}$$

Differentiating (19) with respect to t up to the order $n - 1$ and using the boundary conditions that $u'(0) = u''(0) = \dots = u^{(n-1)}(0) = 0$, we obtain

$$u(t) = -I_{0+}^\rho h(t) + c_0. \tag{20}$$

From the above equation and the condition that $u(1) + pu'(1) = qu'(\xi)$, it follows that

$$c_0 = I_{0+}^\rho h(1) + pI_{0+}^{\rho-1} h(1) - qI_{0+}^{\rho-1} h(\xi). \tag{21}$$

Substituting c_0 into (20), we have

$$\begin{aligned}
 u(t) &= -I^\rho h(t) + I_{0+}^\rho h(1) + pI_{0+}^{\rho-1} h(1) - qI_{0+}^{\rho-1} h(\xi) \\
 &= \int_0^t \frac{(1-s)^{\rho-1} - (t-s)^{\rho-1}}{\Gamma(\rho)} h(s) ds \\
 &\quad + \int_t^1 \frac{(1-s)^{\rho-1}}{\Gamma(\rho)} h(s) ds \\
 &\quad + p \int_0^1 \frac{(1-s)^{\rho-2}}{\Gamma(\rho-1)} h(s) ds - q \int_0^\xi \frac{(\xi-s)^{\rho-2}}{\Gamma(\rho-1)} h(s) ds \\
 &= \int_0^1 G(t, s) h(s) ds, \tag{22}
 \end{aligned}$$

where $G(t, s)$ is defined by (16). The proof is completed. □

We now give some properties of the functions $G_i(t, s)$.

Lemma 9. *If condition (A2) holds, then*

$$G_i(s, s) \geq G_i(t, s) \geq \eta(t) G_i(s, s) \geq 0, \tag{23}$$

for all s, t in $[0, 1]$ and $i = 1, 2$, where $\eta(t) = 1 - t^{\rho-1}$.

Proof. It follows from the definition of $G_1(t, s)$ that, for $t \geq s$,

$$\begin{aligned} G_1(s, s) &\geq G_1(t, s) = \frac{(1-s)^{\rho-1} - (t-s)^{\rho-1}}{\Gamma(\rho)} \\ &\geq \frac{(1-s)^{\rho-1} - (t-ts)^{\rho-1}}{\Gamma(\rho)} \quad (24) \\ &= \eta(t) G_1(s, s) \geq 0. \end{aligned}$$

On the other hand, for $t \leq s$, the assertion for $G_1(t, s)$ is obvious.

As for the assertion for $G_2(t, s)$, it is sufficient to verify that $G_2(s, s) \geq 0$ for each s in $[0, 1]$. In fact, the definition of $G_2(t, s)$ and condition (A2) directly imply

$$\begin{aligned} G_2(s, s) &= \frac{1}{\Gamma(\rho-1)} [p(1-s)^{\rho-2} - q(\xi-s)^{\rho-2} \chi_{E_\xi}(s)] \\ &\geq \frac{1}{\Gamma(\rho-1)} [p(1-s)^{\rho-2} - q(1-s)^{\rho-2}] \\ &= \frac{p-q}{\Gamma(\rho-1)} (1-s)^{\rho-2} \geq 0. \end{aligned} \quad (25)$$

The proof is completed. □

The following results establish the connection between the monotonicity and concavity of a function and its Caputo fractional derivatives under some conditions.

Lemma 10. *Let u be a function defined on $[0, 1]$. Assume $u^{(k)}(0) = 0$ for $k = 1, 2, \dots, n-1$. Suppose that ${}^c D_{0+}^\rho u(t)$ is continuously differentiable on $[0, 1]$; if ${}^c D_{0+}^\rho u(t) \leq 0$ on $[0, 1]$, then $u^{(k)}(t) \leq 0$ on $[0, 1]$ for $k = 1, 2, \dots, n-1$.*

Proof. Set $h(t) = {}^c D_{0+}^\rho u(t)$. Then, as in the proof of Lemma 8, the assumptions made on ${}^c D_{0+}^\rho u(t)$ and $u^{(k)}(0)$ yield

$$u(t) = I_{0+}^\rho h(t) + c_0. \quad (26)$$

This implies

$$u^{(k)}(t) = I_{0+}^{\rho-k} h(t), \quad (27)$$

for $k = 1, 2, \dots, n-1$. Thus the desired results follow from the nonpositivity of $h(t)$. The proof is completed. □

Lemmas 8–10 yield the following important properties of the solution of BVP (14), which is easy to check.

