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Research Article

Positive Definite Solutions of the Matrix Equation $X^r - \sum_{i=1}^{m} A_i^* X^{-\delta_i} A_i = I$

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We investigate the nonlinear matrix equation $X^r - \sum_{i=1}^{m} A_i^* X^{-\delta_i} A_i = I$, where r is a positive integer and $\delta_i \in (0,1]$, for $i = 1, 2, \ldots, m$. We establish necessary and sufficient conditions for the existence of positive definite solutions of this equation. A sufficient condition for the equation to have a unique positive definite solution is established. An iterative algorithm is provided to compute the positive definite solutions for the equation and error estimate. Finally, some numerical examples are given to show the effectiveness and convergence of this algorithm.

1. Introduction

We consider the nonlinear matrix equation:

$$X^r - \sum_{i=1}^{m} A_i^* X^{-\delta_i} A_i = I, \tag{1}$$

where I is an $n \times n$ identity matrix, r is a positive integer, A_i are $n \times n$ complex matrices, and $\delta_i \in (0,1]$, for $i = 1, 2, \ldots, m$. This type of equations arises in solving a large-scale system of linear equations. In many physical applications, we must solve the system of linear equation

$$Mx = f, \tag{2}$$

where M arises from a finite difference approximation to an elliptic partial differential equation [1]. For example, let

$$M = \begin{pmatrix} I & 0 & \cdots & 0 & A_1 \\ 0 & I & \cdots & 0 & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & A_m \\ A_1^* & A_2^* & \cdots & A_m^* & -I \end{pmatrix}. \tag{3}$$

We can rewrite M as $M = \widetilde{M} + D$, where

$$\widetilde{M} = \begin{pmatrix} X^{\delta_1} & 0 & \cdots & 0 & A_1 \\ 0 & X^{\delta_2} & \cdots & 0 & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X^{\delta_m} & A_m \\ A_1^* & A_2^* & \cdots & A_m^* & -I \end{pmatrix},$$
$$D = \begin{pmatrix} I - X^{\delta_1} & 0 & \cdots & 0 & 0 \\ 0 & I - X^{\delta_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I - X^{\delta_m} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4}$$

We can decompose the matrix $\widetilde{M}$ to the LU decomposition

$$\widetilde{M} = \begin{pmatrix} -I & 0 & \cdots & 0 & 0 \\ 0 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & 0 \\ -A_1^* X^{-\delta_1} & -A_2^* X^{-\delta_2} & \cdots & -A_m^* X^{-\delta_m} & -I \end{pmatrix} \begin{pmatrix} -X^{\delta_1} & 0 & \cdots & 0 & -A_1 \\ 0 & -X^{\delta_2} & \cdots & 0 & -A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -X^{\delta_m} & -A_m \\ 0 & 0 & \cdots & 0 & X^r \end{pmatrix} \quad (5)$$

if and only if X is a solution of (1). In addition, similar types of (1) have many applications in various areas, including control systems, ladder networks, dynamic programming, control theory, stochastic filtering, statistics, and optimal interpolation problem [2–5].

During the last few years, many researchers worked to develop the theory and numerical approaches for positive definite solutions to the nonlinear matrix equations of the form (1) [6–9]. Recently, Duan et al. [10] showed that the equation $X - \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i = Q$ $(0 < |\delta_i| < 1)$ always has a unique positive definite solution by using the fixed point theorem of mixed monotone operators. They proposed an iterative method for computing the unique positive definite solution. Lim [11] presented a new proof for the existence and uniqueness of the positive definite solution for the last equation. Yin and Liu [12] derived necessary conditions and sufficient conditions for the existence of positive definite solutions for the matrix equations $X \pm A^* X^{-q} A = Q$ with $q \geq 1$; they also proposed iterative methods for obtaining positive definite solutions of these equations. He and Long [13] obtained some conditions for the existence of positive definite solution of the equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$. They also presented two iterative algorithms to find the maximal positive definite solution of this equation. Duan et al. [14] used the Thompson metric to prove that the matrix equation $X - \sum_{i=1}^{m} N_i^* X^{-1} N_i = I$ always has a unique positive definite solution and they gave a precise perturbation bound for the unique positive definite solution. Similar kinds of equations have been investigated by many authors [15–20].

In this paper, we study the positive definite solutions of (1). The paper is organized as follows. In Section 2, we derive necessary and sufficient conditions for the existence of positive definite solutions of (1). We give a sufficient condition for the existence of a unique positive definite solution of this equation. In Section 3, we propose an iterative algorithm to obtain the positive definite solutions of (1). Moreover, we investigate convergence rate of the proposed algorithm. Finally, we give some numerical examples in Section 4 to ensure the performance and the effectiveness of the suggested algorithm.

