Research Article

Best Proximity Points for Generalized Proximal Weak Contractions Satisfying Rational Expression on Ordered Metric Spaces

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We introduce a generalized proximal weak contraction of rational type for the non-self-map and proved results to ensure the existence and uniqueness of best proximity point for such mappings in the setting of partially ordered metric spaces. Further, our results provides an extension of a result due to Luong and Thuan (2011) and also it provides an extension of Harjani (2010) to the case of self-mappings.

1. Introduction and Preliminaries

The fixed point theory of partially ordered metric space was introduced by Ran and Reurings [1], where they extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. Subsequently, Nieto and Rodríguez-López [2] extended the result of Ran and Reurings and apply their results to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. The following notion of an altering distance function was introduced by Khan et al. in [3].

Definition 1. A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is said to be an altering distance function if it satisfies the following conditions.

(i) ψ is continuous and nondecreasing.

(ii) $\psi(t) = 0$ if and only if $t = 0$.

In [3], Khan et al. proved the fixed point theorems by using altering distance function together with contractive type condition.

Motivated by the interesting paper of Jaggi [4], in [5] Harjani et al. proved the following fixed point theorem in partially ordered metric spaces.

Theorem 2 (see [5]). *Let $(X, \leq)$ be an ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space.*

Let $T : X \rightarrow X$ be a nondecreasing mapping such that

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y) \tag{1}$$

$$\text{for } x, y \in X, \quad x \geq y, \quad x \neq y.$$

Also, assume either T is continuous or X has the property that

$$\{x_n\} \text{ is a nondecreasing sequence in } X$$
$$\text{such that } x_n \longrightarrow x, \text{ then } x = \sup \{x_n\}. \tag{2}$$

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

In [6] Luong and Thuan proved the following theorem.

Theorem 3 (see [6]). *Let $(X, \leq)$ be an ordered set and suppose that there exists a metric d in X such that (X, d) is a complete*

metric space. Let $T : X \rightarrow X$ be a nondecreasing mapping such that

$$d(Tx, Ty) \leq m(x, y) - \phi(m(x, y))$$
$$\text{for } x, y \in X, \quad x \geq y, \quad x \neq y, \tag{3}$$

where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a lower semicontinuous function with $\phi(t) = 0$ if and only if $t = 0$, and $m(x, y) = \max\{d(x, Tx)d(y, Ty)/d(x, y), d(x, y)\}$.

Also, assume either T is continuous or X has the property (2).

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

In this article, we attempt to give a generalization of Theorem 3 by considering a non-self-map T. Before getting into the details of our main theorem, let us give a brief discussion of best proximity point results.

1.1. Best Proximity Point. Let $A \neq \emptyset$ be a subset of a metric space (X, d). A mapping $T : A \rightarrow X$ has a fixed point in A if the fixed point equation $Tx = x$ has at least one solution. That is, $x \in A$ is a fixed point of T if $d(x, Tx) = 0$. If the fixed point equation $Tx = x$ does not possess a solution, then $d(x, Tx) > 0$ for all $x \in A$. In such a situation, it is our aim to find an element $x \in A$ such that $d(x, Tx)$ is minimum in some sense. The best approximation theory and best proximity pair theorems are studied in this direction. Here we state the following well-known best approximation theorem due to Fan [7].

Theorem 4 (see [7]). *Let A be a nonempty compact convex subset of a normed linear space X and $T : A \rightarrow X$ be a continuous function. Then there exists $x \in A$ such that $\|x - Tx\| = d(Tx, A) := \inf\{\|Tx - a\| : a \in A\}$.*

Such an element $x \in A$ in Theorem 4 is called a best approximant of T in A. Note that if $x \in A$ is a best approximant, then $\|x - Tx\|$ need not be the optimum. Best proximity point theorems have been explored to find sufficient conditions so that the minimization problem $\min_{x \in A} \|x - Tx\|$ has at least one solution. To have a concrete lower bound, let us consider two nonempty subsets A, B of a metric space X and a mapping $T : A \rightarrow B$. The natural question is whether one can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\{d(x, Tx) : x \in A\}$. Since $d(x, Tx) \geq d(A, B)$, the optimal solution to the problem of minimizing the real valued function $x \rightarrow d(x, Tx)$ over the domain A of the mapping T will be the one for which the value $d(A, B)$ is attained. A point $x_0 \in A$ is called a best proximity point of T if $d(x_0, Tx_0) = d(A, B)$. Note that if $d(A, B) = 0$, then the best proximity point is nothing but a fixed point of T.