Lemma 11. *Let $h \in C^1([0, 1], [0, +\infty))$. If condition (A2) holds, then the solution of BVP (14) is nonnegative, monotone, and concave on $[0, 1]$.*

Lemma 12 (see [17]). *If a function u is nonnegative and concave on $[0, 1]$ and $u'(0) = 0$, then*

- (i) $u(t) \geq (1-t)\|u\|$ for each t in $[0, 1]$, where $\|u\| = \sup\{|u(t)| : t \in [0, 1]\}$;
- (ii) $(1-s)u(t) \geq (1-t)u(s)$ for all t, s in $[0, 1]$ with $t \geq s$.

Now, denote by $\mathcal{E} = C[0, 1]$ the classical Banach space with the norm $\|u\| = \sup_{t \in [0, 1]} |u(t)|$, where $u \in \mathcal{E}$. Furthermore, define a cone, denoted by \mathcal{P} , through

$$\mathcal{P} = \left\{ u \in \mathcal{E} \mid u(t) \geq 0, u'(t) \leq 0, u''(t) \leq 0 \right. \\ \left. \text{for each } t \text{ in } [0, 1] \right\}. \quad (28)$$

Also, for a given positive real number r , define a function set \mathcal{P}_r by

$$\mathcal{P}_r = \{u \in \mathcal{P} \mid \|u\| < r\}. \quad (29)$$

Naturally, we denote that $\overline{\mathcal{P}}_r = \{u \in \mathcal{P} \mid \|u\| \leq r\}$ and that $\partial \mathcal{P}_r = \{u \in \mathcal{P} \mid \|u\| = r\}$.

Next, define the operator $\mathcal{A} : \mathcal{P} \rightarrow \mathcal{E}$ by

$$[\mathcal{A}u](t) = \int_0^1 G(t, s) f(s, u(s)) ds, \quad (30)$$

for any $u \in \mathcal{P}$. We now show some important properties on this map.

Lemma 13. *Assume that hypotheses (A1)-(A2) are all fulfilled. Then $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$ and $\mathcal{A} : \overline{\mathcal{P}}_r \rightarrow \mathcal{P}$ is completely continuous.*

Proof. It is easy to check that $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$. Moreover, analysis similar to that in [6] shows that $\mathcal{A} : \overline{\mathcal{P}}_r \rightarrow \mathcal{P}$ is completely continuous. The proof is completed. □

Lemma 14. *If (A1)-(A2) hold, then a function u in $C[0, 1]$ is a solution of BVP (2) if and only if it is a fixed point of \mathcal{A} in \mathcal{P} .*

Proof. If u is a solution of BVP (2), then Lemma 11 implies $u \in \mathcal{P}$. Furthermore, replacing $h(t)$ in Lemma 8 by $f(t, u(t))$, we get $\mathcal{A}u = u$. Hence u is a fixed point of \mathcal{A} in \mathcal{P} .

On the other hand, if $u \in \mathcal{P}$ and $\mathcal{A}u = u$, then

$$\begin{aligned} u(t) &= \int_0^1 G(t, s) f(s, u(s)) ds \\ &= -I_{0+}^\rho f(t, u(t)) + I_{0+}^\rho f(1, u(1)) \\ &\quad + pI_{0+}^{\rho-1} f(1, u(1)) - qI_{0+}^{\rho-1} f(\xi, u(\xi)). \end{aligned} \quad (31)$$

The above equation and Lemma 7 imply

$$D_{0+}^\rho u(t) + f(t, u(t)) = 0. \quad (32)$$

Moreover, it is easy to check that all the boundary conditions in BVP (2) are satisfied. Therefore u is a positive solution of BVP (2). We consequently complete the proof. □

3. Two Positive Solutions in Boundary Value Problems

In this section, we aim to adopt the well-known Avery-Henderson fixed point theorem to prove the existence of at least two positive solutions in BVP (2). For the sake of self-containment, we first state the Avery-Henderson fixed point theorem as follows.

