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Research Article

Approximate Solutions of the Generalized Abel's Integral Equations Using the Extension Khan's Homotopy Analysis Transformation Method

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User friendly algorithm based on the optimal homotopy analysis transform method (OHATM) is proposed to find the approximate solutions to generalized Abel's integral equations. The classical theory of elasticity of material is modeled by the system of Abel integral equations. It is observed that the approximate solutions converge rapidly to the exact solutions. Illustrative numerical examples are given to demonstrate the efficiency and simplicity of the proposed method. Finally, several numerical examples are given to illustrate the accuracy and stability of this method. Comparison of the approximate solution with the exact solutions shows that the proposed method is very efficient and computationally attractive. We can use this method for solving more complicated integral equations in mathematical physical.

1. Introduction

An integral equation is defined as an equation in which the unknown function $y(x)$ to be determined appears under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. Abel's equation is one of the integral equations derived directly from a concrete problem of physics, without passing through a differential equation. This integral equation occurs in the mathematical modeling of several models in physics, astrophysics, solid mechanics, and applied sciences. The great mathematician Niels Abel gave the initiative of integral equations in 1823 in his study of mathematical physics [1–4]. In 1924, generalized Abel's integral equation on a finite segment was studied by Zeilon [5]. The different types of Abel integral equation in physics have been solved by Pandey et al. [6], Kumar and Singh [7], Kumar et al. [8], Dixit et al. [9], Yousefi [10], Khan and Gondal [11], and Li and Zhao [12] by applying various kinds of analytical and numerical methods.

The development of science has led to the formation of many physical laws, which, when restated in mathematical form, often appear as differential equations. Engineering problems can be mathematically described by differential equations, and thus differential equations play very important roles in the solution of practical problems. For example, Newton's law, stating that the rate of change of the momentum of a particle is equal to the force acting on it, can be translated into mathematical language as a differential equation. Similarly, problems arising in electric circuits, chemical kinetics, and transfer of heat in a medium can all be represented mathematically as differential equations.

The main aim of this paper is to present analytical and approximate solution of integral equations by using new mathematical tool like optimal homotopy analysis transform method. The proposed method is coupling of the homotopy analysis method HAM and Laplace transform method. The HAM, first proposed in 1992 by Liao, has been successfully applied to solve many problems in physics and science [13–18]. In recent years many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods of approximate solutions such as HAM, HPM, and ADM combined with the Laplace transform [19–27].

A typical form of an integral equation in $y(x)$ is of the form

$$\mathbf{y}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \boldsymbol{\lambda} \int_{\alpha(x)}^{\beta(x)} \mathbf{K}(\mathbf{x}, \mathbf{t}) \mathbf{y}(\mathbf{t}) \, \mathbf{dt}, \tag{1}$$

where $K(x, t)$ is called the kernel of integral equation (1) and $\alpha(x)$ and $\beta(x)$ are the limits of integration. It can be easily observed that the unknown function $y(x)$ appears under the integral sign. It is to be noted here that both the kernel $K(x, t)$ and the function $f(x)$ in (1) are given functions; and λ is a constant parameter. The prime objective of this text is to determine the unknown function $y(x)$ that will satisfy (1) using a number of solution techniques. We will devote considerable efforts in exploring these methods to find solutions of the unknown function, so that we can deduce the exact solution of the integral equation (1).

2. Basic Idea of Optimal Homotopy Analysis Transform Method

In order to elucidate the solution procedure of the optimal homotopy analysis transform method, we consider the following integral equations of second kind:

$$\mathbf{y}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \int_{\mathbf{0}}^{\mathbf{x}} \mathbf{K}(\mathbf{x}, \mathbf{t}) \mathbf{y}(\mathbf{t}) \, \mathbf{dt}, \quad \mathbf{0} \le \mathbf{x} \le \mathbf{1}. \tag{2}$$

Now operating the Laplace transform on both sides in (2), we get

$$\mathbf{L}[\mathbf{y}(\mathbf{x})] = \mathbf{L}[\mathbf{f}(\mathbf{x})] + \mathbf{L}\left\{\int_{\mathbf{0}}^{\mathbf{x}} \mathbf{K}(\mathbf{x}, \mathbf{t}) \mathbf{y}(\mathbf{t}) \, \mathbf{dt}\right\}. \tag{3}$$