The existence and convergence of best proximity points is an interesting topic of optimization theory which recently attracted the attention of many authors [8–16]. Also one can find the existence of best proximity point in the setting of partially ordered metric space in [17–24].

The purpose of this article is to present best proximity point theorems for non-self-mappings in the setting of partially ordered metric spaces, thereby producing optimal approximate solutions for $Tx = x$, where T is a non-self-mapping. When the map T is considered to be a self-map and ψ is defined as identity function, then our result reduces to the fixed point theorem of Luong and Thuan [6].

Given nonempty subsets A and B of a metric space X, the following notions are used subsequently:

$$d(A, B) := \inf\{d(x, y) : x \in A, y \in B\},$$
$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \tag{4}$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

In [14], the authors discussed sufficient conditions which guarantee the nonemptiness of A_0 and B_0. Moreover, in [12], the authors proved that A_0 is contained in the boundary of A in the setting of normed linear spaces.

Definition 5 (see [17]). A mapping $T : A \rightarrow B$ is said to be proximally increasing if it satisfies the condition that

$$\left.\begin{array}{r} y_1 \leq y_2 \\ d(x_1, Ty_1) = d(A, B) \\ d(x_2, Ty_2) = d(A, B) \end{array}\right\} \implies x_1 \leq x_2, \tag{5}$$

where $x_1, x_2, y_1, y_2 \in A$.

One can see that, for a self-mapping, the notion of proximally increasing mapping reduces to that of increasing mapping.

Definition 6. A mapping $T : A \rightarrow B$ is said to be generalized proximal weak contraction of rational type if it satisfies the condition that

$$\left.\begin{array}{r} x \leq y, \quad x \neq y \\ d(u_1, Tx) = d(A, B) \\ d(u_2, Ty) = d(A, B) \end{array}\right\} \implies \psi(d(u_1, u_2))$$
$$\leq \psi(m(x, y)) - \phi(m(x, y)), \tag{6}$$

where $u_1, u_2, x, y \in A$, ψ is an altering distance function, $\phi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function with $\phi(t) = 0$ if and only if $t = 0$, and $m(x, y) = \max\{d(x, u_1)d(y, u_2)/d(x, y), d(x, y)\}$.

One can see that, for a self-mapping, the notion of generalized proximal weak contraction of rational type reduces to generalized weak contraction of rational type.

2. Main Results

Now, let us state our main result.

Theorem 7. *Let X be a nonempty set such that $(X, \leq)$ is a partially ordered set and (X, d) is a complete metric space. Let A and B be nonempty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $T : A \rightarrow B$ satisfy the following conditions.*

(i) *T is continuous, proximally increasing, and generalized proximal weak contraction of rational type such that $T(A_0) \subseteq B_0$.*

(ii) *There exist x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \leq x_1. \tag{7}$$

Then, there exists an element x in A such that

$$d(x, Tx) = d(A, B). \tag{8}$$

Further, the sequence $\{x_n\}$, defined by

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \text{for } n \geq 1, \tag{9}$$

converges to the element x.

Proof. By hypothesis there exist elements x_0 and x_1 in A_0 such that

$$d(x_1, Tx_0) = d(A, B), \quad x_0 \leq x_1. \tag{10}$$

Because of the fact that $T(A_0) \subseteq B_0$, there exists an element x_2 in A_0 such that

$$d(x_2, Tx_1) = d(A, B). \tag{11}$$

Since T is proximally increasing, we get $x_1 \leq x_2$.

Continuing this process, we can construct a sequence (x_n) in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \forall n \in \mathbb{N}$$
$$\text{with } x_0 \leq x_1 \leq x_2 \leq \cdots x_n \leq x_{n+1} \cdots. \tag{12}$$

If there exist n_0 such that $x_{n_0} = x_{n_0+1}$, then $d(x_{n_0+1}, Tx_{n_0}) = d(x_{n_0}, Tx_{n_0}) = d(A, B)$. This means that x_{n_0} is a best proximity point of T and the proof is finished. Thus, we can suppose that $x_n \neq x_{n+1}$ for all n.