Theorem 15 (see [18]). *Let \mathcal{P} be a cone in a real Banach space \mathcal{E} . For each $d > 0$, set $\mathcal{P}(\psi, d) = \{x \in \mathcal{P} \mid \psi(x) < d\}$. Let α and γ be increasing, nonnegative continuous functional on \mathcal{P} , and let θ be a nonnegative continuous functional on \mathcal{P} with $\theta(0) = 0$ such that, for some $c > 0$ and $H > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \quad \|x\| \leq H\gamma(x), \quad (33)$$

for all $x \in \overline{\mathcal{P}(\gamma, c)}$. Suppose that there exist a completely continuous operator $\mathcal{A} : \mathcal{P}(\gamma, c) \rightarrow \mathcal{P}$ and three positive numbers $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x), \quad 0 \leq \lambda \leq 1, \quad x \in \partial\mathcal{P}(\theta, b), \quad (34)$$

and (i) $\gamma(\mathcal{A}x) > c$ for all $x \in \partial\mathcal{P}(\gamma, c)$; (ii) $\theta(\mathcal{A}x) < b$ for all $x \in \partial\mathcal{P}(\theta, b)$; (iii) $\mathcal{P}(\alpha, a) \neq \emptyset$ and $\alpha(\mathcal{A}x) > a$ for all $x \in \partial\mathcal{P}(\alpha, a)$. Then, the operator \mathcal{A} has at least two fixed points, denoted by x_1 and x_2 , belonging to $\overline{\mathcal{P}(\gamma, c)}$ and satisfying $a < \alpha(x_1)$ with $\theta(x_1) < b$ and $b < \theta(x_2)$ with $\gamma(x_2) < c$.

Now, select t_* and t^* such that $0 < t_* < 1/2 < t^* < 1$. Let

$$\begin{aligned} M &= \frac{(1-t^*)(1-(1-t^*)^\rho)}{\Gamma(\rho+1)}, \\ N &= \frac{p\rho+1}{\Gamma(\rho+1)}, \\ L &= \frac{(1-t_*)^{\rho+1}}{\Gamma(\rho+1)}, \\ L_0 &= \frac{1+p\rho-t_*^\rho}{\Gamma(\rho+1)}. \end{aligned} \quad (35)$$

We are now in a position to obtain the following result.

Theorem 16. *Assume that hypotheses (A1)-(A2) all hold and that there exist positive real numbers a, b , and c such that*

$$0 < a < b < c, \quad a < \frac{L}{N}b < \frac{L(1-t^*)}{N}c. \quad (36)$$

Furthermore, assume that f satisfies the following conditions:

- (C1) $f(t, u) > c/M$ for (t, u) in $[0, t^*] \times [c, (1/(1-t^*))c]$;
- (C2) $f(t, u) < b/N$ for (t, u) in $[0, 1] \times [0, (1/(1-t^*))b]$;
- (C3) $f(t, u) > a/L$ for (t, u) in $[t_*, 1] \times [0, a]$.

Then BVP (2) has at least two positive solutions u_1 and u_2 such that

$$\begin{aligned} a < \max_{t \in [t_*, 1]} u_1(t) \quad \text{with} \quad \max_{t \in [t^*, 1]} u_1(t) < b, \\ b < \max_{t \in [t^*, 1]} u_1(t) \quad \text{with} \quad \min_{t \in [t_*, t^*]} u_2(t) < c. \end{aligned} \quad (37)$$

Proof. Let the cone \mathcal{P} and the operator \mathcal{A} be defined by (28) and (30), respectively. Furthermore, define the increasing,

nonnegative, and continuous functionals γ, θ , and α on \mathcal{P} , respectively, by

$$\begin{aligned} \gamma(u) &= \min_{t \in [t_*, t^*]} u(t) = u(t_*), \\ \theta(u) &= \max_{t \in [t^*, 1]} u(t) = u(t^*), \\ \alpha(u) &= \max_{t \in [t_*, 1]} u(t) = u(t_*). \end{aligned} \quad (38)$$

Evidently, $\gamma(u) = \theta(u) \leq \alpha(u)$ for each u in \mathcal{P} .

