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Research Article

Some Inequalities for the Omori-Yau Maximum Principle

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We generalize A. Borbély's condition for the conclusion of the Omori-Yau maximum principle for the Laplace operator on a complete Riemannian manifold to a second-order linear semielliptic operator L with bounded coefficients and no zeroth order term. Also, we consider a new sufficient condition for the existence of a tamed exhaustion function. From these results, we may remark that the existence of a tamed exhaustion function is more general than the hypotheses in the version of the Omori-Yau maximum principle that was given by A. Ratto, M. Rigoli, and A. G. Setti.

1. Introduction

Let (M, g) be a smooth complete Riemannian manifold of dimension n. For a smooth real-valued function h on M, a second-order linear differential operator $L : C^{\infty}(M) \rightarrow C^{\infty}(M)$ without zeroth-order term can be written as

$$Lh = \operatorname{Tr}\left(A \circ \operatorname{Hess}_h\right) + g\left(V, \nabla h\right), \tag{1}$$

where $A \in \Gamma(\operatorname{End}(\mathrm{TM}))$ is self-adjoint with respect to g, $\operatorname{Hess}_h \in \Gamma(\operatorname{End}(\mathrm{TM}))$ is the Hessian of h in the form defined by $\operatorname{Hess}_h(X) = \nabla_X \nabla h$ for $X \in \Gamma(\mathrm{TM})$, and finally $V \in \Gamma(\mathrm{TM})$. In this paper, we will deal with the semielliptic case, that is, A is positive semidefinite at each point, and we always assume that

$$\sup_M \operatorname{Tr}(A) + \sup_M |V| < \infty. \tag{2}$$

Definition 1. A smooth complete Riemannian manifold M is said to satisfy the Omori-Yau maximum principle for the Laplace operator Δ (the above semielliptic operator L) if for any C^2 function $h : M \rightarrow \mathbb{R}$ which is bounded from above and for any $\epsilon > 0$ there is a point $x_\epsilon \in M$ such that $|h(x_\epsilon) - \sup_M h| < \epsilon$, $\|\nabla h(x_\epsilon)\| < \epsilon$, and $\Delta h(x_\epsilon) < \epsilon$ $(Lh(x_\epsilon) < \epsilon)$.

The Omori-Yau maximum principle is a useful substitute of the usual maximum principle in noncompact settings. For the operator Δ, Definition 1 is the well-known Omori-Yau maximum principle for the Laplacian, which was first proven by Omori [1] and Yau [2] when the Ricci curvature is bounded below. This was improved upon by Chen and Xin [3] and Ratto et al. [4] when the Ricci curvature decays were slower than a certain decreasing function tending to minus infinity. For instance, we have the following.

Theorem 2 (Ratto-Rigoli-Setti's condition [4, Theorem 2.3]). *Let $o \in M$ be a fixed point and $r(x)$ be the distance function from o. Let one assumes that away from the cut locus of o one has*

$$Ricc(\nabla r, \nabla r) \geq -(n-1) BG^2(r), \tag{3}$$

where $B > 0$ is some constant and $G(t)$ on $[0, \infty)$ satisfies

$$\int_0^\infty \frac{1}{G(t)} dt = \infty, \quad G(0) = 1, \; G' \geq 0, \qquad \sqrt{G}^{(2k+1)}(0) = 0, \quad \forall k \geq 0, \tag{4}$$

$$\limsup_{t \to \infty} \frac{t\sqrt{G\left(\sqrt{t}\right)}}{\sqrt{G(t)}} < \infty. \tag{5}$$

Then M satisfies the Omori-Yau maximum principle for the Laplacian Δ.

Borbély [5, Theorem] has given an elegant proof of the validity of the Omori-Yau maximum principle where the Ricci curvature condition (3) is replaced by the assumption $\Delta r(x) \le G(r(x))$ without (4) and (5). Also, Bessa et al. [6, Theorem 5.6] proved Borbély's theorem [5, Theorem] for the f-Laplacian Δ_f for a selected smooth function on M. In this paper, we first show that Borbély's theorem [5, Theorem] is also true for our semielliptic operator L by following his method in [5] (see Theorem 5).

To state other results, we need the following definitions.

Definition 3. Let u be a real-valued continuous function on M and let a point $p \in M$.

(i) A function u is called proper, if the set $\{p : u(p) \le r\}$ is compact for every real number r.

(ii) A function v defined on a neighborhood U_p of p is called an upper-supporting function for u at p, if the conditions $v(p) = u(p)$ and $v \ge u$ hold in U_p.

Definition 4. A proper continuous function $u : M \to \mathbb{R}$ is called a Δ-tamed exhaustion, if the following condition holds:

(1) $u \ge 0$.

