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Research Article

Explicit Solutions for the Solomon-Wilson-Alexiades's Mushy Zone Model with Convective or Heat Flux Boundary Conditions

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We complete the Solomon-Wilson-Alexiades's mushy zone model (Solomon, 1982) for the one-phase Lamé-Clapeyron-Stefan problem by obtaining explicit solutions when a convective or heat flux boundary condition is imposed on the fixed face for a semi-infinite material. We also obtain the necessary and sufficient condition on data in order to get the explicit solutions for both cases which is new with respect to the original model. Moreover, when these conditions are satisfied, the two phase-change problems are equivalent to the same problem with a temperature boundary condition on the fixed face and therefore an inequality for the coefficient which characterized one of the two free interfaces of the model is also obtained.

1. Introduction

Heat transfer problems with a phase-change such as melting and freezing have been studied in the last century due to their wide scientific and technological applications [1–9]. A review of a long bibliography on moving and free boundary problems for phase-change materials (PCM) for the heat equation is shown in [10]. The importance of obtaining explicit solutions to some free boundary problems was given in the works [11–26].

We consider a semi-infinite material, with constant thermal coefficients, that is initially assumed to be liquid at its melting temperature which is assumed to be equal to 0°C. At time $t = 0$, a heat flux or a convective boundary condition is imposed at the fixed face $x = 0$, and a solidification process begins where three regions can be distinguished [27, 28]:

- (H1) liquid region at the temperature 0°C, in $x > r(t)$, $t > 0$;
- (H2) solid region at the temperature $T(x,t) < 0$, in $0 < x < s(t)$, $t > 0$ (with $s(t) < r(t)$);
- (H3) mushy region at the temperature $T(x,t) = 0$, in $s(t) \le x \le r(t)$, $t > 0$. The mushy region is considered isothermal and we make the following assumptions on its structure:
  - (H3i) the material contains a fixed portion $\varepsilon\ell$ (with $0 < \varepsilon < 1$) of the total latent heat ℓ (see condition (3) in below);
  - (H3ii) the width of the mushy region is inversely proportional to the gradient of temperature (see condition (4) below).

Following the methodology given in [27–29] and the recent one in [30], we consider a convective boundary condition in Sections 2 to 4 and a heat flux condition in Sections 5 and 6 at the fixed face $x = 0$, respectively. In both cases, we obtain explicit solutions for the temperature and the two free boundaries which define the mushy region. We also obtain, for both cases, the necessary and sufficient condition on data in order to get these explicit solutions given in Sections 2 and 5, respectively, which is new with respect to the original model when a temperature boundary condition at the face $x = 0$ was imposed. Moreover, these two problems are equivalent to the same phase-change process with a temperature boundary condition on the fixed face $x = 0$ studied in [27] and therefore an inequality for the coefficient which characterized one of the two free interfaces is also obtained in Sections 4 and 6. Moreover, in Section 3, we obtain the convergence of the solution of the phase-change process with a convective boundary condition to the solution given in [27] for a temperature boundary condition at the fixed face $x = 0$ when the heat transfer coefficient goes to infinity, and we also give the order of the corresponding convergence when the coefficient that characterized the transient heat transfer at $x = 0$ goes to infinity.

This paper completes the model given in [27] by considering two new boundary conditions (convective and heat flux) at the fixed face of the PCMs and obtaining explicit solutions for both cases when a restriction on data is satisfied.

2. Explicit Solution with a Convective Boundary Condition

The phase-change process consists in finding the free boundaries $x = s(t)$ and $x = r(t)$ and the temperature $T = T(x,t)$ such that the following conditions must be verified (problem (P_1)):

$$T_t - \alpha T_{xx} = 0, \quad 0 < x < s(t), \ t > 0 \ \left(\alpha = \frac{k}{\rho c}\right), \tag{1}$$

$$T(s(t),t) = 0, \quad t > 0, \tag{2}$$

$$kT_x(s(t),t) = \rho\ell\left[\varepsilon\dot{s}(t) + (1-\varepsilon)\dot{r}(t)\right], \quad t > 0, \tag{3}$$

$$T_x(s(t),t)(r(t)-s(t)) = \gamma > 0, \quad t > 0 \ (\text{with } \gamma > 0), \tag{4}$$

$$s(0) = r(0) = 0, \tag{5}$$

$$kT_x(0,t) = \frac{h_0}{\sqrt{t}}\left(T(0,t) + D_\infty\right), \quad t > 0 \ (h_0 > 0, D_\infty > 0). \tag{6}$$

Condition (6) represents a convective boundary condition (Robin condition) at the fixed face $x = 0$ [31–33] with a heat transfer coefficient which is inversely proportional to the square root of the time [29, 30, 34, 35]. Now, we will obtain the solution of problem (1)–(6) when data satisfy the restriction (7).