Since $x_{n-1} < x_n$, we get

$$\psi(d(x_n, x_{n+1}))$$
$$\leq \psi\left(\max\left\{\frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\}\right)$$
$$- \phi\left(\max\left\{\frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\}\right)$$
$$= \psi(\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\})$$
$$- \phi(\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}). \tag{13}$$

Suppose that there exists m_0 such that $d(x_{m_0}, x_{m_0+1}) > d(x_{m_0-1}, x_{m_0})$, and from (13), we have

$$\psi(d(x_{m_0}, x_{m_0+1}))$$
$$\leq \psi(d(x_{m_0}, x_{m_0+1})) - \phi(d(x_{m_0}, x_{m_0+1})) \tag{14}$$
$$< \psi(d(x_{m_0}, x_{m_0+1})).$$

Hence, the sequence $\{d(x_n, x_{n+1})\}$ is monotone, nonincreasing and bounded. Thus, there exists $r \geq 0$ such that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = r \geq 0. \tag{15}$$

Since $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence, from (13), we get

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)),$$
$$\forall x_{n-1} < x_n, \quad n \geq 1. \tag{16}$$

Suppose that $\lim_{n \to \infty} d(x_n, x_{n+1}) = r > 0$. Then the inequality (16)

$$\psi(d(x_n, x_{n+1})) \leq \psi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n))$$
$$\leq \psi(d(x_{n-1}, x_n)) \tag{17}$$

implies that

$$\lim_{n \to \infty} \phi(d(x_n, x_{n+1})) = 0. \tag{18}$$

But, as $0 < r \leq d(x_n, x_{n+1})$ and ϕ is nondecreasing function,

$$0 < \phi(r) \leq \phi(d(x_n, x_{n+1})), \tag{19}$$

and this gives us $\lim_{n \to \infty} \phi(d(x_n, x_{n+1})) \geq \phi(r) > 0$ which contradicts (18). Hence,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{20}$$

Now to prove that $\{x_n\}$ is a Cauchy sequence, suppose $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is smallest index for which

$$n(k) > m(k) > k, \quad d(x_{m(k)}, x_{n(k)}) \geq \epsilon. \tag{21}$$

This means that

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon,$$
$$\epsilon \leq d(x_{m(k)}, x_{n(k)})$$
$$\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \tag{22}$$
$$< \epsilon + d(x_{n(k)-1}, x_{n(k)}).$$

Letting $k \to \infty$ and using (20) we can conclude that

$$\lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{23}$$

By triangle inequality

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1})$$
$$+ d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}),$$
$$d(x_{n(k)-1}, x_{m(k)-1}) \leq d(x_{n(k)-1}, x_{n(k)})$$
$$+ d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}). \tag{24}$$

Letting $k \to \infty$ in the above two inequalities, using (20) and (23), we get

$$\lim_{k \to \infty} d\left(x_{n(k)-1}, x_{m(k)-1}\right) = \epsilon. \tag{25}$$

Since $m(k) < n(k)$, $x_{m(k)-1} \le x_{n(k)-1}$, from (16), we have

$$0 < \psi(\epsilon) \le \psi\left(d\left(x_{m(k)}, x_{n(k)}\right)\right)$$
$$\le \psi\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) - \phi\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) \tag{26}$$
$$\le \psi\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right).$$

Using (25) and continuity of ψ in the above inequality we can obtain

$$\lim_{k \to \infty} \phi\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) = 0. \tag{27}$$

But, from $\lim_{k \to \infty} d(x_{n(k)-1}, x_{m(k)-1}) = \epsilon$ we can find $k_0 \in \mathbb{N}$ such that for any $k \ge k_0$

$$\frac{\epsilon}{2} \le d\left(x_{n(k)-1}, x_{m(k)-1}\right) \tag{28}$$

and consequently,

$$0 < \phi\left(\frac{\epsilon}{2}\right) \le \phi\left(d\left(x_{n(k)-1}, x_{m(k)-1}\right)\right) \quad \text{for } k \ge k_0. \tag{29}$$

Therefore, $0 < \phi(\epsilon/2) \le \phi(d(x_{n(k)-1}, x_{m(k)-1}))$ and this contradicts (27). Thus, $\{x_n\}$ is a Cauchy sequence in A and hence converges to some element x in A. Since T is continuous, we have $Tx_n \to Tx$.