(2) At all points $p \in M$ it has a C^2 smooth, upper-supporting function v at p defined on an open neighborhood U_p such that $\|\nabla v|_p\| \le 1$ and $\Delta v|_p \le 1$.

Royden [7] showed that every complete Riemannian manifold satisfying Omori-Yau's condition (i.e., the Ricci curvature is bounded from below) admits a Δ-tamed exhaustion function. Inspired by Royden's article [7], Kim and Lee [8, Theorem 2] proved the Omori-Yau maximum principle for the Laplacian Δ when there exists a Δ-tamed exhaustion function. Moreover, they proved that every complete Riemannian manifold satisfying Ratto-Rigoli-Setti's condition admits a Δ-tamed exhaustion function [8]. Similar to Definition 4, we define an L-tamed exhaustion function (i.e., we replace Δ with L) [9, Definition 1.4]. Then, using the existence of an L-tamed exhaustion function, Hong and Sung [9, Theorem 2.1] generalized the Omori-Yau maximum principle for the Laplacian Δ to the operator L. In this paper, we give a new sufficient condition for the existence of an L-tamed exhaustion function (see Theorem 6). We prove this result using the ideas adapted from [8]. Note that Theorem 6, together with [9, Theorem 2.1], implies the maximum principle of Omori and Yau for the operator L. As a corollary, we prove that the existence of a Δ-tamed exhaustion is more general than Ratto-Rigoli-Setti's condition. Unfortunately, for the operator L, the relation between Borbély's condition (or the existence of an L-tamed exhaustion) and Ratto-Rigoli-Setti's condition remains for further study.

Now, we formulate our main results. From (1), A is diagonalizable at each point on an orthonormal basis, since A is symmetric. Then one can take a normal coordinate $(x_1, \dots, x_n)$ around $x_\epsilon \in M$ such that A at x_ϵ is represented as a diagonal matrix. Thus, we have

$$Lh|_{x_\epsilon} = \sum_l a_{ll}(x_\epsilon) \frac{\partial^2}{\partial x_l^2} h\Big|_{x_\epsilon} + \sum_l a_l(x_\epsilon) \frac{\partial}{\partial x_l} h\Big|_{x_\epsilon}, \quad (6)$$

for a real-valued function h on M, where each $a_{ll}(x_\epsilon)$ is nonnegative; the entries $a_{ll}(x_\epsilon)$ and $|a_l(x_\epsilon)|$ are bounded above as x_ϵ varies by (2). We introduce a locally defined differential operator for convenience as follows:

$$\begin{aligned}\widetilde{\Delta}_{x_\epsilon} &:= a_{11}(x_\epsilon) \frac{\partial^2}{\partial x_1^2} + \cdots + a_{nn}(x_\epsilon) \frac{\partial^2}{\partial x_n^2}, \\ \widetilde{\nabla}^1_{x_\epsilon} &:= a_1(x_\epsilon) \frac{\partial}{\partial x_1} + \cdots + a_n(x_\epsilon) \frac{\partial}{\partial x_n}, \\ \widetilde{\nabla}_{x_\epsilon} &:= \left(a_{11}(x_\epsilon) \frac{\partial}{\partial x_1}, \dots, a_{nn}(x_\epsilon) \frac{\partial}{\partial x_n} \right).\end{aligned} \quad (7)$$

Put $d_l = a_{ll}(x_\epsilon)$ and $e_l = |a_l(x_\epsilon)|$ for $1 \le l \le n$. We may assume that d_1 and e_1 are the largest of $\{d_1, \dots, d_n\}$ and $\{e_1, \dots, e_n\}$, respectively.

Then we have the following.

Theorem 5. *Let $o \in M$ be a fixed point and $r(x)$ be the distance function from o. Assume that for all $x \in M$*

$$\widetilde{\Delta}_x r(x) \le G(r(x)), \quad (8)$$

where r is smooth, $r(x) > 1$, and $G(t)$ on $[0, \infty)$ satisfies

$$\int_0^\infty \frac{dt}{G(t)} = \infty, \quad G \ge 1, \; G' \ge 0. \quad (9)$$

Then M satisfies the Omori-Yau maximum principle for the operator L.

Theorem 6. *Let $o \in M$ be a fixed point and $r(x)$ be the distance function from o. Assume that for all $x \in M$*

$$\widetilde{\Delta}_x r(x) \le G(r(x)), \quad (10)$$

where r is smooth, $r(x) > 1$, and $G(t)$ on $[0, \infty)$ satisfies

$$\int_0^\infty \frac{dt}{G(t)} = \infty, \quad G \ge 1, \; G' \ge 0, \quad (11)$$

$$\limsup_{t \to +\infty} \frac{t\sqrt{G(\sqrt{t})}}{\sqrt{G(t)}} < +\infty. \quad (12)$$

Then M admits an L-tamed exhaustion function.