Theorem 1. *If the coefficient h_0 satisfies the inequality*

$$h_0 > \frac{1}{D_\infty}\sqrt{\frac{\gamma(1-\varepsilon)\rho\ell k}{2}} = h_0^*, \tag{7}$$

then the solution of problem (1)–(6) is given by

$$T(x,t) = -\frac{\left(h_0 D_\infty\sqrt{\pi\alpha}/k\right)\operatorname{erf}(\xi)}{1 + \left(h_0\sqrt{\pi\alpha}/k\right)\operatorname{erf}(\xi)}\left[1 - \frac{\operatorname{erf}\left(x/2\sqrt{\alpha t}\right)}{\operatorname{erf}(\xi)}\right], \quad 0 < x < s(t), \ t > 0, \tag{8}$$

$$s(t) = 2\xi\sqrt{\alpha t}, \quad t > 0, \tag{9}$$

$$r(t) = 2\mu\sqrt{\alpha t}, \quad t > 0, \tag{10}$$

with

$$\mu = \xi + \frac{\gamma k}{2D_\infty h_0\sqrt{\alpha}}e^{\xi^2}\left[1 + \frac{h_0\sqrt{\pi\alpha}}{k}\operatorname{erf}(\xi)\right], \tag{11}$$

and the coefficient ξ is given as the unique solution of the equation

$$\frac{D_\infty c}{\ell\sqrt{\pi}}F(x) = G(x), \quad x > 0, \tag{12}$$

where the real functions G and F are defined by

$$F(x) = \frac{e^{-x^2}}{k/h_0\sqrt{\pi\alpha} + \operatorname{erf}(x)}, \qquad G(x) = x + \frac{\gamma(1-\varepsilon)\sqrt{\pi}}{2D_\infty}\frac{1}{F(x)}, \quad x > 0. \tag{13}$$

Proof. Taking into account that $\operatorname{erf}(x/2\sqrt{\alpha t})$ is a solution of the heat equation (3) [3], we propose as a solution of problem (1)–(6) the following expression:

$$T(x,t) = C_1 + C_2\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right), \quad 0 < x < s(t), \ t > 0, \tag{14}$$

where the two coefficients C_1 and C_2 must be determined.

From condition (4), we deduce the expression (9) for the free boundary $s(t)$, where the coefficient ξ must be determined. From conditions (6) and (2), we deduce the system of equations

$$C_2 = \frac{h_0\sqrt{\pi\alpha}}{k}\left(C_1 + D_\infty\right), \qquad C_1 + C_2\operatorname{erf}(\xi) = 0, \tag{15}$$

whose solution is given by

$$C_1 = -\frac{\left(h_0\sqrt{\pi\alpha}/k\right)D_\infty\operatorname{erf}(\xi)}{1 + \left(h_0\sqrt{\pi\alpha}/k\right)\operatorname{erf}(\xi)}, \qquad C_2 = \frac{h_0 D_\infty\sqrt{\pi\alpha}}{k}\frac{1}{1 + \left(h_0\sqrt{\pi\alpha}/k\right)\operatorname{erf}(\xi)}, \tag{16}$$

and then we get expression (8) for the temperature.

From condition (4), we deduce expression (10) for the interface $r(t)$ and expression (11) for μ. From condition (3), we deduce (12) for the coefficient ξ. Functions F_3 and G have the following properties:

$$\begin{aligned} &F(0^+) = \frac{h_0\sqrt{\pi\alpha}}{k} > 0, \\ &F(+\infty) = 0^+, \\ &F'(x) < 0, \quad \forall x > 0, \\ &G(0^+) = \frac{(1-\varepsilon)\gamma k}{2D_\infty h_0\sqrt{\alpha}} > 0, \\ &G(+\infty) = +\infty, \\ &G'(x) > 0, \quad \forall x > 0. \end{aligned} \tag{17}$$

