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Research Article

On the Boundary of Self-Affine Sets

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This paper is devoted to studying the boundary behavior of self-affine sets. We prove that the boundary of an integral self-affine set has Lebesgue measure zero. In addition, we consider the variety of the boundary of a self-affine set when some other contractive maps are added. We show that the complexity of the boundary of the new self-affine set may be the same, more complex, or simpler; any one of the three cases is possible.

1. Introduction

Let (X, ρ) be a complete matric space. Recall that a map $S : X \to X$ is contractive if there exists a constant $0 < r < 1$ such that $\rho(S(x), S(y)) \le r\rho(x, y)$. We call a finite set of contractive maps $\{S_j\}_{j=1}^m$ an *iterated function system* (IFS). It is well known [1] that there exists a unique nonempty compact subset $K \subset X$ such that $K = \bigcup_{j=1}^m S_j(K)$. We call K the *invariant set* or *attractor* of the IFS. Moreover, if we associate the IFS with a set of probability weights $\{p_i > 0 : i = 1, \ldots, m\}$, then there exists a unique probability measure μ supported on K satisfying the equation

$$\mu(\cdot) = \sum_{j=1}^m p_j \mu\left(S_j^{-1}(\cdot)\right). \tag{1}$$

We call μ the *invariant measure*.

Let A be a $d \times d$ expanding real matrix; that is, all its eigenvalues have modules larger than one. Let λ be the smallest absolute value of A's eigenvalues, choose $c \in (1, \lambda)$, and define $\|x\|$ for each $x \in \mathbb{R}^d$ as

$$\|x\| = \sum_{n=1}^\infty c^n \left|A^{-n}x\right|, \tag{2}$$

where $|\cdot|$ is the Euclidian norm in $\mathbb{R}^d$. Then $\|\cdot\|$ is a norm in $\mathbb{R}^d$. Let $\rho(x, y) = \|x - y\|$ be the induced metric. It is easy to check that the map $S(x) = A^{-1}(x + c)$ with $x, c \in \mathbb{R}^d$ is contractive under the metric ρ.

Let A be a $d \times d$ expanding real matrix and $\mathscr{D} = \{d_1, d_2, \ldots, d_m\} \subset \mathbb{R}^d$. We call the family of maps on $\mathbb{R}^d$

$$S_i(x) = A^{-1}(x + d_i), \quad i = 1, 2, \ldots, m \tag{3}$$

a *self-affine* IFS. The corresponding invariant set K and invariant measure μ are called a *self-affine set* and a *self-affine measure* of the IFS, respectively. Furthermore, if the matrix A in (3) is an orthonormal matrix multiple a constant, then such IFS is called *self-similar*, and the invariant set and invariant measure are called *self-similar set* and *self-similar measure* of the IFS, respectively.

Our main interests in this note are the structures and properties of the boundary ∂K of a self-affine set K. For self-similar IFS, Lau and Xu [2] showed that $\dim_H(\partial K) < d$ provided that the self-similar IFS satisfies the *open set condition* (OSC). He et al. [3] studied the calculation of $\dim_H(\partial K)$ for integral self-similar IFS. Furthermore, the overlapping cases were considered by Lau and Ngai in [4]. For self-affine sets, however, less is known about K and ∂K (see [5–7]). There is no method to compute the Hausdorff dimension and the Lebesgue measure $\mathscr{L}(\partial K)$ of ∂K for overlapping self-affine set.

Motivated by these results, we consider the Lebesgue measures of the boundaries of integral self-affine sets. We prove that they have Lebesgue measure zero.

Theorem 1. *Let $\{A^{-1}(x+d_j)\}_{j=1}^m$ be a self-affine IFS defined on $\mathbb{R}^d$. Assume that A and d_j are all integral. Let K be the self-affine set of the IFS; then $\mathscr{L}(\partial K)=0$.*

Consider two IFSs $\{S_j\}_{j=1}^m$ and $\{S_j\}_{j=1}^n$, $m < n$ (they may not be self-affine). Let K_1 and K_2 be the invariant sets, respectively; then $K_1 \subseteq K_2$, so $\dim(K_1) \leq \dim(K_2)$. We think about the natural question: what is the relationship between ∂K_1 and ∂K_2?