Moreover, for each u in \mathcal{P} , Lemma 12 implies that $u(t^*) \geq (1-t^*)\|u\|$. Observing $\gamma(u) = u(t_*)$, we have

$$\|u\| \leq \frac{1}{1-t^*}\gamma(u), \quad (39)$$

for each u in \mathcal{P} . Also, notice that $\theta(\lambda u) = \lambda\theta(u)$ for each λ in $[0, 1]$ and u in $\partial\mathcal{P}(\theta, b)$. In addition, Lemma 13 guarantees that the operator $\mathcal{A} : \mathcal{P}(\gamma, c) \rightarrow \mathcal{P}$ is completely continuous.

Next, we are to verify that all the conditions of Theorem 15 are satisfied with respect to the operator \mathcal{A} .

Let $u \in \partial\mathcal{P}(\gamma, c)$. Then $\gamma(u) = \min_{t \in [t_*, t^*]} u(t) = u(t^*) = c$. This implies that $u(t) \geq c$ for each t in $[0, t^*]$, which, combined with (39), yields that

$$c \leq u(t) \leq \frac{1}{1-t^*}c, \quad (40)$$

for each t in $[0, t^*]$. This inequality and assumption (C1) imply

$$f(t, u(t)) > \frac{c}{M}, \quad (41)$$

for each t in $[0, t^*]$. Now, from the definition of the operator \mathcal{A} and Lemmas 8 and 12, we obtain that

$$\begin{aligned} \gamma(\mathcal{A}u) &= [\mathcal{A}u](t^*) \geq (1-t^*)\|\mathcal{A}u\| = (1-t^*)[\mathcal{A}u](0) \\ &= (1-t^*) \int_0^1 G(0, s) f(s, u(s)) ds \\ &> (1-t^*) \cdot \frac{c}{M} \cdot \int_0^{t^*} G_1(0, s) ds \\ &> (1-t^*) \cdot \frac{c}{M} \cdot \frac{1-(1-t^*)^\rho}{\Gamma(\rho+1)} = c. \end{aligned} \quad (42)$$

Thus condition (i) in Theorem 15 is satisfied.

We now claim that condition (ii) in Theorem 15 is satisfied. To this end, let $u \in \partial\mathcal{P}(\theta, b)$. Then, $\gamma(u) = \theta(u) = \max_{t \in [t^*, 1]} u(t) = u(t^*) = b$, from which we have $0 \leq u(t) \leq b$ for each t in $[t^*, 1]$. Analogously, it follows from inequality (39) that, for each u in \mathcal{P} ,

$$\|u\| \leq \frac{1}{1-t^*}\gamma(u) = \frac{1}{1-t^*}b, \quad (43)$$

which implies $0 \leq u(t) \leq (1/(1-t^*))b$ for each t in $[0, 1]$. This, combined with assumption (C2), yields $f(t, u(t)) < b/N$ for each t in $[0, 1]$. Thus we have

$$\begin{aligned} \theta(\mathcal{A}u) &= \max_{t \in [t^*, 1]} [\mathcal{A}u](t) = [\mathcal{A}u](t^*) \leq [\mathcal{A}u](0) \\ &= \int_0^1 G(0, s) f(s, u(s)) ds \\ &= \int_0^1 G_1(0, s) f(s, u(s)) ds \\ &\quad + \int_0^1 G_2(0, s) f(s, u(s)) ds \\ &< \frac{b}{N} \left[\int_0^1 \frac{1}{\Gamma(\rho)} (1-s)^{\rho-1} ds \right. \\ &\quad \left. + \int_0^1 \frac{1}{\Gamma(\rho-1)} P(1-s)^{\rho-2} ds \right] \\ &= \frac{b}{N} \cdot \frac{1+p\rho}{\Gamma(\rho+1)} = b, \end{aligned} \tag{44}$$

which consequently implies the validity of condition (ii) in Theorem 15.