We start with some notations which will be used in this paper. The symbol $C^{n\times n}$ denotes the set of $n \times n$ complex matrices. A^* denotes the complex conjugate transpose of A. We write $A > 0$ $(A \geq 0)$, if matrix A is positive definite (positive semidefinite). If $A - B$ is positive definite (positive semidefinite), then we write $A > B$ $(A \geq B)$. Moreover, we denote $\rho(A)$ by the spectral radius of A. $\|\cdot\|$, $\|\cdot\|_\infty$, and $\|\cdot\|_F$ denote the spectral, infinity, and Frobenius norm. For $A = (a_1, a_2, \ldots, a_n) = (a_{ij})$ and a matrix B, $A \otimes B = (a_{ij}B)$ is a Kronecker product. $\mathrm{vec}(A)$ is a vector defined by $\mathrm{vec}(A) = (a_1^T, a_2^T, \ldots, a_n^T)^T$.

2. Conditions for the Existence of the Solutions

In this section, we will derive the necessary and sufficient conditions for existence of the positive definite solutions of (1). We give a sufficient condition for the existence of a unique positive definite solution of (1). We begin with some lemmas, which will play a key role to establish our results.

Lemma 1 (see [21]). *If $A > B > 0$ (or $A \geq B > 0$), then $A^\alpha > B^\alpha > 0$ (or $A^\alpha \geq B^\alpha > 0$), for all $\alpha \in (0, 1]$, and $0 < A^\alpha < B^\alpha$ (or $0 < A^\alpha \leq B^\alpha$), for all $\alpha \in [-1, 0)$.*

Lemma 2 (see [22]). *If A and B are $n \times n$ matrices, then $\|AB\|_F \leq \|A\|_F \|B\|$ and $\|AB\|_F \leq \|A\| \|B\|_F$.*

Lemma 3 (see [23]). *Let A and B be positive operators such that $A \geq aI \geq 0$ and $B \geq bI \geq 0$; then, $|||f(A) - f(B)||| \leq C(a,b)|||A - B|||$, for every nonnegative operator monotone function f on $(0, \infty)$, where $C(a,b) = (f(a) - f(b))/(a - b)$ if $a \neq b$ and $C(a,b) = f'(a)$ if $a = b$. Here, $|||\cdot|||$ stands for a unitarily invariant norm on matrices.*

Theorem 4. *If (1) has a positive definite solution X, then*

$$I \leq X \leq \left(I + \sum_{i=1}^{m} A_i^* A_i \right)^{1/r}. \quad (6)$$

Proof. Since X is a positive definite solution of (1), then

$$X^r = I + \sum_{i=1}^{m} A_i^* X^{-\delta_i} A_i \geq I; \quad (7)$$

by Lemma 1, we have $X \geq I$. Also, from inequality $X \geq I$ and Lemma 1, we have $A_i^* X^{-\delta_i} A_i \leq A_i^* A_i$. Therefore,

$$X^r = I + \sum_{i=1}^{m} A_i^* X^{-\delta_i} A_i \leq I + \sum_{i=1}^{m} A_i^* A_i. \quad (8)$$

Hence, $X \leq (I + \sum_{i=1}^{m} A_i^* A_i)^{1/r}$ implies that $I \leq X \leq (I + \sum_{i=1}^{m} A_i^* A_i)^{1/r}$. □

Theorem 5. *Equation (1) has a positive definite solution X if and only if the coefficient matrices A_i, $i = 1, 2, \ldots, m$, can be factored as*

$$A_i = (W^*W)^{\delta_i/2} Y_i (W^*W)^{r/2}, \tag{9}$$

where W is a nonsingular matrix, $Z = (W^*W)^{-r/2}$, and $\begin{pmatrix} Z \\ Y_1 \\ \vdots \\ Y_m \end{pmatrix}$ is column-orthonormal. In this case, $X = W^*W$ is a solution of (1).

Proof. Let X be a positive definite solution of (1); then, $X = W^*W$, where W is a nonsingular matrix. Furthermore, (1) can be rewritten as

$$(W^*W)^r - \sum_{i=1}^{m} A_i^* (W^*W)^{-\delta_i} A_i = I, \tag{10}$$

which implies that

$$\begin{aligned}&(W^*W)^{-r/2} (W^*W)^{-r/2} + \sum_{i=1}^{m} (W^*W)^{-r/2} \\ &\quad \cdot A_i^* (W^*W)^{-\delta_i/2} (W^*W)^{-\delta_i/2} A_i (W^*W)^{-r/2} = I.\end{aligned} \tag{11}$$

Let $Z = (W^*W)^{-r/2}, Y_i = (W^*W)^{-\delta_i/2} A_i (W^*W)^{-r/2}$; then, $A_i = (W^*W)^{\delta_i/2} Y_i (W^*W)^{r/2}$ and (11) turns into

$$Z^*Z + \sum_{i=1}^{m} Y_i^* Y_i = I; \tag{12}$$

that is,

$$\begin{pmatrix} Z \\ Y_1 \\ \vdots \\ Y_m \end{pmatrix}^* \begin{pmatrix} Z \\ Y_1 \\ \vdots \\ Y_m \end{pmatrix} = I, \tag{13}$$

which means that $\begin{pmatrix} Z \\ Y_1 \\ \vdots \\ Y_m \end{pmatrix}$ is column-orthonormal.