Therefore, we deduce that (12) has a unique solution when the coefficient h_0 satisfies the inequality

$$\frac{D_\infty c}{\ell\sqrt{\pi}} F(0^+) > G(0^+) \Longleftrightarrow h_0^2 > \frac{\gamma(1-\varepsilon)\rho\ell k}{2D_\infty^2}; \tag{18}$$

that is, inequality (7) holds. □

Corollary 2. *If the coefficient h_0 satisfies inequality (7), then the temperature, defined by (8), verifies the following inequalities:*

$$-D_\infty < T(0,t) \le T(x,t) < 0, \quad 0 < x < s(t), \ t > 0. \tag{19}$$

Proof. From (8), we obtain

$$\begin{aligned} T(0,t) &= -\frac{\left(h_0 D_\infty\sqrt{\pi\alpha}/k\right)\operatorname{erf}(\xi)}{1+\left(h_0\sqrt{\pi\alpha}/k\right)\operatorname{erf}(\xi)} \\ &= -\frac{D_\infty}{1+k/h_0\sqrt{\pi\alpha}\operatorname{erf}(\xi)} > -D_\infty, \quad \forall t > 0. \end{aligned} \tag{20}$$

Moreover, from (8) and (20), we also get

$$\begin{aligned} &T(x,t) + D_\infty \\ &= \frac{D_\infty}{1+\left(h_0\sqrt{\pi\alpha}/k\right)\operatorname{erf}(\xi)}\left[1+\frac{h_0\sqrt{\pi\alpha}}{k}\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right] \\ &\ge \frac{D_\infty}{1+\left(h_0\sqrt{\pi\alpha}/k\right)\operatorname{erf}(\xi)} = T(0,t) + D_\infty > 0, \\ &\qquad 0 < x < s(t), \ t > 0; \end{aligned} \tag{21}$$

that is, (19) holds. □

3. Asymptotic Behavior When the Coefficient $h_0 \to +\infty$

Now, we will obtain the asymptotic behaviour of solution (8)–(12) of problem (1)–(6) when the heat transfer coefficient is large, that is, when $h_0 \to +\infty$. From a physical point of view, it must be convergent to the solution of the same problem with a temperature boundary condition at the fixed face $x = 0$ given by (23).

For any coefficient h_0 satisfying inequality (7), we will denote the temperature T and the two free boundaries s and r (defined in (8), (9), and (10), resp.) by $T = T(x,t,h_0)$, $x = s(t,h_0)$, and $x = r(t,h_0)$, respectively, with coefficients $\xi = \xi(h_0)$ and $\mu = \mu(h_0)$. We will also denote by $F(x,h_0)$ and $G(x,h_0)$ the functions defined in (13). We have the following result.

Theorem 3. *One obtains the following limits:*

$$\begin{aligned} &\lim_{h_0\to\infty} T(x,t,h_0) = T_\infty(x,t), \\ &\lim_{h_0\to\infty} s(t,h_0) = s_\infty(t), \\ &\lim_{h_0\to\infty} r(t,h_0) = r_\infty(t), \end{aligned} \tag{22}$$

where $T_\infty(x,t)$, $s_\infty(t)$, and $r_\infty(t)$ are the solutions of the following phase-change process with mushy region: (1)–(5) and

$$T(0,t) = -D_\infty, \quad t > 0, \tag{23}$$

instead of the boundary condition (6).

Proof. The solution of problem (1)–(5) and (23) is given by [27]

$$\begin{aligned} T_\infty(x,t) &= -D_\infty\left[1-\frac{\operatorname{erf}\left(x/2\sqrt{\alpha t}\right)}{\operatorname{erf}(\xi_\infty)}\right], \\ &\qquad 0 < x < s_\infty(t), \ t > 0, \end{aligned} \tag{24}$$

$$s_\infty(t) = 2\xi_\infty\sqrt{\alpha t}, \quad t > 0, \tag{25}$$

$$r_\infty(t) = 2\mu_\infty\sqrt{\alpha t}, \quad t > 0, \tag{26}$$

with

$$\mu_\infty = \xi_\infty + \frac{\gamma\sqrt{\pi}}{2D_\infty}e^{\xi_\infty^2}\operatorname{erf}(\xi_\infty), \tag{27}$$

and the coefficient ξ_∞ given as the unique solution of the equation

$$G_1(x) = \frac{D_\infty c}{\ell\sqrt{\pi}}, \quad x > 0, \tag{28}$$

where the real function G_1 is defined by

$$G_1(x) = \frac{G_\infty(x)}{F_\infty(x)}, \quad x > 0, \tag{29}$$

with

$$\begin{aligned} G_\infty(x) &= x + \frac{\gamma(1-\varepsilon)\sqrt{\pi}}{2D_\infty}\frac{1}{F_\infty(x)} = \lim_{h_0\to\infty} G(x,h_0), \\ &\qquad x > 0, \end{aligned} \tag{30}$$

$$F_\infty(x) = \frac{e^{-x^2}}{\operatorname{erf}(x)} = \lim_{h_0\to\infty} F(x,h_0), \quad x > 0. \tag{31}$$

Then,

$$\lim_{h_0\to\infty}\xi(h_0)=\xi_\infty,$$
$$\lim_{h_0\to\infty}\mu(h_0)=\mu_\infty. \quad (32)$$

And, therefore, the limits (22) hold.