Finally, notice that the constant function $(1/2)a \in \mathcal{P}(\alpha, a)$ so that $\mathcal{P}(\alpha, a) \neq \emptyset$. Letting $u \in \partial\mathcal{P}(\alpha, a)$, we get $\alpha(u) = \max_{t \in [t^*, 1]} u(t) = u(t_*) = a$. This with assumption (C3) implies that $0 \leq u(t) \leq a$ and $f(t, u) > a/L$ for each t in $[t_*, 1]$. Similarly, we have

$$\begin{aligned} \alpha(\mathcal{A}u) &= [\mathcal{A}u](t_*) \geq (1-t_*) [\mathcal{A}u](0) \\ &= (1-t_*) \int_0^1 G(0, s) f(s, u(s)) ds \\ &> (1-t_*) \frac{a}{L} \cdot \int_{t_*}^1 G_1(0, s) ds \\ &> (1-t_*) \frac{a}{L} \cdot \frac{(1-t_*)^\rho}{\Gamma(\rho+1)} = a. \end{aligned} \tag{45}$$

Thus condition (iii) in Theorem 15 is satisfied.

Consequently, an application of Theorem 15 implies that BVP (2) has at least two positive solutions, denoted by u_1 and u_2 , satisfying $a < \alpha(u_1)$ with $\theta(u_1) < b$ and $b < \theta(u_2)$ with $\gamma(u_2) < c$, respectively. \square

4. Three Positive Solutions in Boundary Value Problems

In this section, we are to prove the existence of at least three positive solutions in BVP (2) by using the five-functional fixed point theorem which is attributed to Avery [19].

Let γ, β, θ be nonnegative continuous convex functionals on \mathcal{P} . α and ψ are supposed to be nonnegative continuous concave functionals on \mathcal{P} . Thus, for nonnegative

real numbers h, a, b, c , and d , define five convex sets, respectively, by

$$\begin{aligned} \mathcal{P}(\gamma, c) &= \{x \in \mathcal{P} \mid \gamma(x) < c\}, \\ \mathcal{P}(\gamma, \alpha, a, c) &= \{x \in \mathcal{P} \mid a \leq \alpha(x), \gamma(x) \leq c\}, \\ \mathcal{Q}(\gamma, \beta, d, c) &= \{x \in \mathcal{P} \mid \beta(x) \leq d, \gamma(x) \leq c\}, \\ \mathcal{P}(\gamma, \theta, \alpha, a, b, c) &= \{x \in \mathcal{P} \mid a \leq \alpha(x), \theta(x) \leq b, \gamma(x) \leq c\}, \\ \mathcal{Q}(\gamma, \beta, \psi, h, d, c) &= \{x \in \mathcal{P} \mid h \leq \psi(x), \beta(x) \leq d, \gamma(x) \leq c\}. \end{aligned} \tag{46}$$

Theorem 17 (see [19]). *Let \mathcal{P} be a cone in a real Banach space \mathcal{E} . Suppose that α and ψ are nonnegative continuous concave functionals on \mathcal{P} and that γ, β , and θ are nonnegative continuous convex functionals on \mathcal{P} such that, for some positive numbers c and M ,*

$$\alpha(x) \leq \beta(x), \quad \|x\| \leq M\gamma(x), \tag{47}$$

for all $x \in \overline{\mathcal{P}(\gamma, c)}$. In addition, suppose that $\mathcal{A} : \overline{\mathcal{P}(\gamma, c)} \mapsto \overline{\mathcal{P}(\gamma, c)}$ is a completely continuous operator and that there exist nonnegative real numbers h, d, a, b with $0 < d < a$ such that

- (i) $\{x \in \mathcal{P}(\gamma, \theta, \alpha, a, b, c) \mid \alpha(x) > a\} \neq \emptyset$ and $\alpha(\mathcal{A}x) > a$ for $x \in \mathcal{P}(\gamma, \theta, \alpha, a, b, c)$;
- (ii) $\{x \in \mathcal{Q}(\gamma, \beta, \psi, h, d, c) \mid \beta(x) < d\} \neq \emptyset$ and $\beta(\mathcal{A}x) < d$ for $x \in \mathcal{Q}(\gamma, \beta, \psi, h, d, c)$;
- (iii) $\alpha(\mathcal{A}x) > a$ for $x \in \mathcal{P}(\gamma, \alpha, a, c)$ with $\theta(\mathcal{A}x) > b$;
- (iv) $\beta(\mathcal{A}x) < d$ for $x \in \mathcal{Q}(\gamma, \beta, d, c)$ with $\psi(\mathcal{A}x) < h$.