Conversely, suppose that A_i has the decomposition (9). Let $X = W^*W$; then, X is a positive definite matrix, and we have

$$\begin{aligned}X^r - \sum_{i=1}^{m} A_i^* X^{-\delta_i} A_i &= (W^*W)^r - \sum_{i=1}^{m} A_i^* (W^*W)^{-\delta_i} A_i \\ &= (W^*W)^r - \sum_{i=1}^{m} (W^*W)^{r/2} Y_i^* (W^*W)^{\delta_i/2} \\ &\quad \cdot (W^*W)^{-\delta_i} (W^*W)^{\delta_i/2} Y_i (W^*W)^{r/2} = (W^*W)^r \\ &\quad - \sum_{i=1}^{m} (W^*W)^{r/2} Y_i^* Y_i (W^*W)^{r/2} = (W^*W)^{r/2} \Big[I \\ &\quad - \sum_{i=1}^{m} Y_i^* Y_i \Big] (W^*W)^{r/2} = (W^*W)^{r/2} \\ &\quad \cdot Z^*Z (W^*W)^{r/2} = I.\end{aligned} \tag{14}$$

Hence, $X = W^*W$ is a positive definite solution of (1). □

Theorem 6. *If $A_1, A_2, \ldots, A_m$ are invertible, then (1) has a positive definite solution X if and only if A_i, $i = 1, 2, \ldots, m$, can be factored as*

$$A_i = P^* \Gamma^{\delta_i/2} V_i \Theta P, \tag{15}$$

where P is unitary, $\Gamma > 0, \Theta > 0$ are diagonal matrices such that $\Gamma^r - \Theta^2 = I$, and $\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{pmatrix}$ is column-orthonormal. In this case, $X = P^*\Gamma P$ is a solution of (1).

Proof. Let X be a positive definite solution of (1); then, by spectral decomposition of X, we have $X = P^*\Gamma P$, where P is unitary and $\Gamma > 0$ is diagonal. Therefore, (1) can be rewritten as

$$(P^*\Gamma P)^r - \sum_{i=1}^{m} A_i^* (P^*\Gamma P)^{-\delta_i} A_i = I, \tag{16}$$

which implies that

$$\Gamma^r - I = \sum_{i=1}^{m} P A_i^* P^* \Gamma^{-\delta_i} P A_i P^*, \tag{17}$$

$$\sum_{i=1}^{m} (\Gamma^r - I)^{-1/2} P A_i^* P^* \Gamma^{-\delta_i} P A_i P^* (\Gamma^r - I)^{-1/2} = I. \tag{18}$$

Let $V_i = \Gamma^{-\delta_i/2} P A_i P^* (\Gamma^r - I)^{-1/2}$ and $\Theta = (\Gamma^r - I)^{1/2}$. Then, $A_i = P^* \Gamma^{\delta_i/2} V_i \Theta P, \Theta > 0$, is a diagonal matrix and $\Gamma^r - \Theta^2 = I$. Moreover, (18) turns into $\sum_{i=1}^{m} V_i^* V_i = I$, which means that $\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_m \end{pmatrix}$ is column-orthonormal.

Conversely, suppose that A_i has the decomposition (15). Let $X = P^*\Gamma P$; then X is a positive definite matrix, and we have

$$\begin{aligned}X^r - \sum_{i=1}^{m} A_i^* X^{-\delta_i} A_i &= (P^*\Gamma P)^r - \sum_{i=1}^{m} A_i^* (P^*\Gamma P)^{-\delta_i} A_i \\ &= P^* \Gamma^r P - \sum_{i=1}^{m} A_i^* P^* \Gamma^{-\delta_i} P A_i \\ &= P^* \Gamma^r P - \sum_{i=1}^{m} P^* \Theta V_i^* V_i \Theta P \\ &= P^* \left[\Gamma^r - \Theta^2 \right] P = I.\end{aligned} \tag{19}$$

Hence, $X = P^*\Gamma P$ is a positive definite solution of (1). □

Theorem 7. *If $q = \sum_{i=1}^{m} \delta_i \|A_i\|^2 < 1$, then (1) has a unique positive definite solution.*

Proof. We consider the mapping $f(X) = [I + \sum_{i=1}^{m} A_i^* X^{-\delta_i} A_i]^{1/r}$.

Let $\Omega = \{X = X^* : I \leq X \leq [I + \sum_{i=1}^{m} A_i^* A_i]^{1/r}\}$. Obviously, Ω is a bounded closed convex set and f is continuous on Ω.

If $X \in \Omega$, then by Lemma 1, $X^{-\delta_i} \leq I$ and

$$I \leq \left[I + \sum_{i=1}^{m} A_i^* X^{-\delta_i} A_i\right]^{1/r} \leq \left[I + \sum_{i=1}^{m} A_i^* A_i\right]^{1/r}; \tag{20}$$

that is,

$$I \leq f(X) \leq \left[I + \sum_{i=1}^{m} A_i^* A_i\right]^{1/r}, \tag{21}$$

which means that $f(\Omega) \subseteq \Omega$. By Brouwer's fixed point theorem, f has a fixed point in Ω; thus, (1) has a solution in Ω.