We prove that any one case of $\dim_H(\partial K_2)=\dim_H(\partial K_1)$, $\dim_H(\partial K_2) < \dim_H(\partial K_1)$, and $\dim_H(\partial K_2) > \dim_H(\partial K_1)$ may occur.

2. Proofs of Results

For an IFS $\{S_j\}_{j=1}^n$ on $\mathbb{R}^d$, we use the following notations throughout the paper. Let $\Sigma_m=\{1,\ldots,m\}$ (or Σ if there is no confusion), and $\Sigma^*=\bigcup_{n\geq 1}\Sigma^n$. For any $I=i_1i_2\cdots i_n\in\Sigma^n$ and $J=j_1j_2\cdots j_k\in\Sigma^k$, let $IJ=i_1i_2\cdots i_nj_1j_2\cdots j_k$ and

$$\begin{gathered} p_I = p_{i_1}p_{i_2}\cdots p_{i_n}, \qquad S_I=S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_n},\\ d_I = d_{i_n}+Ad_{i_{n-1}}+\cdots+A^{n-1}d_{i_1}, \\ \mathscr{D}_n=\mathscr{D}+A\mathscr{D}+\cdots+A^{n-1}\mathscr{D}. \end{gathered} \tag{4}$$

Also, we use $\mathscr{L}(E)$, E^o, and ∂E to denote the Lebesgue measure, the interior, and the boundary of a subset $E\subset\mathbb{R}^d$, respectively.

Theorem 2. *Let $\{\phi_j\}_{j=1}^m$ and $\{\psi_i\}_{i=1}^k$ be two contractive IFSs on $\mathbb{R}^d$ under some norm $\|\cdot\|$ with the invariant sets K_1 and K_2, respectively. If the invariant set K_1 contains interior points, then there exist $a,n\in\mathbb{N}$ and $\alpha\in\mathbb{Z}^d$ such that the IFSs $\mathscr{F}=\{\varphi_{i_1i_2\cdots i_n}: 1\leq i_j\leq m\}$ and $\mathscr{F}\cup\mathscr{G}$ generate the same attractor $aK_1+\alpha$, where $\mathscr{G}=\{\psi_{j_1j_2\cdots j_n}: 1\leq j_i\leq k\}$ and $\varphi_j(x)=a\phi_j(a^{-1}(x-\alpha))+\alpha$, $j=1,\ldots,m$.*

Proof. Observe that

$$\begin{aligned} \bigcup_{j=1}^m \varphi_j(aK_1+\alpha) &= \bigcup_{j=1}^m \left(a\phi_j(K_1)+\alpha\right) \\ &= a\left(\bigcup_{j=1}^m \phi_j(K_1)\right)+\alpha = aK_1+\alpha. \end{aligned} \tag{5}$$

This means that $aK_1+\alpha$ is the invariant set of $\{\varphi_j\}_{j=1}^m$ for any $a>0$ and $\alpha\in\mathbb{R}^d$. Hence it is also the invariant set of the IFS $\mathscr{F}$. Now we need only to prove that $aK_1+\alpha$ is the invariant set of $\mathscr{F}\cup\mathscr{G}$ for some $a,n\in\mathbb{N}$ and $\alpha\in\mathbb{Z}^d$.

Note that K_1 contains interior points; we can find a constant $r>0$ and a point $x_0\in K_1$ with rational entries such that $B_{2r}(x_0)\subset K_1$. Hence $B_{2ar}(0)\subset aK_1-ax_0$ for all positive real number $a>0$. Since $\{\psi_i\}_{i=1}^k$ are contractive in the norm $\|\cdot\|$, we can choose integers $a,n\in\mathbb{N}$ large enough such that $K_2\subset B_{ar}(0)$ and $|\psi_J(aK_1+\alpha)|<ar$ for all $J\in\Sigma_k^*$ with $|J|\geq n$, where $|E|$ is the diameter of the set $E\subset\mathbb{R}^d$ under the norm $\|\cdot\|$. Also, we can assume that $\alpha=-ax_0\in\mathbb{Z}^d$. Noting $K_2\subseteq B_{ar}(0)\subseteq B_{2ar}(0)\subseteq aK_1+\alpha$, $|\psi_{j_1j_2\cdots j_n}(aK_1+\alpha)|<ar$ and observing