Then the operator \mathcal{A} admits at least three fixed points x_1, x_2 , and $x_3 \in \overline{\mathcal{P}(\gamma, c)}$ satisfying $\beta(x_1) < d, a < \alpha(x_2)$, and $d < \beta(x_3)$ with $\alpha(x_3) < a$, respectively.

With this theorem, we are now in a position to establish the following result on the existence of at least three positive solutions in BVP (2).

Theorem 18. *Suppose that hypotheses (A1)-(A2) are all fulfilled. Assume that there exist positive real numbers a, b , and c such that*

$$\begin{aligned} 0 < a < b < c, \quad a < (1-t_*)b < (1-t_*)(1-t^*)c, \\ Nb < Mc. \end{aligned} \tag{48}$$

Furthermore, assume that f satisfies the following conditions:

- (H1) $f(t, u) < c/N$ for (t, u) in $[0, 1] \times [0, (1/(1-t^*))c]$;
- (H2) $f(t, u) > b/M$ for (t, u) in $[0, t^*] \times [b, (1/(1-t^*))^2b]$;
- (H3) $f(t, u) < a/L_0$ for (t, u) in $[0, 1] \times [0, (1/(1-t_*))a]$.

Then BVP (2) admits at least three positive solutions $u_1(t), u_2(t)$, and $u_3(t)$, defined on $[0, 1]$, satisfying, respectively,

$$\begin{aligned} \max_{t \in [t_*, 1]} u_1(t) < a, \quad b < \min_{t \in [0, t_*]} u_2(t), \\ a < \max_{t \in [t_*, 1]} u_3(t) \quad \text{with} \quad \min_{t \in [0, t_*]} u_3(t) < b. \end{aligned} \tag{49}$$

Proof. Let the cone \mathcal{P} and the operator \mathcal{A} be defined by (28) and (30), respectively. Define, respectively, the nonnegative continuous concave functionals on the \mathcal{P} as follows:

$$\begin{aligned} \gamma(u) &= \theta(u) = \max_{t \in [t^*, 1]} u(t) = u(t^*), \\ \alpha(u) &= \min_{t \in [0, t_*]} u(t) = u(t_*), \\ \beta(u) &= \max_{t \in [t_*, 1]} u(t) = u(t_*), \\ \psi(u) &= \min_{t \in [0, t^*]} u(t) = u(t^*). \end{aligned} \tag{50}$$

It is obvious that $\alpha(u) = \beta(u)$ for u in \mathcal{P} . Moreover, from Lemma 12, it follows that

$$\|u\| \leq \frac{1}{1-t^*} \gamma(u), \tag{51}$$

for each u in \mathcal{P} .

Next, we intend to verify that all the conditions in Theorem 17 hold with respect to the operator \mathcal{A} . We first claim that the operator $\mathcal{A} : \mathcal{P}(\gamma, c) \mapsto \mathcal{P}(\gamma, c)$ is completely continuous. By Lemma 13, we only need to show that $\mathcal{A}u \subset \mathcal{P}(\gamma, c)$ for each u in $\mathcal{P}(\gamma, c)$. To this end, let $u \in \mathcal{P}(\gamma, c)$. Then, $\gamma(u) = \max_{t \in [t^*, 1]} u(t) = u(t^*) \leq c$, which, combined with (51), implies that $0 \leq u(t) \leq (1/(1-t^*))c$ for t in $[0, 1]$ and u in \mathcal{P} . Thus, it follows from assumption (H1) that, for $t \in [0, 1]$, $f(t, u(t)) < c/N$, from which we have the following estimations:

$$\begin{aligned} \|\gamma(\mathcal{A}u)\| &= [\mathcal{A}u](t^*) \leq [\mathcal{A}u](0) \\ &= \int_0^1 G(0, s) f(s, u(s)) ds \\ &= \int_0^1 G_1(0, s) f(s, u(s)) ds \\ &\quad + \int_0^1 G_2(0, s) f(s, u(s)) ds \\ &< \frac{c}{N} \left[\int_0^1 \frac{1}{\Gamma(\rho)} (1-s)^{\rho-1} ds \right. \\ &\quad \left. + \int_0^1 \frac{1}{\Gamma(\rho-1)} P(1-s)^{\rho-2} ds \right] \\ &= \frac{c}{N} \cdot \frac{1+p\rho}{\Gamma(\rho+1)} = c. \end{aligned} \tag{52}$$

Hence we obtain the desired result. Now, it remains to verify that conditions (i)–(iv) in Theorem 17 are satisfied.