Now, we prove that f is a contraction mapping on Ω. For arbitrary $X, Y \in \Omega$, we get

$$\begin{aligned}
&\|f(X)^r - f(Y)^r\|_F \\
&= \left\|\sum_{k=0}^{r-1} f(X)^k (f(X) - f(Y)) f(Y)^{r-1-k}\right\|_F \\
&= \left\|\text{vec}\left[\sum_{k=0}^{r-1} f(X)^k (f(X) - f(Y)) f(Y)^{r-1-k}\right]\right\| \\
&= \left\|\sum_{k=0}^{r-1} \text{vec}\left[f(X)^k (f(X) - f(Y)) f(Y)^{r-1-k}\right]\right\| \\
&= \left\|\sum_{k=0}^{r-1} \left(f(Y)^{r-1-k} \otimes f(X)^k\right) \text{vec}\left(f(X) - f(Y)\right)\right\| \\
&\geq r \left\|\text{vec}\left(f(X) - f(Y)\right)\right\| = r \|f(X) - f(Y)\|_F.
\end{aligned} \tag{22}$$

According to the definition of the mapping f, we have

$$f(X)^r - f(Y)^r = \sum_{i=1}^{m} A_i^* \left(X^{-\delta_i} - Y^{-\delta_i}\right) A_i. \tag{23}$$

From (22) and (23), and by Lemmas 2 and 3, we have

$$\begin{aligned}
\|f(X) - f(Y)\|_F &\leq \frac{1}{r} \|f(X)^r - f(Y)^r\|_F \\
&= \frac{1}{r} \left\|\sum_{i=1}^{m} A_i^* \left(X^{-\delta_i} - Y^{-\delta_i}\right) A_i\right\|_F \\
&\leq \frac{1}{r} \sum_{i=1}^{m} \|A_i\|^2 \left\|X^{-\delta_i} - Y^{-\delta_i}\right\|_F \\
&\leq \frac{1}{r} \sum_{i=1}^{m} \|A_i\|^2 \left\|X^{-\delta_i}\right\| \left\|Y^{-\delta_i}\right\| \left\|Y^{\delta_i} - X^{\delta_i}\right\|_F \\
&\leq \frac{1}{r} \sum_{i=1}^{m} \delta_i \|A_i\|^2 \|X - Y\|_F = \frac{q}{r} \|X - Y\|_F.
\end{aligned} \tag{24}$$

Since $q = \sum_{i=1}^{m} \delta_i \|A_i\|^2 < 1$, then f is a contraction mapping on Ω. By Banach's fixed point theorem, the mapping f has a unique fixed point in Ω. This means that (1) has a unique positive definite solution in Ω. □

3. The Iterative Algorithm

In this section, we consider the fixed point iteration method and its speed of convergence. We present the following iterative algorithm to compute the positive definite solution of (1).

Algorithm 8. Consider

$$\begin{aligned}
&X_0 = I, \\
&X_{k+1} = \left[I + \sum_{i=1}^{m} A_i^* X_k^{-\delta_i} A_i\right]^{1/r}, \quad \text{for } k = 0, 1, 2, \ldots.
\end{aligned} \tag{25}$$

Theorem 9. *If $q = \sum_{i=1}^{m} \delta_i \|A_i\|^2 < 1$, then (1) has a unique positive definite solution X and satisfies*

$$X_{2k} \leq X \leq X_{2k+1}, \tag{26}$$

$$\|X_{2k+1} - X_{2k}\| \leq q^{2k} \left\|\left(I + \sum_{i=1}^{m} A_i^* A_i\right)^{1/r} - I\right\|, \tag{27}$$

where the sequence $\{X_k\}$, $k = 0, 1, 2, \ldots$, is determined by Algorithm 8.

Proof. According to Theorem 7, (1) has a unique positive definite solution. Consider the sequence $\{X_k\}$ generated from Algorithm 8 and using Lemma 1.

Since $A_i^* A_i \geq 0$, then we have

$$X_1 = \left[I + \sum_{i=1}^{m} A_i^* A_i\right]^{1/r} \geq I = X_0, \tag{28}$$

which implies that

$$\sum_{i=1}^{m} A_i^* X_1^{-\delta_i} A_i \leq \sum_{i=1}^{m} A_i^* A_i; \tag{29}$$

then,

$$\begin{aligned}
I \leq X_2 &= \left[I + \sum_{i=1}^{m} A_i^* X_1^{-\delta_i} A_i\right]^{1/r} \leq \left[I + \sum_{i=1}^{m} A_i^* A_i\right]^{1/r} \\
&= X_1;
\end{aligned} \tag{30}$$

that is,

$$X_0 \leq X_2 \leq X_1. \tag{31}$$

We find the relation between X_2, X_3, X_4, and X_5. Using $X_0 \leq X_2 \leq X_1$, we obtain

$$\begin{aligned} X_3 &= \left[I + \sum_{i=1}^{m} A_i^* X_2^{-\delta_i} A_i\right]^{1/r} \leq \left[I + \sum_{i=1}^{m} A_i^* A_i\right]^{1/r} \\ &= X_1, \\ X_3 &= \left[I + \sum_{i=1}^{m} A_i^* X_2^{-\delta_i} A_i\right]^{1/r} \geq \left[I + \sum_{i=1}^{m} A_i^* X_1^{-\delta_i} A_i\right]^{1/r} \\ &= X_2. \end{aligned} \tag{32}$$