$$\psi_{j_1j_2\cdots j_n}(aK_1+\alpha)\cap K_2 \supseteq \psi_{j_1j_2\cdots j_n}(K_2)\neq\emptyset, \tag{6}$$

we have

$$\psi_{j_1j_2\cdots j_n}(aK_1+\alpha)\subseteq aK_1+\alpha. \tag{7}$$

Therefore

$$aK_1+\alpha=\bigcup_{f\in\mathscr{F}} f(aK_1+\alpha)\subseteq \bigcup_{f\in\mathscr{F}\cup\mathscr{G}} f(aK_1+\alpha)\subseteq aK_1+\alpha. \tag{8}$$

We see that $aK_1+\alpha$ is the invariant set of $\mathscr{F}\cup\mathscr{G}$. This completes the proof. □

In Theorem 2, IFS $\mathscr{F}$ is a subset of IFS $\mathscr{F}\cup\mathscr{G}$ and they have the same invariant set $aK_1+\alpha$. So do the same boundary of the invariant set. On the other hand, the invariant set of $\mathscr{G}$ is K_2. Obviously, either $\dim_H(\partial(aK_1+\alpha))<\dim_H(\partial K_2)$ or $\dim_H(\partial(aK_1+\alpha))>\dim_H(\partial K_2)$ may occur.

In the following, we consider the Lebesgue measure of ∂K for the self-affine IFS (3). We will prove Theorem 1; that is, $\mathscr{L}(\partial K)=0$ if A and d_j are all integral. For this, we first prove some lemmas.

Lemma 3. *Let the IFS in (3) be integral; that is, all entries of A and d_j are integers. Assume that the self-affine set K has positive Lebesgue measure; then $K^o\neq\emptyset$.*

Proof. Note that the fact that A and d_j are all integral implies that the IFS is uniformly discrete, and the assertion follows from [7, Theorem 3.1]. □

Lemma 4. *Let the IFS in (3) be integral. Suppose that $\{d_j\}_{j=1}^m$ contains a complete set of residues $(\operatorname{mod} A\mathbb{Z}^d)$. Then the self-affine measure μ in (1) is absolutely continuous with respect to the Lebesgue measure provided that*

$$\sum_{j:(d_i-d_j)\in A\mathbb{Z}^d} p_j=\frac{1}{|\det(A)|},\quad i=1,\ldots,m. \tag{9}$$

Proof. Without loss of generality, assume that $\widetilde{\mathscr{D}}=\{d_1,\ldots,d_\ell\}$ is a complete set of residues $(\operatorname{mod} A\mathbb{Z}^d)$ with $|\det(A)|=\ell$. Then $\widetilde{\mathscr{D}}_n:=\widetilde{\mathscr{D}}+A\widetilde{\mathscr{D}}+\cdots+A^{n-1}\widetilde{\mathscr{D}}$ is a complete set of residues (mod $A^n\mathbb{Z}^d$).

For each $i\in\{1,\ldots,\ell\}$, let $I_i=\{j: 1\leq j\leq m, (d_j-d_i)\in A\mathbb{Z}^d\}$ and $p_j=1/\ell\# I_i$ if $j\in I_i$; then we have

$$\sum_{j\in I_i} p_j=\frac{1}{|\det(A)|},\quad i=1,\ldots,\ell. \tag{10}$$

Hence such probability weights $\{p_i\}_{i=1}^m$ satisfying (9) always exist.

To prove the absolute continuity of μ, by making use of [8, Theorem 3.5], we need only to show that

$$\sum_{J\in\Sigma^n, d_J=z} p_J \le |\det(A)|^{-n}, \quad \forall n>0,\ z\in\mathbb{Z}^d. \tag{11}$$