Remark 7. By [5, Corollary] and Theorem 6, Ratto-Rigoli-Setti's condition without $\sqrt{G}^{(2k+1)}(0) = 0\ \forall k \ge 0$ implies the existence of a Δ-tamed exhaustion function. Therefore, the existence of a Δ-tamed exhaustion function for the conclusion of the Omori-Yau maximum principle for the Laplacian Δ is more general than the hypothesis in Theorem 2.

There are some other sufficient conditions under which the Omori-Yau maximum principle for the Laplacian Δ holds [10–12]. Also, [13] deals with the general setting of semielliptic operators (trace type operators). Recently, Bessa and Pessoa [14, Theorem 1] present a sufficient condition for the conclusion of the Omori-Yau maximum principle for a second-order linear semielliptic operator with bounded first-order coefficients and no zeroth-order term. However, they will not consider the existence of a tamed exhaustion function as sufficient conditions for the conclusion of the Omori-Yau maximum principle.

2. Proof of Theorem 5

The proof is similar to the method in [5]. Let $U = \sup h$. We may assume that $h < U$ at every point of M; otherwise, h has its maximum at some point and that point directly satisfies the Omori-Yau maximum principle for a semielliptic operator L.

Define the function $F(t)$ as

$$F(t) = e^{\int_0^t (1/G(s))ds}. \tag{13}$$

Then

$$F' = \frac{F}{G}. \tag{14}$$

Since $G \geq 1$ on $[0, \infty)$, we have $F \geq 1$, and $F' > 0$. Hence the function F is strictly increasing, and $\lim_{t\to\infty} F(t) = \infty$. Since the set $\{x \in M : r(x) \leq 1\}$ is compact, we have

$$U - \sup\{h(x) : r(x) \leq 1\} > 0. \tag{15}$$

For any positive constant $\epsilon < \min\{1, U - \sup\{h(x) : r(x) \leq 1\}\}$, we define the function $h_\lambda : M \to \mathbb{R}$ as

$$h_\lambda(x) = \lambda F(r(x)) + U - \epsilon. \tag{16}$$

Then

$$h_\lambda(x) > h(x) \quad \text{if } r(x) \leq 1,\ \lambda \geq 0. \tag{17}$$

Because, for all $x \in M$, $F(r(x)) \geq 1$ and $U > h(x)$. If $\lambda > \epsilon$, then we have

$$h_\lambda(x) > h(x), \quad \forall x \in M. \tag{18}$$

Define λ_0 as

$$\lambda_0 = \inf\{\lambda : h_\lambda(x) > h(x),\ \forall x \in M\}. \tag{19}$$

Then, clearly, $\lambda_0 > 0$. Furthermore, we can obtain $h_{\lambda_0}(x) \geq h(x)$ for all $x \in M$; that is, there is a point $x_\epsilon \in M$ such that $h_{\lambda_0}(x_\epsilon) = h(x_\epsilon)$. Assume that to the contrary $h_{\lambda_0}(x) > h(x)$ for all $x \in M$. Then we will show that there is a constant λ' with $\lambda_0 > \lambda'$ such that $h_{\lambda'}(x) > h(x)$ for all $x \in M$. This is a contradiction to the definition of λ_0.

Let $\lambda_0 > \lambda_1$. Because $\lim_{r\to\infty} F(r) = \infty$, there is a sufficiently large positive number r_0 such that $h_{\lambda_1}(x) > U > h(x)$ for $r(x) > r_0$. Also, because the set $\{x \in M : r(x) \leq r_0\}$ is compact, the statement $h_{\lambda_0}(x) > h(x)$ for all $x \in M$ implies that there is a constant λ_2 with $\lambda_0 > \lambda_2$ such that $h_{\lambda_2}(x) > h(x)$ for $r(x) \leq r_0$. Now, let $\lambda' = \max\{\lambda_1, \lambda_2\}$. Then, for $\lambda_0 > \lambda'$, we have $h_{\lambda'}(x) > h(x)$ for all $x \in M$. Moreover, by (17) and $\lambda_0 > 0$, we have $r(x_\epsilon) > 1$.