Hence, $X_2 \leq X_3 \leq X_1$. By the same way, we can prove that

$$X_0 \leq X_2 \leq X_4 \leq X_5 \leq X_3 \leq X_1. \tag{33}$$

So, assume that the inequalities

$$X_0 \leq X_{2k} \leq X_{2k+2} \leq X_{2k+3} \leq X_{2k+1} \leq X_1 \tag{34}$$

hold for k.

Now, for $k+1$ and using inequalities (34), we have

$$\begin{aligned} X_{2k+4} &= \left[I + \sum_{i=1}^{m} A_i^* X_{2k+3}^{-\delta_i} A_i\right]^{1/r} \\ &\leq \left[I + \sum_{i=1}^{m} A_i^* X_{2k+2}^{-\delta_i} A_i\right]^{1/r} = X_{2k+3}, \\ X_{2k+4} &= \left[I + \sum_{i=1}^{m} A_i^* X_{2k+3}^{-\delta_i} A_i\right]^{1/r} \\ &\geq \left[I + \sum_{i=1}^{m} A_i^* X_{2k+1}^{-\delta_i} A_i\right]^{1/r} = X_{2k+2}. \end{aligned} \tag{35}$$

Similarly

$$\begin{aligned} X_{2k+5} &= \left[I + \sum_{i=1}^{m} A_i^* X_{2k+4}^{-\delta_i} A_i\right]^{1/r} \\ &\leq \left[I + \sum_{i=1}^{m} A_i^* X_{2k+2}^{-\delta_i} A_i\right]^{1/r} = X_{2k+3}, \\ X_{2k+5} &= \left[I + \sum_{i=1}^{m} A_i^* X_{2k+4}^{-\delta_i} A_i\right]^{1/r} \\ &\geq \left[I + \sum_{i=1}^{m} A_i^* X_{2k+3}^{-\delta_i} A_i\right]^{1/r} = X_{2k+4}. \end{aligned} \tag{36}$$

Therefore, inequalities (34) are true, for all $k = 0, 1, 2, \ldots$; that is, the subsequences $\{X_{2k}\}$ and $\{X_{2k+1}\}$ are monotonic and bounded. Hence, these are convergent to positive definite matrices.

To prove the subsequences $\{X_{2k}\}$, $\{X_{2k+1}\}$ have a common limit, we have

$$\begin{aligned} \left\|X_{2k+1}^r - X_{2k}^r\right\| &= \left\|\sum_{i=1}^{m} A_i^* \left(X_{2k}^{-\delta_i} - X_{2k-1}^{-\delta_i}\right) A_i\right\| \\ &\leq \sum_{i=1}^{m} \left\|A_i\right\|^2 \left\|X_{2k}^{-\delta_i} - X_{2k-1}^{-\delta_i}\right\| \\ &= \sum_{i=1}^{m} \left\|A_i\right\|^2 \left\|X_{2k}^{-\delta_i} \left(X_{2k-1}^{\delta_i} - X_{2k}^{\delta_i}\right) X_{2k-1}^{-\delta_i}\right\| \\ &\leq \sum_{i=1}^{m} \left\|A_i\right\|^2 \left\|X_{2k}^{-\delta_i}\right\| \left\|X_{2k-1}^{-\delta_i}\right\| \left\|X_{2k-1}^{\delta_i} - X_{2k}^{\delta_i}\right\|. \end{aligned} \tag{37}$$

Since $I \leq X_{2k}, I \leq X_{2k-1}$, for each $k = 1, 2, 3, \ldots$, and $\delta_i \in (0, 1]$, then $\|X_{2k}^{-\delta_i}\|, \|X_{2k-1}^{-\delta_i}\| \leq 1$. By using Lemma 3, we have

$$\begin{aligned} \left\|X_{2k+1}^r - X_{2k}^r\right\| &\leq \sum_{i=1}^{m} \left\|A_i\right\|^2 \left\|X_{2k}^{\delta_i} - X_{2k-1}^{\delta_i}\right\| \\ &\leq \sum_{i=1}^{m} \delta_i \left\|A_i\right\|^2 \left\|X_{2k} - X_{2k-1}\right\|. \end{aligned} \tag{38}$$

Since

$$\begin{aligned} &\left(X_{2k+1}^r - X_{2k}^r\right) - \left(X_{2k} - X_{2k+1}\right) \\ &\quad \geq \left(X_{2k}^r - X_{2k}^r\right) - \left(X_{2k} - X_{2k+1}\right) = X_{2k+1} - X_{2k} \\ &\quad \geq X_{2k} - X_{2k} = 0, \end{aligned} \tag{39}$$

from (38) and (39), we have

$$\left\|X_{2k+1} - X_{2k}\right\| \leq \sum_{i=1}^{m} \delta_i \left\|A_i\right\|^2 \left\|X_{2k} - X_{2k-1}\right\|. \tag{40}$$