Hence the continuity of the metric function d implies that $d(x_{n+1}, Tx_n) \to d(x, Tx)$. But (12) shows that the sequence $d(x_{n+1}, Tx_n)$ is a constant sequence with the value $d(A, B)$. Therefore, $d(x, Tx) = d(A, B)$. This completes the proof. □

Corollary 8. *Let X be a nonempty set such that $(X, \le)$ is a partially ordered set and (X, d) is a complete metric space. Let A be a nonempty closed subset of the metric space (X, d). Let $T : A \to A$ satisfy the following conditions.*

   (i) *T is continuous, proximally increasing, and generalized proximal weak contraction of rational type.*

   (ii) *There exist elements x_0 and x_1 in A such that $d(x_1, Tx_0) = 0$ with $x_0 \le x_1$.*

Then, there exist an element x in A such that $d(x, Tx) = 0$.

In what follows we prove that Theorem 7 is still valid for T which is not necessarily continuous, assuming the following hypothesis in A. A has the property that

$$\{x_n\} \text{ is a nondecreasing sequence in } A$$
$$\text{such that } x_n \longrightarrow x, \text{ then } x = \sup\{x_n\}. \tag{30}$$

Theorem 9. *Assume the conditions (30) and A_0 is closed in X instead of continuity of T in Theorem 7; then the conclusion of Theorem 7 holds.*

Proof. Following the proof of Theorem 7, there exists a sequence $\{x_n\}$ in A satisfying the following condition:

$$d\left(x_{n+1}, Tx_n\right) = d(A, B) \quad \forall n \in \mathbb{N}$$
$$\text{with } x_0 \le x_1 \le x_2 \le \cdots x_n \le x_{n+1} \cdots. \tag{31}$$

and x_n converges to x in A. Note that the sequence $\{x_n\}$ in A_0 and A_0 is closed. Therefore, $x \in A_0$. Since $T(A_0) \subseteq B_0$, we get $Tx \in B_0$.

Since $Tx \in B_0$, there exist $y_1 \in A$ such that

$$d\left(y_1, Tx\right) = d(A, B). \tag{32}$$

Since $\{x_n\}$ is a nondecreasing sequence and $x_n \to x$, then $x = \sup\{x_n\}$. Particularly, $x_n \le x$ for all n. Since T is proximally increasing and from (31) and (32), we obtain $x_{n+1} \le y_1$. But $x = \sup\{x_n\}$ which implies $x \le y_1$. Therefore, we get that there exist elements x and y_1 in A_0 such that

$$d\left(y_1, Tx\right) = d(A, B), \quad x \le y_1. \tag{33}$$

Consider the sequence $\{y_n\}$ that is constructed as follows:

$$d\left(y_{n+1}, Ty_n\right) = d(A, B) \quad \forall n \in \mathbb{N}.$$
$$\text{with } x = y_0 \le y_1 \le y_2 \le \cdots y_n \le y_{n+1} \cdots. \tag{34}$$

Arguing like above Theorem 7, we obtain that $\{y_n\}$ is a nondecreasing sequence and $y_n \to y$ for certain $y \in A$. From (30), we have $y = \sup\{y_n\}$. Since $x_n < x = y_0 \le y_1 \le y_n \le y$ for all n, suppose that $x \ne y$, T is generalized proximal weak contraction of rational type; from (31) and (34) we have

$$\psi\left(d\left(x_{n+1}, y_{n+1}\right)\right)$$
$$\le \psi\left(\max\left\{\frac{d\left(x_n, x_{n+1}\right)d\left(y_n, y_{n+1}\right)}{d\left(x_n, y_n\right)}, d\left(x_n, y_n\right)\right\}\right)$$
$$- \phi\left(\max\left\{\frac{d\left(x_n, x_{n+1}\right)d\left(y_n, y_{n+1}\right)}{d\left(x_n, y_n\right)}, d\left(x_n, y_n\right)\right\}\right). \tag{35}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$\psi\left(d\left(x, y\right)\right) \le \psi\left(\max\{0, d\left(x, y\right)\}\right)$$
$$- \phi\left(\max\{0, d\left(x, y\right)\}\right) \tag{36}$$
$$< \psi\left(d\left(x, y\right)\right)$$

which is a contradiction. Hence, $x = y$. We have $x = y_0 \le y_1 \le y_n = x$, and therefore $y_n = x$, for all n. From (34), we obtain x is a best proximity point for T. The proof is complete. □

Corollary 10. *Assume the condition (30) instead of continuity of T in the Corollary 8; then the conclusion of Corollary 8 holds.*

Now, we present an example where it can be appreciated that hypotheses in Theorems 7 and 9 do not guarantee uniqueness of the best proximity point.