Next, we have to show that h_{λ_0} is smooth at x_ϵ. Since $h_\lambda(x) = \lambda F(r(x)) + U - \epsilon$, it is enough to show that r is smooth at x_ϵ. To avoid confusion, the point o, in the statement of Theorem 5, is switched to p. Note that r is a Lipschitz function and is smooth on $M \setminus \{p, C_p\}$, where C_p is the cut locus of p. Suppose that $x_\epsilon \in C_p$. Then we have two possibilities (Petersen [15, Lemma 8.2]); either there are two distinct minimizing geodesic segments $\gamma_1, \gamma_2 : [0, t_0] \to M$ joining p to x_ϵ, or there is a geodesic segment $\gamma : [0, t_0] \to M$ from p to x_ϵ along which x_ϵ is conjugate to p. Notice that

$$t_0 = r(\gamma_i(t_0)) = r(x_\epsilon) \quad \text{for } i = 1 \text{ or } 2. \tag{20}$$

We consider the first case. Let $w = \gamma_1'(t_0)$ and $v = \gamma_2'(t_0)$. Since γ_1 and γ_2 are distinct segments, we have $w \neq v$. For $i = 1$ or 2, the functions $t \to r(\gamma_i(t))$ are differentiable on $(0, t_0)$ and they have a left-derivative at t_0. Note that h is C^2 smooth on M. From the definition of λ_0, $h_{\lambda_0} \geq h$, and $h_{\lambda_0}(x_\epsilon) = h(x_\epsilon)$ we obtain

$$\liminf_{s\to 0^+} \frac{h_{\lambda_0}(\gamma_2(t_0+s)) - h_{\lambda_0}(\gamma_2(t_0))}{s} \geq D_v h(x_\epsilon), \tag{21}$$

where $D_v h(x_\epsilon)$ denotes the directional derivative of h at the point x_ϵ in the direction of v. Furthermore, since h_{λ_0} has a directional derivative at x_ϵ in the direction of $-v$, we have

$$\begin{aligned} -\lambda_0 F'(t_0) &= -\lambda_0 F'(r(x_\epsilon)) = D_{-v} h_{\lambda_0}(x_\epsilon) \\ &\geq D_{-v} h(x_\epsilon) = -D_v h(x_\epsilon). \end{aligned} \tag{22}$$

This yields

$$D_v h(x_\epsilon) \geq \lambda_0 F'(r(x_\epsilon)). \tag{23}$$

Hence, by (21) and (23), we get the following inequality:

$$\begin{aligned} &\liminf_{s\to 0^+} \frac{h_{\lambda_0}(\gamma_2(t_0+s)) - h_{\lambda_0}(\gamma_2(t_0))}{s} \\ &\quad \geq \lambda_0 F'(r(x_\epsilon)). \end{aligned} \tag{24}$$

Note that $(h_{\lambda_0}(\gamma_2))' = \lambda_0 F'(r(\gamma_2)) r'(\gamma_2)$ and $r(\gamma_2(t_0)) = r(x_\epsilon)$. Recall that $\lambda_0 > 0$. Then, from (24), we can get

$$\liminf_{s\to 0^+} \frac{r(\gamma_2(t_0+s)) - r(\gamma_2(t_0))}{s} \geq 1. \tag{25}$$

The inequality (25) will lead to a contradiction. Since γ_1 and γ_2 are different segments, by connecting from the point $\gamma_1(t_0 - s)$ to the point $\gamma_2(t_0 + s)$ with a geodesic segment, there is a constant c with $0 < c < 1$ such that, for a sufficiently small $s > 0$, the distance $d(\gamma_1(t_0 - s), \gamma_2(t_0 + s)) < c2s$. Thus there is a constant c' with $0 < c' < 1$ depending only on the angle of v and w such that

$$r(\gamma_2(t_0+s)) < t_0 + c's, \tag{26}$$

for a sufficiently small $s > 0$. Note that $r(\gamma_2(t_0)) = t_0$. By plugging (26) to (25), we have a contradiction.

From now, let us consider the second case. Since γ is distance minimizing between p and x_ϵ, r is smooth at $\gamma(t)$ for $0 < t < t_0$. Let $m(t) = \Delta r(\gamma(t))$. Then $m(t)$ is also smooth for $0 < t < t_0$. Because $\gamma(t_0)$ is conjugate to $p = \gamma(0)$ along γ, by a simple calculation, we get

$$\lim_{t \to t_0^-} m(t) = -\infty. \tag{27}$$

Because $\lambda_0 F'(r(x_\epsilon)) > 0$, by (23), we get $D_\gamma h(x_\epsilon) > 0$; that is, $\nabla h(x_\epsilon) \neq 0$. Hence the level surface $H = \{x \in M : h(x) = h(x_\epsilon)\}$ is a C^2 smooth hypersurface near x_ϵ. Denote by H_s the surface parallel to H and passing through the point $\gamma(t_0 - s)$ for some $s > 0$. Since H is C^2 smooth near x_ϵ, the surface H_s is also C^2 smooth near $\gamma(t_0 - s)$ for a sufficiently small $s > 0$. Therefore, by (27), for some sufficiently small s, the trace of the second fundamental form of H_s at $\gamma(t_0 - s)$ in the direction of $\gamma'(t_0 - s)$ is greater than $m(t_0 - s)$, where $m(t_0 - s)$ is the trace of the second fundamental form of the geodesic sphere $B(p, t_0 - s)$ at $\gamma(t_0 - s)$ with respect to the normal vector $\gamma'(t_0 - s)$. This implies that there has to be a point $q_s \in H_s$ sufficiently close to $\gamma(t_0 - s)$, which lies inside $B(p, t_0 - s)$; that is,