Now, by studying the real functions $F(x,h_0)$ and $G(x,h_0)$ as functions of the variable h_0, we can obtain the order of the convergence of solution (8)–(12) of problem (1)–(6) to solution (24)–(28) of problem (1)–(5) and (23) when $h_0\to\infty$. □

Theorem 4. *When the variable $h_0\to\infty$, one obtains the following estimations:*

$$0<\xi_\infty-\xi(h_0)=O\left(\frac{1}{h_0}\right)\quad \text{when } h_0\longrightarrow\infty, \quad (33)$$

$$0<s_\infty(t)-s(t,h_0)=O\left(\frac{1}{h_0}\right),\quad \forall t\geq 0 \text{ when } h_0\longrightarrow\infty, \quad (34)$$

$$\left|\mu(h_0)-\mu_\infty\right|=O\left(\frac{1}{h_0}\right)\quad \text{when } h_0\longrightarrow\infty, \quad (35)$$

$$\left|r(t,h_0)-r_\infty(t)\right|=O\left(\frac{1}{h_0}\right),\quad \forall t\geq 0 \text{ when } h_0\longrightarrow\infty, \quad (36)$$

$$\left|T(x,t,h_0)-T_\infty(x,t)\right|=O\left(\frac{1}{h_0}\right),\quad \forall x\geq 0,\ \forall t>0 \text{ when } h_0\longrightarrow\infty. \quad (37)$$

Proof. As the variable $h_0\to\infty$, we can consider that $h_0>h_0^*$ and then solution (8)–(12) of problem (1)–(6) is well defined.

Function $F(x,h_0)$ is an increasing function in variable h_0; therefore, function $G(x,h_0)$ is a decreasing function in variable h_0. Then, function $G(x,h_0)/F(x,h_0)$ is a decreasing function in variable h_0, $\forall x>0$, which is convergent to $G_\infty(x)/F_\infty(x)$ as $h_0\to\infty$ because (30) and (31) hold.

By using (13) and (31), we have

$$\begin{aligned}0&<F_\infty(x)-F(x,h_0)=F_\infty(x)\frac{k/h_0\sqrt{\pi\alpha}}{\operatorname{erf}(x)+k/h_0\sqrt{\pi\alpha}}\\&<\frac{e^{-x^2}}{\operatorname{erf}^2(x)}\frac{k}{h_0\sqrt{\pi\alpha}}=O\left(\frac{1}{h_0}\right),\quad \forall x>0 \text{ when } h_0\longrightarrow\infty.\end{aligned} \quad (38)$$

By using (13) and (30), we have

$$\begin{aligned}0&<G(x,h_0)-G_\infty(x)\\&=\frac{\gamma(1-\varepsilon)\sqrt{\pi}}{2D_\infty}\frac{F_\infty(x)-F(x,h_0)}{F_\infty(x)F(x,h_0)}\\&<\frac{\gamma(1-\varepsilon)\sqrt{\pi}}{2D_\infty}\frac{F_\infty(x)-F(x,h_0)}{F_\infty(x)F(x,h_0^*)}=O\left(\frac{1}{h_0}\right),\quad \forall x>0 \text{ when } h_0\longrightarrow\infty.\end{aligned} \quad (39)$$

Therefore, we have

$$\begin{aligned}0&<\frac{G(x,h_0)}{F(x,h_0)}-\frac{G_\infty(x)}{F_\infty(x)}<\left[F_\infty(x)-F(x,h_0)\right]\\&\quad\cdot\frac{x+(\gamma(1-\varepsilon)\sqrt{\pi}/D_\infty)(1/F(x,h_0^*))}{F_\infty(x)F(x,h_0^*)}\\&=O\left(\frac{1}{h_0}\right),\quad \forall x>0 \text{ when } h_0\longrightarrow\infty.\end{aligned} \quad (40)$$