Since the constant function $u(t) \equiv ((2-t^*)/2(1-t^*))b$ belongs to the set

$$\left\{ u \in \mathcal{P} \left(\gamma, \theta, \alpha, b, \frac{1}{1-t^*} b, c \right) \mid \alpha(u) > b \right\}, \tag{53}$$

the set

$$\left\{ u \in \mathcal{P} \left(\gamma, \theta, \alpha, b, \frac{1}{1-t^*} b, c \right) \mid \alpha(u) > b \right\} \tag{54}$$

is not empty. Analogously, since

$$\begin{aligned} u(t) &\equiv \frac{1+t^*}{2} a \in \{u \in \mathcal{Q}(\gamma, \beta, \psi, (1-t^*)a, a, c) \mid \beta(u) < a\}, \end{aligned} \tag{55}$$

the set

$$\{u \in \mathcal{Q}(\gamma, \beta, \psi, (1-t^*)a, a, c) \mid \beta(u) < a\}, \tag{56}$$

is nonempty

In addition, for u in $\mathcal{P}(\gamma, \theta, \alpha, b, (1/(1-t^*))b, c)$, inequality (51) implies

$$\begin{aligned} b &\leq \min_{t \in [0, t^*]} u(t) = u(t^*) \leq u(t) \leq \frac{1}{1-t^*} \gamma(u) \\ &= \frac{1}{(1-t^*)} \theta(u) \leq \frac{1}{(1-t^*)^2} b, \end{aligned} \tag{57}$$

for each t in $[0, t^*]$. From assumption (H2), we thus obtain

$$f(t, u(t)) > \frac{b}{M} \quad \text{for } t \in [0, t^*]. \tag{58}$$

Hence, it follows from (58) and Lemma 12 that

$$\begin{aligned} \alpha(\mathcal{A}u) &= [\mathcal{A}u](t_*) \geq (1-t_*) [\mathcal{A}u](0) \\ &= (1-t_*) \int_0^1 G(0, s) f(s, u(s)) ds \\ &> \frac{b}{M} \cdot (1-t_*) \int_0^{t^*} G_1(0, s) ds = b. \end{aligned} \tag{59}$$

Therefore condition (i) in Theorem 17 is satisfied.

We next claim that condition (ii) in Theorem 17 is satisfied. To see this, letting $u \in \mathcal{Q}(\gamma, \beta, \psi, (1-t^*)a, a, c)$, then we get

$$0 \leq u(t) \leq \frac{1}{1-t_*} a, \tag{60}$$

for each t in $[0, 1]$. Thus assumption (H3) yields $f(t, u(t)) < a/L_0$. Furthermore, we have

$$\begin{aligned} \beta(\mathcal{A}u) &= \int_0^1 G(t_*, s) f(s, u(s)) ds \\ &< \frac{a}{L_0} \left[\int_0^1 G_1(s, s) ds - \int_0^{t_*} \frac{(t_*-s)^{\rho-1}}{\Gamma(\rho)} ds \right. \\ &\quad \left. + \int_0^1 G_2(t_*, s) ds \right] \\ &< \frac{a}{L_0} \left[\int_0^1 \frac{(1-s)^{\rho-1}}{\Gamma(\rho)} ds - \int_0^{t_*} \frac{(t_*-s)^{\rho-1}}{\Gamma(\rho)} ds \right. \\ &\quad \left. + \int_0^1 \frac{P(t_*-s)^{\rho-2}}{\Gamma(\rho-1)} ds \right] \\ &= \frac{a}{L_0} \cdot \frac{1+p\rho-t_*^\rho}{\Gamma(\rho+1)} = a. \end{aligned} \tag{61}$$

Accordingly, the validity of condition (ii) in Theorem 17 is verified.