$$r(q_s) < t_0 - s. \tag{28}$$

Since H_s is parallel to H, we also have a point on $q \in H$ such that the distance $d(q_s, q) = s$. By (28), we have

$$r(q) < t_0 = r(x_\epsilon). \tag{29}$$

Since F is strictly increasing, we get

$$\begin{aligned} h_{\lambda_0}(q) &= \lambda_0 F(r(q)) + U - \epsilon < \lambda_0 F(r(x_\epsilon)) + U - \epsilon \\ &= h_{\lambda_0}(x_\epsilon) = h(x_\epsilon) = h(q). \end{aligned} \tag{30}$$

This is a contradiction to the fact that $h_{\lambda_0}(x) \geq h(x)$ for all $x \in M$. Therefore, the function r must be smooth at x_ϵ.

By the definition of F, $F \geq 1$, $G \geq 1$, and $G' \geq 0$, we have

$$\begin{aligned} 0 &< F' = \frac{F}{G}, \\ F'' &= \frac{F'}{G} - \frac{FG'}{G^2} = \frac{F}{G^2} - \frac{FG'}{G^2} \leq \frac{F}{G^2}. \end{aligned} \tag{31}$$

Because $\lambda_0 > 0$, $F \geq 1$, and $h(x_\epsilon) = \lambda_0 F(r(x_\epsilon)) + U - \epsilon < U$, we have

$$0 < -\lambda_0 F(r(x_\epsilon)) + \epsilon = U - h(x_\epsilon) < \epsilon. \tag{32}$$

Hence

$$\lambda_0 < \frac{\epsilon}{F(r(x_\epsilon))} \leq \epsilon. \tag{33}$$

Recall notations (6) and (7). Since

$$\begin{aligned} h_{\lambda_0}(x) &\geq h(x), \quad \forall x \in M, \\ h_{\lambda_0}(x_\epsilon) &= h(x_\epsilon), \end{aligned} \tag{34}$$

we have

$$\begin{aligned} \nabla h_{\lambda_0}(x_\epsilon) &= \nabla h(x_\epsilon), \\ L h_{\lambda_0}(x_\epsilon) &\geq L h(x_\epsilon). \end{aligned} \tag{35}$$

Note that $\|\nabla r\| = 1$. By (31), (33), and $G \geq 1$, the first equality of (35) yields

$$\begin{aligned} \|\nabla h(x_\epsilon)\| &= \|\lambda_0 F'(r(x_\epsilon)) \nabla r(x_\epsilon)\| \\ &< \frac{\epsilon}{F(r(x_\epsilon))} \frac{F(r(x_\epsilon))}{G(r(x_\epsilon))} \leq \epsilon. \end{aligned} \tag{36}$$

Also, by (2), (31), (33), (36), $G \geq 1$, and $\widetilde{\Delta}_{x_\epsilon} r \leq G$, the second inequality of (35) yields

$$\begin{aligned} L h(x_\epsilon) &\leq L h_{\lambda_0}(x_\epsilon) = \sum_l a_{ll}(x_\epsilon) \frac{\partial^2}{\partial x_l^2} h_{\lambda_0}\Big|_{x_\epsilon} \\ &+ \sum_l a_l(x_\epsilon) \frac{\partial}{\partial x_l} h_{\lambda_0}\Big|_{x_\epsilon} \leq \lambda_0 \Big(F'(r(x_\epsilon)) \widetilde{\Delta}_{x_\epsilon} r(x_\epsilon) \\ &+ F''(r(x_\epsilon)) \widetilde{\nabla}_{x_\epsilon} r(x_\epsilon) \cdot \nabla r(x_\epsilon) \Big) + e_1 \epsilon \\ &< \frac{\epsilon}{F(r(x_\epsilon))} \left(\frac{F(r(x_\epsilon))}{G(r(x_\epsilon))} G(r(x_\epsilon)) \right. \\ &\left. + d_1 \frac{F(r(x_\epsilon))}{G(r(x_\epsilon))^2} \right) + e_1 \epsilon \leq \epsilon (1 + d_1 + e_1). \end{aligned} \tag{37}$$

If we replace ϵ with $\epsilon(1 + d_1 + e_1)$, then the above inequality, (32), and (36) show that the point x_ϵ satisfies the conditions in Definition 1.

3. Proof of Theorem 6

The proof is similar to the method in [8]. Let $o \in M$ be a fixed point and $r(x)$ be the distance function from o. Define a function $u : M \to \mathbb{R}$ by

$$u(x) = \int_0^{r(x)^2} G(s)^{-1} ds. \tag{38}$$

Assume that a smooth complete Riemannian manifold satisfies assumption (10). Then we will prove that u is an L-tamed exhaustion function. We consider two cases.