Then, the estimation (33) holds and

$$\begin{aligned}0&<s_\infty(t)-s(t,h_0)=2\sqrt{\alpha t}\left(\xi_\infty-\xi(h_0)\right)\\&=O\left(\frac{1}{h_0}\right),\quad \forall t\geq 0 \text{ when } h_0\longrightarrow\infty.\end{aligned} \quad (41)$$

By using (11) and (27), we get

$$\begin{aligned}&\mu_\infty-\mu(h_0)\\&=\left(\xi_\infty-\xi(h_0)\right)-\frac{\gamma\sqrt{\pi}}{2D_\infty\sqrt{\alpha}}\frac{e^{\xi^2(h_0)}}{h_0}\\&\quad+\frac{\gamma\sqrt{\pi}}{2D_\infty}\left[e^{\xi_\infty^2}\operatorname{erf}(\xi_\infty)-e^{\xi^2(h_0)}\operatorname{erf}(\xi(h_0))\right]\\&=O\left(\frac{1}{h_0}\right)\quad\text{when } h_0\longrightarrow\infty,\end{aligned} \quad (42)$$

$$\left|r(t,h_0)-r_\infty(t)\right|=2\sqrt{\alpha t}\left|\mu(h_0)-\mu_\infty\right|=O\left(\frac{1}{h_0}\right),\quad \forall t\geq 0 \text{ when } h_0\longrightarrow\infty.$$

Finally, by using (8) and (24), we get

$$\begin{aligned}&T(x,t,h_0)-T_\infty(x,t)\\&=\frac{D_\infty}{1+(h_0\sqrt{\pi\alpha}/k)\operatorname{erf}(\xi(h_0))}\left\{1\right.\\&\quad+\frac{\operatorname{erf}\left(x/2\sqrt{\alpha t}\right)}{\operatorname{erf}(\xi_0)}\left[\frac{h_0\sqrt{\pi\alpha}}{k}\left(\operatorname{erf}(\xi_\infty)\right.\right.\\&\quad\left.\left.\left.-\operatorname{erf}(\xi(h_0))\right)\right]\right\}=O\left(\frac{1}{h_0}\right),\quad \forall x>0,\ \forall t>0 \text{ when } h_0\longrightarrow\infty,\end{aligned} \quad (43)$$

and the thesis holds. In the particular case, when $x = 0$, we have

$$
\begin{aligned}
0 &< T(0,t,h_0) - T_\infty(0,t) \\
&= \frac{D_\infty}{1 + (h_0\sqrt{\pi\alpha}/k)\operatorname{erf}(\xi(h_0))} \\
&< \frac{kD_\infty}{h_0\sqrt{\pi\alpha}\operatorname{erf}(\xi(h_0^*))} = O\left(\frac{1}{h_0}\right), \\
&\qquad \forall t > 0 \text{ when } h_0 \longrightarrow \infty.
\end{aligned} \tag{44}
$$

□

4. Equivalence between the Mushy Zone Models with Convective and Temperature Boundary Conditions

We consider problem (P_2) defined by conditions (1)–(5) and temperature boundary condition

$$T(0,t) = -D_0 < 0, \quad t > 0, \tag{45}$$

at the fixed face $x = 0$, whose solution was given in [27]. We have the following property.

Theorem 5. *If the coefficient h_0 satisfies inequality (7), then problem (P_1), defined by conditions (1)–(6), is equivalent to problem (P_2), defined by conditions (1)–(5) and (45), when the parameter D_0 in problem (P_2) is related to parameters h_0 and D_0 in problem (P_1) by the following expression:*

$$D_0 = \frac{D_\infty \operatorname{erf}(\xi)}{k/h_0\sqrt{\pi\alpha} + \operatorname{erf}(\xi)} > 0, \tag{46}$$

where the coefficient ξ is given as the unique solution of (12) for problem (P_1) or as the unique solution of equation

$$G_2(x) = \frac{D_0 c}{\ell\sqrt{\pi}}, \quad x > 0, \tag{47}$$

for problem (P_2), where the real function G_2 is defined by

$$
\begin{aligned}
G_2(x) &= \frac{G_0(x)}{F_\infty(x)}, \\
G_0(x) &= \left[x + \frac{\gamma(1-\varepsilon)\sqrt{\pi}}{2D_0}\frac{1}{F_\infty(x)}\right], \\
&\qquad x > 0.
\end{aligned} \tag{48}
$$