Hence,

$$\begin{aligned} \left\|X_{2k+1} - X_{2k}\right\| &\leq q \left\|X_{2k} - X_{2k-1}\right\| \leq \cdots \\ &\leq q^{2k} \left\|\left(I + \sum_{i=1}^{m} A_i^* A_i\right)^{1/r} - I\right\|; \end{aligned} \tag{41}$$

that is,

$$\left\|X_{2k+1} - X_{2k}\right\| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{42}$$

Consequently, the subsequences $\{X_{2k}\}$ and $\{X_{2k+1}\}$ are convergent and have a common limit X, which is a positive definite solution of (1). □

From Theorem 9, we can deduce the following corollary.

Corollary 10. *From inequality (27), one has the following upper bound:*

$$\begin{aligned} &\max\left(\left\|X_{2k+1} - X\right\|, \left\|X - X_{2k}\right\|\right) \\ &\quad \leq q^{2k} \left\|\left(I + \sum_{i=1}^{m} A_i^* A_i\right)^{1/r} - I\right\|. \end{aligned} \tag{43}$$

Theorem 11. *If (1) has a positive definite solution and after k iterative steps of Algorithm 8 one has* $\|I - X_k^{-1}X_{k-1}\| < \varepsilon$, *then*

$$\begin{aligned}\left\|X_k^r - \sum_{i=1}^m A_i^* X_k^{-\delta_i} A_i - I\right\| \\ < \varepsilon \sum_{i=1}^m \delta_i \|A_i\|^2 \left\|\left(I + \sum_{i=1}^m A_i^* A_i\right)^{1/r}\right\|.\end{aligned} \tag{44}$$

Proof. By using Algorithm 8, we have

$$\begin{aligned} & X_k^r - \sum_{i=1}^m A_i^* X_k^{-\delta_i} A_i - I \\ &= X_k^r - \sum_{i=1}^m A_i^* X_k^{-\delta_i} A_i - X_k^r + \sum_{i=1}^m A_i^* X_{k-1}^{-\delta_i} A_i \\ &= \sum_{i=1}^m A_i^* \left(X_{k-1}^{-\delta_i} - X_k^{-\delta_i}\right) A_i. \end{aligned} \tag{45}$$

Taking norm of the above equation, we get

$$\begin{aligned} \left\|X_k^r - \sum_{i=1}^m A_i^* X_k^{-\delta_i} A_i - I\right\| &= \left\|\sum_{i=1}^m A_i^* \left(X_{k-1}^{-\delta_i} - X_k^{-\delta_i}\right) A_i\right\| \\ &\le \sum_{i=1}^m \|A_i\|^2 \left\|X_{k-1}^{-\delta_i} - X_k^{-\delta_i}\right\| \\ &\le \sum_{i=1}^m \|A_i\|^2 \left\|X_{k-1}^{-\delta_i}\right\| \left\|X_k^{-\delta_i}\right\| \left\|X_k^{\delta_i} - X_{k-1}^{\delta_i}\right\|. \end{aligned} \tag{46}$$

By Theorem 9, we have $X_{k-1}^{-\delta_i} \le I$, $X_k^{-\delta_i} \le I$, and $X_k \le (I + \sum_{i=1}^m A_i^* A_i)^{1/r}$. By using Lemma 3, we have

$$\begin{aligned} \left\|X_k^r - \sum_{i=1}^m A_i^* X_k^{-\delta_i} A_i - I\right\| &\le \sum_{i=1}^m \|A_i\|^2 \left\|X_k^{\delta_i} - X_{k-1}^{\delta_i}\right\| \\ &\le \sum_{i=1}^m \delta_i \|A_i\|^2 \|X_k - X_{k-1}\| \\ &\le \sum_{i=1}^m \delta_i \|A_i\|^2 \|X_k\| \left\|I - X_k^{-1} X_{k-1}\right\| \\ &< \varepsilon \sum_{i=1}^m \delta_i \|A_i\|^2 \left\|\left(I + \sum_{i=1}^m A_i^* A_i\right)^{1/r}\right\|. \end{aligned} \tag{47}$$

□

4. Numerical Examples

In this section, we give some numerical examples to illustrate the convergence of the proposed algorithm. The solution is computed for different matrices A_i, $i = 1, 2, \dots, m$, with different orders and different values of r and δ_i, $i = 1, 2, \dots, m$. We denote X, the solution obtained by Algorithm 8 and $\epsilon_1(X_k) = \|X_{k+1} - X_k\|_\infty$, $\epsilon_2(X_k) = \|X - X_k\|_\infty$, and $\epsilon_3(X_k) = \|X_k^r - \sum_{i=1}^m A_i^* X_k^{-\delta_i} A_i - I\|_\infty$.