First Case. Assume that o has no cut points in M.

By the definition, the function u is an exhaustion function for M. We have to show that, for certain positive constants C and C_1, $\|\nabla u\| < C$ and $Lu < C_1$ outside a ball of a certain radius with center x_ϵ. Let $\phi(t) = \exp\{\int_0^t G(s)^{-1} ds\}$ and $B(x_\epsilon, r) = \{x \in M \mid \operatorname{dist}(x, x_\epsilon) < r\}$. Then $u(x) = \log \phi(r(x)^2)$. By a direct calculation, one gets

$$\nabla u = \nabla \log \phi(r^2) = 2r \nabla r \frac{\phi'(r^2)}{\phi(r^2)} = 2r \nabla r G(r^2)^{-1}. \tag{39}$$

By (12), there is a positive constant C such that

$$r^2 \frac{G(r)}{G(r^2)} = r^2 G(r) G(r^2)^{-1} < \frac{C}{4}. \quad (40)$$

Then, for $r > 1$, we obtain

$$rG(r) G(r^2)^{-1} < r^2 G(r) G(r^2)^{-1} < \frac{C}{4}. \quad (41)$$

Moreover, by (11), we have

$$\sup_{[0,\infty)} G(r)^{-1} = \left(\inf_{[0,\infty)} G(r)\right)^{-1} \le 1. \quad (42)$$

By plugging (41) to (39), we have

$$\|\nabla u\| < \frac{1}{2} \|\nabla r\| C G(r)^{-1}. \quad (43)$$

Note that $\|\nabla r\| = 1$. Applying (42) gives

$$\|\nabla u\| < \frac{C}{2}. \quad (44)$$

By (2) and (44), one gets

$$\left\|\widetilde{\nabla}^1_{x_\epsilon} u\right\| < e_1 \frac{C}{2}. \quad (45)$$

By assumption (11), we have

$$\left(\frac{\phi'(r^2)}{\phi(r^2)}\right)' = \left(G(r^2)^{-1}\right)' = -G(r^2)^{-2} G'(r^2) \le 0. \quad (46)$$

Because of the above inequality, $\|\widetilde{\nabla}_{x_\epsilon} r\| \le d_1$, (41), and (42), we have for $r > 1$

$$\begin{aligned}
\widetilde{\Delta}_{x_\epsilon} u &= \widetilde{\Delta}_{x_\epsilon} \log \phi(r^2) \\
&= 4r^2 \left(\frac{\phi'(r^2)}{\phi(r^2)}\right)' \left\|\widetilde{\nabla}_{x_\epsilon} r\right\|^2 \\
&\quad + 2G(r^2)^{-1} \left(\left\|\widetilde{\nabla}_{x_\epsilon} r\right\|^2 + r\widetilde{\Delta}_{x_\epsilon} r\right) \\
&\le 2G(r^2)^{-1} \left(\left\|\widetilde{\nabla}_{x_\epsilon} r\right\|^2 + r\widetilde{\Delta}_{x_\epsilon} r\right) \\
&\le 2rG(r^2)^{-1} \left(d_1^2 r^{-1} + \widetilde{\Delta}_{x_\epsilon} r\right) \\
&< \frac{C}{2} G(r)^{-1} \left(d_1^2 r^{-1} + \widetilde{\Delta}_{x_\epsilon} r\right) \\
&< \frac{C}{2} d_1^2 + \frac{C}{2} G(r)^{-1} \widetilde{\Delta}_{x_\epsilon} r.
\end{aligned} \quad (47)$$

By our assumption (10), there exits $r_0 > 1$ such that

$$\widetilde{\Delta}_{x_\epsilon} u < \frac{C}{2} d_1^2 + \frac{C}{2} \quad \text{on } M \setminus B(x_\epsilon, r_0). \quad (48)$$

Thus, by (45) and (48), we have

$$Lu = \widetilde{\Delta}_{x_\epsilon} u + \widetilde{\nabla}^1_{x_\epsilon} u < \frac{C}{2}\left(d_1^2 + 1 + e_1\right) \quad \text{on } M \setminus B(x_\epsilon, r_0). \quad (49)$$

If we replace $(C/2)(d_1^2 + 1 + e_1)$ with C_1, then u satisfies the additional conditions for an L-tamed exhaustion function.

Second Case. Assume that the cut locus of o is nonempty.