Proof. If the coefficient h_0 satisfies inequality (7), then the solution of problem (P_1) is given by (8)–(12). Taking into account that

$$
\begin{aligned}
T(0,t) &= -\frac{(h_0\sqrt{\pi\alpha}/k) D_\infty \operatorname{erf}(\xi)}{1 + (h_0\sqrt{\pi\alpha}/k)\operatorname{erf}(\xi)} \\
&= -\frac{D_\infty \operatorname{erf}(\xi)}{k/h_0\sqrt{\pi\alpha} + \operatorname{erf}(\xi)} < 0, \quad t > 0,
\end{aligned} \tag{49}
$$

we can define problem (P_2) by imposing the temperature boundary condition (45) with data D_0 given in (46). By using this data D_0 in problem (P_2) and the method developed in [30], we can prove that the solutions of both problems (P_1) and (P_2) are the same and then the two problems are equivalent. □

Corollary 6. *If the coefficient h_0 satisfies inequality (7), then the coefficient ξ of the solid-mushy zone interface of problem (P_2) verifies the following inequality:*

$$\operatorname{erf}(\xi) < \frac{D_\infty D_0}{D_\infty - D_0}\sqrt{\frac{2c}{\pi\gamma(1-\varepsilon)\ell}}, \quad \forall D_\infty > D_0. \tag{50}$$

Then,

$$\operatorname{erf}(\xi) < D_0\sqrt{\frac{2c}{\pi\gamma(1-\varepsilon)\ell}}. \tag{51}$$

Remark 7. The real functions G_∞, defined in (30), and G_0, defined in (48), are similar; the difference between them is the parameters D_∞ or D_0 used in each definition.

5. Explicit Solution with a Heat Flux Boundary Condition

Now, we will consider a phase-change process which consists in finding the free boundaries $x = s(t)$ and $x = r(t)$ and the temperature $T = T(x,t)$ such that the following conditions must be verified (problem (P_3)): conditions (1)–(5) and

$$kT_x(0,t) = \frac{q_0}{\sqrt{t}}, \quad t > 0 \ (q_0 > 0). \tag{52}$$

Condition (52) represents the heat flux at the fixed face $x = 0$ characterized by a coefficient which is inversely proportional to the square root of the time [34].

Theorem 8. *If the coefficient q_0 satisfies the inequality*

$$q_0 > \sqrt{\frac{\gamma(1-\varepsilon)\rho\ell k}{2}} = q_0^*, \tag{53}$$

then the solution of problem (1)–(5) and (52) is given by

$$
\begin{aligned}
T(x,t) &= -\frac{q_0\sqrt{\pi\alpha}\operatorname{erf}(\omega)}{k}\left[1 - \frac{\operatorname{erf}\left(x/2\sqrt{\alpha t}\right)}{\operatorname{erf}(\omega)}\right] < 0, \\
&\qquad 0 < x < s(t),\ t > 0,
\end{aligned} \tag{54}
$$

$$s(t) = 2\omega\sqrt{\alpha t}, \quad t > 0, \tag{55}$$

$$r(t) = 2\nu\sqrt{\alpha t}, \quad t > 0, \tag{56}$$

with

$$\nu = \omega + \frac{\gamma k}{2q_0\sqrt{\alpha}}e^{\omega^2}, \tag{57}$$

and the coefficient $\omega > 0$ given as the unique solution of the equation

$$G_3(x) = \frac{q_0}{\rho\ell\sqrt{\alpha}}, \quad x > 0, \tag{58}$$

where the real function G_3 is defined by

$$G_3(x) = \left[x + \frac{\gamma(1-\varepsilon)k}{2q_0\sqrt{\alpha}}e^{x^2}\right]e^{x^2}, \quad x > 0. \tag{59}$$

Proof. Following the proof of Theorem 1, we propose as a solution of problem (1)–(5) and (52) the following expression:

$$T(x,t) = A_1 + A_2 \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right), \qquad 0 < x < s(t), \ t > 0, \tag{60}$$

where the two coefficients A_1 and A_2 must be determined.

From condition (2), we deduce expression (55) for the free boundary $s(t)$, with the coefficient ω to be determined. From conditions (2) and (52), we deduce

$$A_1 = -\frac{q_0\sqrt{\pi\alpha}}{k}\operatorname{erf}(\omega), \qquad A_2 = \frac{q_0\sqrt{\pi\alpha}}{k}, \tag{61}$$

and then we get expression (54) for the temperature.