For computing $Z = Y^{1/p}$ with p an integer, we use the following iterative algorithm.

Algorithm 12 (see [24]). Consider

$$\begin{aligned} Z_0 &= I, \\ Z_{k+1} &= \frac{1}{p}\left[(p-1) Z_k + Z_k^{(1-p)} Y\right], \\ & \text{for } k = 0, 1, 2, \dots. \end{aligned} \tag{48}$$

Example 1. Consider the matrix equation

$$X^3 - A_1^* X^{-1/4} A_1 - A_2^* X^{-1/2} A_2 - A_3^* X^{-1/3} A_3 = I, \tag{49}$$

where A_1, A_2, and A_3 are given by

$$A_1 = \begin{pmatrix} 1 & 0.2 & 0.3 \\ 0.2 & 0.6 & 0 \\ 0.2 & 0.1 & -0.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.3 & -0.4 \\ 0.5 & 0.6 & 0.1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0.2 & 1 & 0.1 \\ 0.4 & 0.1 & 0.6 \\ 0.2 & 0.6 & -0.4 \end{pmatrix}. \tag{50}$$

By using Algorithms 8 and 12, we have after 16 iterations

$$X \approx X_{16} = \begin{pmatrix} 1.32952 & 0.164494 & 0.0955721 \\ 0.164494 & 1.425 & -0.0224966 \\ 0.0955721 & -0.0224966 & 1.28816 \end{pmatrix}. \tag{51}$$

The other results are listed in Table 1.

Example 2. Consider the matrix equation

$$X^7 - A_1^* X^{-1/3} A_1 - A_2^* X^{-1/3} A_2 - A_3^* X^{-1/4} A_3 = I, \tag{52}$$

where A_1, A_2, and A_3 are given by

$$A_1 = \begin{pmatrix} 1 & 0.2 & 0.3 \\ -0.4 & 0.2 & 0.5 \\ -0.2 & 0 & 0.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.5 & -0.2 & -0.5 \\ 0.4 & 0.4 & 0 \\ 0.3 & 0.6 & -0.1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 1.5 & 0.4 \\ -1.5 & 1 & -0.9 \\ 0.2 & 0 & 0.4 \end{pmatrix}. \tag{53}$$

Table 1: Error analysis for Example 1.

k	$\epsilon_1(X_k)$	$\epsilon_2(X_k)$	$\epsilon_3(X_k)$
1	3.96009×10^{-2}	3.71552×10^{-2}	2.78769×10^{-1}
2	2.60028×10^{-3}	2.44577×10^{-3}	1.79779×10^{-2}
3	1.64516×10^{-4}	1.54515×10^{-4}	1.14731×10^{-3}
4	1.06330×10^{-5}	1.00005×10^{-5}	7.43950×10^{-5}
5	6.72778×10^{-7}	6.32454×10^{-7}	4.71572×10^{-6}
6	4.28759×10^{-8}	4.03237×10^{-8}	3.00787×10^{-7}
7	2.71410×10^{-9}	2.55216×10^{-9}	1.90490×10^{-8}

By using Algorithms 8 and 12, we have after 13 iterations

$$X \approx X_{13} = \begin{pmatrix} 1.26207 & -0.0956034 & 0.0322249 \\ -0.0956034 & 1.22629 & 0.00231867 \\ 0.0322249 & 0.00231867 & 1.15893 \end{pmatrix}. \quad (54)$$

The other results are listed in Table 2.

Example 3. Consider the matrix equation

$$X^4 - A_1^* X^{-1/4} A_1 - A_2^* X^{-1/5} A_2 - A_3^* X^{-1/6} A_3 - A_4^* X^{-1/3} A_4 = I, \quad (55)$$

where A_1, A_2, A_3, and A_4 are given by

$$A_1 = \begin{pmatrix} -0.1 & -0.1 & 0.4 & -1.3 \\ 0 & -1.7 & 0.5 & -2 \\ 0.2 & -1 & -0.5 & 0.4 \\ 0.5 & 0.1 & 0.8 & -0.4 \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 2.5 & -4.2 & 0.5 & 1 \\ 0.1 & -0.5 & 0 & -0.4 \\ -1 & 0.5 & 0.5 & 0.1 \\ 0.2 & -2.5 & 0.5 & 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 1.5 & 2 & 0.1 & 0 \\ 0 & -0.2 & 0 & 0.6 \\ 1.2 & 0.1 & 2.6 & -0.4 \\ -0.6 & 2.4 & -0.2 & 0.1 \end{pmatrix}, \quad (56)$$
$$A_4 = \begin{pmatrix} 0.2 & -1.5 & 0.1 & -2 \\ -1.5 & -0.4 & 6.5 & 0.6 \\ 0.2 & 0.1 & -0.1 & 0.4 \\ 0.1 & 2.5 & 0.1 & -1 \end{pmatrix}.$$

Table 2: Error analysis for Example 2.