Let x_ϵ be a cut point of o and let $F(t) = \log \phi(t^2)$ for $t > 0$. We choose a point $\widehat{x_\epsilon}$ outside of cut locus of o such that $\operatorname{dist}(x_\epsilon, \widehat{x_\epsilon}) < 1$ and $r(\widehat{x_\epsilon}) > r(x_\epsilon)$. Denote by $B(y, r) = \{x \in M \mid \operatorname{dist}(x, y) < r\}$. Take $\eta, \delta > 0$ such that $B(x_\epsilon, \eta) \cap B(\widehat{x_\epsilon}, \delta) = \emptyset$ and $B(\widehat{x_\epsilon}, \delta)$ does not have cut point of o.

Now, we present several functions to find an upper-supporting function for u.

For a neighborhood $\mathcal{U} \subset B(x_\epsilon, \eta)$, we define a smooth map $T : \mathcal{U} \to B(\widehat{x_\epsilon}, \delta)$ with $T_{x_\epsilon}(x_\epsilon) = \widehat{x_\epsilon}$, and it is translation sending x_ϵ to $\widehat{x_\epsilon}$ in a coordinate chart including both $B(x_\epsilon, \eta)$ and $B(\widehat{x_\epsilon}, \delta)$ and satisfying $r(T(x)) \ge r(x)$. Also, we define a C^2 function λ such that $\lambda(x_\epsilon) = 1$, $\nabla \lambda(x_\epsilon) = 0$, $\Delta \lambda(x_\epsilon) = 0$, and

$$\lambda(x) r(T(x)) \ge r(x) + r(\widehat{x_\epsilon}) - r(x_\epsilon) \quad \text{on } \mathcal{U}. \quad (50)$$

Since $r(\widehat{x_\epsilon}) > r(x_\epsilon)$ and $r \ge 0$, we get $\lambda(x) > 0$. Finally, for $x \in \mathcal{U}$, we define a function

$$H(x) = \begin{cases}
N(x) + \left(\frac{1}{2}\right) F''(r(x_\epsilon)) \lambda(x) \left(r(T(x)) - r(\widehat{x_\epsilon})\right)^2 & \text{when } F''(r(x_\epsilon)) > 0, \\
N(x) - \left(\frac{1}{2}\right) F''(r(\widehat{x_\epsilon})) \left(r(T(x)) - r(\widehat{x_\epsilon})\right)^2 & \text{when } F''(r(x_\epsilon)) < 0, \\
N(x) + \left(\frac{1}{2}\right) Q(r(x_\epsilon)) \left(r(T(x)) - r(\widehat{x_\epsilon})\right)^2 & \text{when } F''(r(x_\epsilon)) = 0,
\end{cases} \quad (51)$$

where $N(x) = -F'(r(\widehat{x_\epsilon}))(r(T(x)) - r(\widehat{x_\epsilon})) + F'(r(x_\epsilon))(\lambda(x)r(T(x)) - r(\widehat{x_\epsilon}))$ and $Q(r(x_\epsilon)) = \sup|F''(t)|$ for $t \in (r(x_\epsilon)-1, r(x_\epsilon)+1)$. Note that we choose $\widehat{x_\epsilon}$ as close to x_ϵ such that $\operatorname{sign}[F''(r(\widehat{x_\epsilon}))] = \operatorname{sign}[F''(r(x_\epsilon))]$. Therefore, $H(x) - N(x) \geq 0$.

Let $v(x) = F(r \circ T(x)) + F(r(x_\epsilon)) - F(r(\widehat{x_\epsilon})) + H(x)$. Then one gets $v(x_\epsilon) = F(r(x_\epsilon)) = u(x_\epsilon)$. Because of the fact $F'(r(x))\nabla r(x) = \nabla u(x) = G(r(x)^2)^{-1}2r(x)\nabla r(x)$ and the inequality (41), we get

$$0 < F'(r(x)) = G\left(r(x)^2\right)^{-1} 2r(x) < \frac{C}{2} G(r(x))^{-1}. \quad (52)$$

Moreover, we have two inequalities; that is, for $x \in \mathcal{U}$,

$$\begin{aligned} &\text{first order term of } v(x) - u(x) = F'(r(x_\epsilon)) \\ &\quad \cdot \left(\lambda(x) r(T(x)) - r(\widehat{x_\epsilon}) - (r(x) - r(x_\epsilon))\right) \geq 0, \\ &\text{second order term of } v(x) - u(x) = H(x) - N(x) \\ &\quad \geq 0. \end{aligned} \quad (53)$$

Hence v is an upper-supporting function for u at the point x_ϵ.