We will prove this by induction on n. By (9), the inequality (11) holds for $n=1$. Assume that (11) holds for $n=k$. Let $z=d_i+Az_1$ with $d_i\in\widetilde{\mathscr{D}}$ and $z_1\in\mathbb{Z}^d$. If $J\in\Sigma^k$, $j\in\Sigma$, and $d_{Jj}=z$, then $d_j+Ad_J=d_i+Az_1$, so $(d_j-d_i)\in A\mathbb{Z}^d$, and let $d_j=d_i+Ae_j$ with $e_j\in\mathbb{Z}^d$; we have $e_j+d_J=z_1$. Therefore

$$\begin{aligned}\sum_{Jj\in\Sigma^{k+1}, d_{Jj}=z} p_{Jj} &\le \sum_{j\in\Sigma,(d_j-d_i)\in A\mathbb{Z}^d} p_j \sum_{J\in\Sigma^k, d_J=z_1-e_j} p_J \\ &\le |\det(A)|^{-k} \sum_{j\in\Sigma,(d_j-d_i)\in A\mathbb{Z}^d} p_j \le |\det(A)|^{-(k+1)}.\end{aligned} \tag{12}$$

Hence (11) is also true for $n=k+1$. This completes the proof. □

Remark. Lemma 4 gives a sufficient condition for the existence of L^1-solutions of integral refinement equations:

$$f(x)=|\det(A)|\sum_{j=1}^{m} p_j f\left(Ax-d_j\right) \tag{13}$$

provided that $\{d_1,\dots,d_m\}\subset\mathbb{Z}^d$ contains a complete set of residues $(\operatorname{mod} A\mathbb{Z}^d)$. Condition (9) ensures that the refinement equation has a unique (up to a scalar multiple) bounded L^1-solution with compact support if p_j's satisfy (9). Condition (9) is an extension of the "sum role."

Lemma 5. *Let the IFS in (3) be integral. Suppose $\{d_j\}_{j=1}^m$ contains a complete set of residues $(\operatorname{mod} A\mathbb{Z}^d)$; K is the corresponding self-affine set. Then $\mathscr{L}(\partial K)=0$.*

Proof. Lemma 4 implies that there exist probability weights $\{p_j\}_{j=1}^m$ such that the corresponding self-affine measure μ is absolutely continuous with respect to the Lebesgue measure and so $\mathscr{L}(K)>0$.

Lemma 3 implies that $K^o\neq\emptyset$, so K^o is a nonempty invariant open set (i.e., $\bigcup_{j=1}^m S_j(K^o)\subseteq K^o$) and $\mu(K^o)>0$. Then [8, Theorem 4.13] implies that $\mu(\partial K)=0$. On the other hand, [8, Theorem 3.12] implies that the Lebesgue measure restricted on K is also absolutely continuous with respect to μ. Hence $\mathscr{L}(\partial K)=0$. □

Now we can prove the main theorem of the paper.

Proof of Theorem 1. If $K^o=\emptyset$, then $\partial K=K$ and Lemma 3 implies that $\mathscr{L}(\partial K)=0$.

Now we consider the case $K^o\neq\emptyset$. Let $\phi_j(x)=A^{-1}(x+d_j)$, $\varphi_j(x)=a\phi_j(a^{-1}(x-\alpha))+\alpha=A^{-1}(x-\alpha+ad_j+A\alpha)$, $j=1,\dots,m$, and $\psi_i(x)=A^{-1}(x+z_i)$, $i=1,\dots,k$, where $\mathscr{Z}=\{z_1=0,\dots,z_k\}$ is a complete set of residues $(\operatorname{mod} A\mathbb{Z}^d)$. Making use of Theorem 2 and the notations there, there exist $a,n\in\mathbb{N}$ and $\alpha\in\mathbb{Z}^d$ such that the IFSs $\mathscr{F}$ and $\mathscr{F}\cup\mathscr{G}$ have the same attractor $aK+\alpha$. Let $\widetilde{\mathscr{D}}=a\mathscr{D}-\alpha+A\alpha$. Then $\mathscr{F}\cup\mathscr{G}=\{A^{-n}(x+d): d\in\widetilde{\mathscr{D}}_n\cup\mathscr{Z}_n\}$. Note that $\widetilde{\mathscr{D}}_n\cup\mathscr{Z}_n$ contains a complete set $\mathscr{Z}_n$ of residues $(\operatorname{mod} A\mathbb{Z}^d)$; Lemma 5 implies that $\mathscr{L}(\partial K)=a^{-d}\mathscr{L}(\partial(aK+\alpha))=0$. We complete the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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