k	$\epsilon_1(X_k)$	$\epsilon_2(X_k)$	$\epsilon_3(X_k)$
1	7.87438×10^{-3}	7.64002×10^{-3}	2.44198×10^{-1}
2	2.41622×10^{-4}	2.34359×10^{-4}	7.73525×10^{-3}
3	7.47444×10^{-6}	7.26322×10^{-6}	2.43083×10^{-4}
4	2.17530×10^{-7}	2.11217×10^{-7}	6.98136×10^{-6}
5	6.50298×10^{-9}	6.31369×10^{-9}	2.08893×10^{-7}
6	1.94921×10^{-10}	1.89286×10^{-10}	6.27981×10^{-9}

Table 3: Error analysis for Example 3.

k	$\epsilon_1(X_k)$	$\epsilon_2(X_k)$	$\epsilon_3(X_k)$
1	1.80539×10^{-1}	1.73636×10^{-1}	12.8192
2	7.29793×10^{-3}	6.90272×10^{-3}	4.63002×10^{-1}
3	4.16292×10^{-4}	3.95211×10^{-4}	2.66295×10^{-2}
4	2.22090×10^{-5}	2.10808×10^{-5}	1.41947×10^{-3}
5	1.18818×10^{-6}	1.12817×10^{-6}	7.60338×10^{-5}
6	6.32186×10^{-8}	6.00133×10^{-8}	3.98446×10^{-6}
7	3.37643×10^{-9}	3.20528×10^{-9}	2.86995×10^{-7}

By using Algorithms 8 and 12, we have after 18 iterations

$$X \approx X_{18} = \begin{pmatrix} 1.85179 & -0.19784 & -0.0873059 & 0.0249042 \\ -0.19784 & 2.51638 & -0.0882272 & -0.0105432 \\ -0.0873059 & -0.0882272 & 2.50302 & 0.00541474 \\ 0.0249042 & -0.0105432 & 0.00541474 & 1.85343 \end{pmatrix}. \quad (57)$$

The other results are listed in Table 3.

Example 4. Consider the matrix equation

$$X^5 - A_1^* X^{-1/2} A_1 - A_2^* X^{-1/3} A_2 = I, \quad (58)$$

where A_1 and A_2 are given by

$$A_1 = \begin{cases} a_{ij} = \dfrac{i}{n} & \text{if } i = j, \\ a_{ij} = \dfrac{2j}{2n^3 + i} & \text{if } i \neq j, \end{cases}$$
$$A_2 = \begin{cases} a_{ij} = \dfrac{1}{2i + n} & \text{if } i = j, \\ a_{ij} = \dfrac{j}{4n + i} & \text{if } i > j, \\ a_{ij} = \dfrac{i}{n^3 - j} & \text{if } i < j. \end{cases} \quad (59)$$

By using Algorithms 8 and 12, we have solutions when $n = 10, 50, 100$. The results are listed in Table 4.

5. Conclusion

We have presented existence theorems of positive definite solutions for a nonlinear matrix equation (1) under certain

Table 4: Error analysis for Example 4.

k	$\epsilon_1(X_k)$	$\epsilon_2(X_k)$	$\epsilon_3(X_k)$
		$n = 10$	
1	7.81929×10^{-3}	7.45968×10^{-3}	6.72846×10^{-2}
2	3.77022×10^{-4}	3.59607×10^{-4}	3.20257×10^{-3}
3	1.82579×10^{-5}	1.74147×10^{-5}	1.55207×10^{-4}
4	8.84049×10^{-7}	8.43217×10^{-7}	7.51571×10^{-6}
5	4.28088×10^{-8}	4.08315×10^{-8}	3.63970×10^{-7}
6	2.07307×10^{-9}	1.97732×10^{-9}	1.76271×10^{-8}
		$n = 50$	
1	7.79810×10^{-3}	7.43990×10^{-3}	6.69737×10^{-2}
2	3.75515×10^{-4}	3.58193×10^{-4}	3.18340×10^{-3}
3	1.81596×10^{-5}	1.73221×10^{-5}	1.54042×10^{-4}
4	8.77998×10^{-7}	8.37505×10^{-7}	7.44758×10^{-6}
5	4.24507×10^{-8}	4.04929×10^{-8}	3.60087×10^{-7}
6	2.05247×10^{-9}	1.95781×10^{-9}	1.74100×10^{-8}
		$n = 100$	
1	7.86249×10^{-3}	7.50582×10^{-3}	9.92082×10^{-2}
2	3.76538×10^{-4}	3.59253×10^{-4}	3.30164×10^{-3}
3	1.81583×10^{-5}	1.73209×10^{-5}	1.55072×10^{-4}
4	8.77928×10^{-7}	8.37439×10^{-7}	7.46052×10^{-6}
5	4.24470×10^{-8}	4.04894×10^{-8}	3.60045×10^{-7}
6	2.05227×10^{-9}	1.95762×10^{-9}	1.74078×10^{-8}

conditions. Uniqueness theorem of solutions for this matrix equation is proved. An iterative algorithm is introduced to obtain positive definite solutions for this matrix equation. Moreover, convergence and error analysis are studied. Finally, the numerical examples confirmed the efficient convergence and error reduction as predicted and this method is efficient and easy.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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