Since $\nabla H|_{x_\epsilon} = \nabla N|_{x_\epsilon}$, $\|\nabla\lambda|_{x_\epsilon}\| = 0$, $\lambda(x_\epsilon) = 1$, and $\|\nabla(r \circ T)\| = 1$, we have

$$\begin{aligned} \left\|\nabla v|_{x_\epsilon}\right\| &\leq \left|F'(r(x_\epsilon))\right| \\ &\quad \cdot \left(\left\|\nabla\lambda|_{x_\epsilon}\right\| r(\widehat{x_\epsilon}) + \left|\lambda(x_\epsilon)\right| \left\|\nabla(r \circ T)|_{x_\epsilon}\right\|\right) \\ &= \left|F'(r(x_\epsilon))\right| = \left\|\nabla u|_{x_\epsilon}\right\| < \frac{C}{2}. \end{aligned} \quad (54)$$

By our assumption (2), the above inequality implies that

$$\left\|\widetilde{\nabla}^1_{x_\epsilon} v\big|_{x_\epsilon}\right\| < e_1 \frac{C}{2}. \quad (55)$$

Notice that

$$\widetilde{\Delta}_{x_\epsilon}(r \circ T(x))\big|_{x_\epsilon} = \|DT\|^2 \widetilde{\Delta}_{x_\epsilon} r\big|_{\widehat{x_\epsilon}} = n\widetilde{\Delta}_{\widehat{x_\epsilon}} r\big|_{\widehat{x_\epsilon}}, \quad (56)$$

where $\dim M = n$. By a simple calculation, we have

$$\begin{aligned} &F''(r(x))\nabla r(x) \\ &\quad = 2G\left(r(x)^2\right)^{-1}\left(-2r(x)^2 G\left(r(x)^2\right)^{-1} + 1\right)\nabla r(x) \end{aligned} \quad (57)$$

and hence

$$\begin{aligned} &F''(r(x)) \\ &\quad = 2G\left(r(x)^2\right)^{-1}\left(-2r(x)^2 G\left(r(x)^2\right)^{-1} + 1\right) \\ &\quad < 2G\left(r(x)^2\right)^{-1}. \end{aligned} \quad (58)$$

Using $\|\nabla(r \circ T)\| = 1$, $\|\widetilde{\nabla}_{x_\epsilon}(r \circ T)\| \leq d_1$, (52), (56), and (58), we have

$$\begin{aligned} \widetilde{\Delta}_{x_\epsilon} v\big|_{x_\epsilon} &\leq d_1^2 F''(r(\widehat{x_\epsilon})) + F'(r(\widehat{x_\epsilon}))\widetilde{\Delta}_{x_\epsilon}(r \circ T)\big|_{x_\epsilon} + \widetilde{\Delta}_{x_\epsilon} H\big|_{x_\epsilon} \\ &\leq \begin{cases} F'(r(x_\epsilon))\widetilde{\Delta}_{x_\epsilon}(r \circ T)\big|_{x_\epsilon} + d_1^2\left(F''(r(\widehat{x_\epsilon})) + F''(r(x_\epsilon))\right) & \text{if } F''(r(x_\epsilon)) > 0, \\ F'(r(x_\epsilon))\widetilde{\Delta}_{x_\epsilon}(r \circ T)\big|_{x_\epsilon} & \text{if } F''(r(x_\epsilon)) < 0, \\ F'(r(x_\epsilon))\widetilde{\Delta}_{x_\epsilon}(r \circ T)\big|_{x_\epsilon} + d_1^2\left(F''(r(\widehat{x_\epsilon})) + Q(r(x_\epsilon))\right) & \text{if } F''(r(x_\epsilon)) = 0, \end{cases} \quad (59) \\ &< \left(\frac{1}{2}\right) CG(r(x_\epsilon))^{-1} n\widetilde{\Delta}_{\widehat{x_\epsilon}} r\big|_{\widehat{x_\epsilon}} + 4d_1^2 G\left(r(x_\epsilon)^2\right)^{-1}. \quad (60) \end{aligned}$$

Let $2a$ be the distance to a closest cut point of o. Because the point x_ϵ is a cut point of o, by (41) and (42), we get

$$\begin{aligned} 2aG\left(r(x_\epsilon)^2\right)^{-1} &\leq r(x_\epsilon) G\left(r(x_\epsilon)^2\right)^{-1} \\ &< \frac{C}{4} G(r(x_\epsilon))^{-1} \leq \frac{C}{4}, \end{aligned} \quad (61)$$

$$G\left(r(x_\epsilon)^2\right)^{-1} < \frac{C}{8a}. \quad (62)$$

By plugging (62) to (60), our assumption (10) tells us that, for $r > 1$,

$$\widetilde{\Delta}_{x_\epsilon} v\big|_{x_\epsilon} < \frac{C}{2} n + \frac{C}{2a} d_1^2. \quad (63)$$

Therefore, by (55) and (63), we obtain, for $r > 1$,

$$Lv|_{x_\epsilon} < \frac{C}{2}\left(n + \frac{d_1^2}{a} + e_1\right). \quad (64)$$

So u satisfies the conditions for an L-tamed exhaustion function.

Altogether, we can conclude that u must be an L-tamed exhaustion function for M.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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