Example 11. Let $X = \{(0,1), (1,0), (-1,0), (0,-1)\} \subset \mathbb{R}^2$ and consider the usual order $(x,y) \preceq (z,t) \Leftrightarrow x \leq z$ and $y \leq t$.

Thus, $(X, \preceq)$ is a partially ordered set. Besides, (X, d_2) is a complete metric space considering d_2 the euclidean metric. Let $A = \{(0,1), (1,0)\}$ and $B = \{(0,-1), (-1,0)\}$ be a closed subset of X. Then, $d(A, B) = \sqrt{2}$, $A = A_0$ and $B = B_0$. Let $T : A \to B$ be defined as $T(x,y) = (-y, -x)$. Then, it can be seen that T is continuous, proximally increasing mappings such that $T(A_0) \subseteq B_0$. The only comparable pairs of elements in A are $x \preceq x$ for $x \in A$ and there are no elements such that $x \prec y$ for $x, y \in A$. Hence, T is generalized proximal weak contraction of rational type. It can be shown that the other hypotheses of the Theorems 7 and 9 are also satisfied.

However, T has two best proximity points $(0,1)$ and $(1,0)$.

Theorem 12. *In addition to the hypotheses of Theorem 7 (resp., Theorem 9), suppose that*

$$for\ every\ x, y \in A_0,\ there\ exist\ z \in A_0 \tag{37}$$
$$that\ is\ comparable\ to\ x\ and\ y$$

and then T has a unique best proximity point.

Proof. From Theorem 7 (resp., Theorem 9), the set of best proximity points of T is nonempty. Suppose that there exist elements x, y in A which are best proximity points. We distinguish two cases:

Case 1. If x and y are comparable.

Since $d(x, Tx) = d(A, B)$ and $d(y, Ty) = d(A, B)$.

Since T is a generalized proximal weak contraction of rational type, we get

$$\psi(d(x,y))$$
$$\leq \psi\left(\max\left\{\frac{d(x,x)d(y,y)}{d(x,y)}, d(x,y)\right\}\right)$$
$$- \phi\left(\max\left\{\frac{d(x,x)d(y,y)}{d(x,y)}, d(x,y)\right\}\right) \tag{38}$$
$$= \psi(d(x,y)) - \phi(d(x,y))$$

which implies $\phi(d(x,y)) = 0$, and by our assumption about ϕ, we get $d(x,y) = 0$ or $x = y$.

Case 2. If x is not comparable to y.

By the condition (37) there exist $z_0 \in A_0$ comparable to x and y. We define a sequence $\{z_n\}$ as $d(z_{n+1}, Tz_n) = d(A, B)$. Since z_0 is comparable with x, we may assume that $z_0 \preceq x$. Now using T is proximally increasing, it is easy to show that $z_n \preceq x$ for all n.

Suppose that there exist $n_0 > 1$ such that $x = z_{n_0}$, and again by using T which is proximally increasing, we get $x \preceq z_{n_0+1}$. But, $z_n \preceq x$ for all n. Therefore, $x = z_{n_0+1}$. Arguing like above, we obtain $x = z_n$ for all $n \geq n_0$. Hence, $z_n \to x$ as $n \to \infty$.

On the other hand, if $z_{n-1} \neq x$ for all n, now using T is a generalized proximal weak contraction of rational type, we have

$$\psi(d(z_n, x))$$
$$\leq \psi\left(\max\left\{\frac{d(z_{n-1}, z_n)d(x,x)}{d(z_{n-1}, x)}, d(z_{n-1}, x)\right\}\right)$$
$$- \phi\left(\max\left\{\frac{d(z_{n-1}, z_n)d(x,x)}{d(z_{n-1}, x)}, d(z_{n-1}, x)\right\}\right)$$
$$= \psi(d(z_{n-1}, x)) - \phi(d(z_{n-1}, x)) < \psi(d(z_{n-1}, x)). \tag{39}$$