From condition (4), we deduce expression (56) for the interface $r(t)$ and expression (57) for ν. From condition (3), we deduce (58) for the coefficient ω. Since function G_3 has the following properties:

$$G_3(0^+) = \frac{\gamma(1-\varepsilon)k}{2q_0\sqrt{\alpha}} > 0, \qquad G_3(+\infty) = +\infty, \qquad G_3'(x) > 0, \quad \forall x > 0, \tag{62}$$

we can deduce that (58) has a unique solution when the coefficient q_0 satisfies the inequality

$$\frac{q_0}{\rho\ell\sqrt{\alpha}} > G_3(0^+) \Longleftrightarrow \qquad q_0^2 > \frac{\gamma(1-\varepsilon)\rho\ell k}{2}, \tag{63}$$

which is inequality (53). □

Remark 9. We have a relationship between q_0^* (the lower limit for coefficient q_0 in order to have a phase-change process with a mushy region with a heat flux boundary condition at $x = 0$) and h_0^* (the lower limit for the coefficient h_0 in order to have a phase-change process with a mushy region with a convective boundary condition at $x = 0$) given by

$$q_0^* = D_\infty h_0^*. \tag{64}$$

6. Equivalence between the Mushy Zone Models with Heat Flux and Temperature Boundary Conditions

Following Section 4, we will now study the relationship between problems (P_3) and (P_2). We have the following property.

Theorem 10. *If the coefficient q_0 satisfies inequality (53), then problem (P_3), defined by conditions (1)–(5) and (52), is equivalent to problem (P_2), defined by conditions (1)–(5) and (45), when the parameter D_0 in problem (P_2) is related to the parameter q_0 in problem (P_3) by the following expression:*

$$D_0 = \frac{q_0\sqrt{\pi\alpha}}{k}\operatorname{erf}(\omega) > 0, \tag{65}$$

where the coefficient ω is given as the unique solution of (58) for problem (P_3) or as the unique solution of (47) for problem (P_2).

Proof. If the coefficient q_0 satisfies inequality (53), then the solution of problem (P_3) is given by (54)–(58). Taking into account that

$$T(0,t) = -\frac{q_0\sqrt{\pi\alpha}}{k}\operatorname{erf}(\omega) < 0, \quad t > 0, \tag{66}$$

we can define problem (P_2) by imposing the temperature boundary condition (45) with the data D_0 given in (65). By using this data D_0 in problem (P_2) and the method developed in [30], we can prove that the solutions of both problems (P_3) and (P_2) are the same and then the two problems are equivalent. □

Corollary 11. *If the coefficient q_0 satisfies inequality (53), then the coefficient ξ of the solid-mushy zone interface of problem (P_2) verifies inequality (51) which is the same as that we have obtained through the equivalence between problems (P_1) and (P_2).*

Remark 12. At last, for a suggestion of an anonymous referee, we will transform problem (P_1), given by the equations and conditions (1)–(6), and inequality (7) in a dimensionless form. We define the following dimensionless change of variables:

$$\eta = \frac{x}{L}, \qquad \tau = \frac{\alpha t}{L^2}, \qquad S(\tau) = \frac{s(t)}{L}, \qquad R(\tau) = \frac{r(t)}{L}, \qquad \theta(\eta,\tau) = \frac{T(x,t)}{D_\infty}, \tag{67}$$

where L is a characteristic length. Therefore, the equations and conditions (1)–(6) are transformed as

$$\begin{aligned}
&\theta_\tau - \theta_{\eta\eta} = 0, \quad 0 < \eta < S(\tau), \ \tau > 0, \\
&\theta(S(\tau), \tau) = 0, \quad \tau > 0, \\
&\theta_\eta(r(\tau), \tau) = \frac{1}{\mathrm{Ste}}\left[\varepsilon S'(\tau) + (1-\varepsilon) R'(\tau)\right], \quad \tau > 0, \\
&\theta_\eta(r(\tau), \tau)(R(\tau) - S(\tau)) = \frac{\gamma}{D_\infty}, \quad \tau > 0, \\
&S(0) = R(0) = 0, \\
&\theta_\eta(0, \tau) = \frac{B}{\sqrt{\tau}}(\theta(0, \tau) + 1), \quad \tau > 0,
\end{aligned} \tag{68}$$

where Ste is the Stefan number and $B/\sqrt{\tau}$ is the Biot number defined by the following expressions:

$$\mathrm{Ste} = \frac{cD_\infty}{\ell} > 0, \tag{69}$$

$$B = \frac{h_0\sqrt{\alpha}}{k} = \frac{h_0}{\sqrt{\rho k c}} > 0. \tag{70}$$

Moreover, inequality (7) for the physical coefficient h_0, which characterized the heat transfer coefficient in the boundary condition (6), is transformed in the following way:

$$B > \frac{1}{D_\infty}\sqrt{\frac{\gamma(1-\varepsilon)\ell}{2c}} = B^*. \tag{71}$$