Aside from conditions (i) and (ii), we are finally to verify the validity of conditions (iii) and (iv). For this purpose, on the one hand, consider $u \in \mathcal{P}(\gamma, \alpha, b, c)$ with $\theta(\mathcal{A}u) > (1/(1-t^*))b$. Thus we have

$$\alpha(\mathcal{A}u) = [\mathcal{A}u](t_*) \geq [\mathcal{A}u](t^*) = \theta(\mathcal{A}u) > \frac{1}{1-t^*}b > b. \tag{62}$$

On the other hand, consider $u \in \mathcal{Q}(\gamma, \beta, a, c)$ with $\psi(\mathcal{A}u) < (1-t^*)a$. In such a case, we obtain

$$\begin{aligned} \beta(\mathcal{A}u) &= [\mathcal{A}u](t_*) \leq \frac{1-t_*}{1-t^*} [\mathcal{A}u](t^*) \\ &= \frac{1-t_*}{1-t^*} \psi(\mathcal{A}u) < (1-t_*)a < a. \end{aligned} \tag{63}$$

Therefore both conditions (iii) and (iv) in Theorem 17 are satisfied. Consequently, by virtue of Theorem 17, BVP (2) has at least three positive solutions defined on $[0, 1]$ satisfying $\max_{t \in [t_*, 1]} u_1(t) < a$, $b < \min_{t \in [0, t_*]} u_2(t)$, and $a < \max_{t \in [t_*, 1]} u_3(t)$ with $\min_{t \in [0, t_*]} u_3(t) < b$. \square

5. An Illustrative Example

Consider BVP

$$\begin{aligned} D_{0+}^{5/2} u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, \\ u'(0) = u''(0) &= 0, \quad u(1) + u'(1) = \frac{1}{2}u'\left(\frac{1}{8}\right), \end{aligned} \tag{64}$$

where

$$f(t, u) = \frac{\Gamma(\rho + 1)}{\rho + 1} \cdot \frac{cu^2}{2c + t + u + u^2}, \tag{65}$$

for (t, u) in $[0, 1] \times [0, +\infty)$. Here, $\rho = 5/2$, and $c = 5 \times 10^6 \cdot (\rho + 1)/\Gamma(\rho + 1)$.

We claim that the above BVP has at least three positive solutions. To see this, letting $t_* = 1/4$, $t^* = 3/4$, we have

$$\begin{aligned} M &= \frac{1}{224} \cdot \frac{\rho + 1}{\Gamma(\rho + 1)}, \quad N = \frac{\rho + 1}{\Gamma(\rho + 1)} = \frac{28}{15\sqrt{\pi}} \approx 0.5942, \\ L_0 &= \frac{111}{121} \cdot \frac{\rho + 1}{\Gamma(\rho + 1)}. \end{aligned} \tag{66}$$

Selecting $a = 1/10$, $b = 10^3$, and $c = 5 \times 10^6 N$, then it is easy to check that these parameters satisfy

$$0 < a < (1-t_*)b < (1-t_*)(1-t^*)c, \quad Nb < Mc. \tag{67}$$

Now, we can verify that conditions (H1)–(H3) in Theorem 17 are satisfied. Indeed, direct computations produce the following estimations:

$$\begin{aligned} f(t, u) &< \frac{\Gamma(\rho + 1)}{\rho + 1} \cdot c = \frac{c}{N}, \\ &\text{for } (t, u) \text{ in } [0, 1] \times [0, 4c], \\ f(t, u) &\geq \frac{\Gamma(\rho + 1)}{\rho + 1} \cdot \frac{cb^2}{2c + t^* + b + b^2} > \frac{b}{M}, \\ &\text{for } (t, u) \text{ in } [0, t^*] \times [b, 16b], \\ f(t, u) &\leq \frac{\Gamma(\rho + 1)}{\rho + 1} \cdot \frac{1}{2} \cdot \left(\frac{4a}{3}\right)^2 < \frac{a}{L_0}, \\ &\text{for } (t, u) \text{ in } [0, 1] \times \left[0, \frac{4}{3}a\right]. \end{aligned} \tag{68}$$

Thus conditions (H1)–(H3) in Theorem 17 are satisfied for the above specified functions and parameters. Therefore, in light of Theorem 17, the assertion made above is verified.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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