Since ψ is nondecreasing, we get $d(z_n, x) \leq d(z_{n-1}, x)$ Hence, the sequence $\{d(z_n, x)\}$ is monotone, nonincreasing and bounded. Thus, there exist $r \geq 0$ such that

$$\lim_{n \to \infty} d(z_n, x) = r \geq 0. \tag{40}$$

Suppose that $\lim_{n \to \infty} d(z_n, x) = r > 0$. Then the inequality

$$\psi(d(z_n, x)) \leq \psi(d(z_{n-1}, x)) - \phi(d(z_{n-1}, x))$$
$$\leq \psi(d(z_{n-1}, x)) \tag{41}$$

implies that

$$\lim_{n \to \infty} \phi(d(z_{n-1}, x)) = 0. \tag{42}$$

But, as $0 < r \leq d(z_{n-1}, x)$ and ϕ is nondecreasing function, $0 < \phi(r) \leq \phi(d(z_{n-1}, x))$, and this gives $\lim_{n \to \infty} \phi(d(z_{n-1}, x)) \geq \phi(r) > 0$ which contradicts (42). Hence, $\lim_{n \to \infty} d(z_n, x) = 0$. Analogously, it can be proved that $\lim_{n \to \infty} d(z_n, y) = 0$. Finally, the uniqueness of the limit gives us $x = y$. □

Let us illustrate the above theorem with the following example.

Example 13. Let $X = \mathbb{R}^2$ and consider the order $(x,y) \preceq (z,t) \Leftrightarrow x \leq z$ and $y \leq t$, where $\leq$ is usual order in $\mathbb{R}$.

Thus, $(X, \preceq)$ is a partially ordered set. Besides, (X, d_1) is a complete metric space where the metric is defined as $d_1((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Let $A = \{(0, x) : x \in [0, \infty)\}$ and $B = \{(1, x) : x \in [0, \infty)\}$ be a closed subset of X. Then, $d(A, B) = 1$, $A = A_0$ and $B = B_0$. Let $T : A \to B$ be defined as $T(0, x) = (1, x/2)$. Then, it can be seen that T is continuous, proximally increasing mappings and proximally weak increasing such that $T(A_0) \subseteq B_0$. Now, we have to prove T is a generalized proximal weak contraction of rational type. That is to prove

$$\left. \begin{array}{c} x \leq y, \quad x \neq y \\ d\left(\left(0, \dfrac{x}{2}\right), T(0,x)\right) = 1 \\ d\left(\left(0, \dfrac{y}{2}\right), T(0,y)\right) = 1 \end{array} \right\} \implies \psi\left(d\left(\left(0, \dfrac{x}{2}\right), \left(0, \dfrac{y}{2}\right)\right)\right)$$

$$\leq \psi(m((0,x), (0,y)))$$
$$- \phi(m((0,x), (0,y))), \tag{43}$$

where $m((0,x), (0,y)) = \max\{xy/4(y-x), y-x\}$.

Note that $d((0, x/2), (0, y/2)) = (1/2)(y - x)$ and

$$
m((0, x), (0, y)) = \begin{cases} \dfrac{xy}{4(y - x)} & \text{if } 9xy \geq x^2 + y^2, \\ y - x & \text{if } 9xy \leq x^2 + y^2. \end{cases} \tag{44}
$$

For $\psi(t) = 2t$ and $\phi(t) = t$, we have $\psi(d((0, x/2), (0, y/2))) = y - x$ and

$$
\psi(m((0, x), (0, y))) - \phi(m((0, x), (0, y)))
$$
$$
= \begin{cases} \dfrac{3xy}{8(y - x)} & \text{if } 9xy \geq x^2 + y^2, \\ y - x & \text{if } 9xy \leq x^2 + y^2. \end{cases} \tag{45}
$$

Now, we easily conclude that the mapping T is a generalized proximal weak contraction of rational type. Hence all the hypotheses of the Theorem 12 are satisfied. Also, it can be observed that $(0, 0)$ is the unique best proximity point of the mapping T.

The following result, due to Luong and Thuan [6], is a corollary from the above Theorem 12, by taking $A = B$.

Corollary 14. *In addition to the hypothesis of Corollary 8 (resp., Corollary 10), suppose that*

$$
\text{for every } x, y \in A, \text{ there exist } z \in A
$$
$$
\text{that is comparable to } x \text{ and } y \tag{46}
$$

and then T has a unique fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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