Therefore, limit $h_0 \to \infty$ in problem (1)–(6) in physical variables is equivalent to limit $B \to \infty$ in problem (68) in dimensionless variables.

By using the results of this work, we can now obtain new explicit expression for the determination of one or two unknown thermal coefficients through a phase-change process with a mushy zone by imposing an overspecified convective boundary condition at the fixed face $x = 0$. This will complete and improve the results obtained previously in [28].

7. Conclusions

The goal of this paper is to complete the solution of Solomon-Wilson-Alexiades's model for a mushy zone model for phase-change materials when a convective or a heat flux boundary condition at the fixed face $x = 0$ is imposed. In both cases, explicit solutions for the temperature and the two free boundaries which define the mushy region were obtained and, for both cases, the necessary and sufficient conditions on data in order to get these explicit solutions are also obtained which is new with respect to the original model when a temperature boundary condition at the face $x = 0$ was imposed. Moreover, the equivalence of this two phase-change process with the one with a temperature boundary condition on the fixed face $x = 0$ was obtained and an inequality for the dimensionless coefficient that characterizes the first free boundary is also given. On the other hand, the convergence of the phase-change process with mushy zone when the heat transfer coefficient goes to infinity was also obtained and the order of the convergence is also shown.

Nomenclature

$B(B^*)$:	Dimensionless coefficient that characterizes the transient heat transfer at $x = 0$ (Biot number), defined in (70)
c:	Specific heat, J/(kg°C)
$-D_0(< 0)$:	Temperature at the fixed face $x = 0$, °C
$-D_\infty(< 0)$:	Bulk temperature at the fixed face $x = 0$, °C
$h_0(h_0^*)$:	Coefficient that characterizes the transient heat transfer at $x = 0$, kg/(C°s$^{5/2}$)
k:	Thermal conductivity, W/(m°C)
ℓ:	Latent heat of fusion by unit of mass, J/kg
L:	Characteristic length, m
P_1:	Phase-change process defined by conditions (1)–(6)
P_2:	Phase-change process defined by conditions (1)–(5) and (45)
P_3:	Phase-change process defined by conditions (1)–(5) and (52)
$q_0(q_0^*)$:	Coefficient that characterizes the transient heat flux at $x = 0$, kg/s$^{5/2}$
$r = r(t)\ (> s(t))$:	Position of the liquid-mushy zone interface at time t, m
$R = R(\tau)\ (> S(\tau))$:	Dimensionless position of the liquid-mushy zone interface at time τ
$s = s(t)$:	Position of the solid-mushy zone interface at time t, m
$S = S(\tau)$:	Dimensionless position of the solid-mushy zone interface at time τ
Ste:	Stefan number, defined in (69)
t:	Time, s
T:	Temperature of the solid phase, °C
x:	Spatial coordinate, m.

Greek Symbols

$\alpha = k/\rho c$:	Diffusivity coefficient, m^2/s
$\gamma > 0$:	One of the two coefficients that characterizes the mushy zone, °C
$\varepsilon \in (0, 1)$:	One of the two coefficients that characterizes the mushy zone, being dimensionless
$\nu(> \omega)$:	Coefficient that characterizes the free boundary $r(t)$ in (45), being dimensionless
ρ:	Density of mass, kg/m^3
$\mu(> \xi)$:	Coefficient that characterizes the free boundary $r(t)$ in (10), being dimensionless
$\mu_\infty(> \xi_\infty)$:	Coefficient that characterizes the free boundary $r(t)$ in (26), being dimensionless

$\omega > 0$: Coefficient that characterizes the free boundary $s(t)$ in (43), being dimensionless
$\xi > 0$: Coefficient that characterizes the free boundary $s(t)$ in (9), being dimensionless
$\xi_\infty > 0$: Coefficient that characterizes the free boundary $s(t)$ in (25), being dimensionless
τ: Dimensionless time
θ: Dimensionless temperature of the solid phase
η: Dimensionless spatial coordinate.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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