We define the nonlinear operator

$$\begin{aligned} \mathbf{N}[\boldsymbol{\phi}(\mathbf{x}; \mathbf{q})] &= \mathbf{L}[\boldsymbol{\phi}(\mathbf{x}; \mathbf{q})] - \mathbf{L}[\mathbf{f}(\mathbf{x})] \\ &\quad - \mathbf{L}\left\{\int_{\mathbf{0}}^{\mathbf{x}} \mathbf{K}(\mathbf{x}, \mathbf{t}) \boldsymbol{\phi}(\mathbf{x}; \mathbf{q}) \, \mathbf{dt}\right\}, \end{aligned} \tag{4}$$

where $q \in [0, 1]$ is an embedding parameter and $\boldsymbol{\phi}(\mathbf{x}; \mathbf{q})$ is the real function of x and q. By means of generalizing the traditional homotopy methods, the great mathematician Liao [13, 14] constructed the zero order deformation equation

$$(\mathbf{1} - \mathbf{q}) \mathbf{L}[\boldsymbol{\phi}(\mathbf{x}; \mathbf{q}) - \mathbf{y}_0(x)] = \hbar \mathbf{q} \mathbf{H}(x) \mathbf{N}[\boldsymbol{\phi}(\mathbf{x}; \mathbf{q})], \tag{5}$$

where is a nonzero auxiliary parameter, $\mathbf{H}(x) \neq 0$ is an auxiliary function, $\mathbf{y_0}(\mathbf{x})$ is an initial guess of $\mathbf{y}(\mathbf{x})$, and $\boldsymbol{\phi}(\mathbf{x}; \mathbf{q})$ is an unknown function. It is important that one has great freedom to choose auxiliary thing in OHATM. Obviously, when $q = 0$ and $q = 1$, it holds

$$\boldsymbol{\phi}(\mathbf{x}; \mathbf{0}) = \mathbf{y}_0(x), \qquad \boldsymbol{\phi}(\mathbf{x}; \mathbf{1}) = \mathbf{y}(x), \tag{6}$$

respectively. Thus, as q increases from 0 to 1, the solution varies from the initial guess to the solution. Expanding $\phi(x; q)$ in Taylor's series with respect to q, we have

$$\boldsymbol{\phi}(\mathbf{x}; \mathbf{q}) = \mathbf{y}_0(\mathbf{x}, \mathbf{t}) + \sum_{\mathbf{m}=\mathbf{1}}^{\infty} \mathbf{q}^m \mathbf{y}_m(\mathbf{x}), \tag{7}$$

where

$$\mathbf{y}_m(\mathbf{x}) = \frac{1}{m!} \left.\frac{\partial^m \boldsymbol{\phi}(\mathbf{x}; \mathbf{q})}{\partial q^m}\right|_{q=0}. \tag{8}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, series (7) converges at $q = 1$ and we have

$$\mathbf{y}(\mathbf{x}) = \mathbf{y}_0(\mathbf{x}) + \sum_{\mathbf{m}=\mathbf{1}}^{\infty} \mathbf{y}_m(\mathbf{x}), \tag{9}$$

which must be one of the solutions of the original integral equations. Define the vectors

$$\vec{y}_n = \{\mathbf{y}_0(x), \mathbf{y}_1(x), \ldots, \mathbf{y}_n(x)\}. \tag{10}$$

Differentiating (6) m-times with respect to the embedding parameter q, then setting $q = 0$, and finally dividing them by $m!$, we obtain the mth order deformation equation

$$\mathbf{L}[\mathbf{y}_m(\mathbf{x}) - \boldsymbol{\chi}_m \mathbf{y}_{m-1}(\mathbf{x})] = \hbar \mathbf{q} \mathbf{H}(\mathbf{x}) \mathbf{R}_m(\vec{\mathbf{y}}_{m-1}, x), \tag{11}$$

where

$$\begin{gathered} \mathbf{R}_m(\vec{\mathbf{y}}_{m-1}, x) = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1} \phi(x; q)}{\partial q^{m-1}}\right|_{\mathbf{q}=\mathbf{0}}, \\ \boldsymbol{\chi}_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases} \end{gathered} \tag{12}$$

In this way, it is easy to obtain $\mathbf{y}_m(\mathbf{x})$ for $m \ge 1$, at mth order; we have

$$\mathbf{y}(\mathbf{x}) = \sum_{\mathbf{m}=\mathbf{0}}^{M} \mathbf{y}_m(\mathbf{x}). \tag{13}$$

When $M \to \infty$ we get an accurate approximation of original equation (2).

Many recently references such as [28–30] have applied the homotopy analysis method to nonlinear ODEs and discussed the optimization method to find out the optimal convergence control parameters by minimum of the square residual error integrated in the whole region. Their approach is based on the square residual error. Let $\Delta(h)$ denote the square residual error of governing equation (2) and it is expressed as

$$\Delta(h) = \int_{\Omega} (N[\widetilde{u_n}(t)])^2 \, d\Omega, \tag{14}$$

where

$$\widetilde{u_m}(t) = u_0(t) + \sum_{k=1}^{m} u_k(t) \tag{15}$$

and the optimal value of h is given by a nonlinear algebraic equation as

$$\frac{d\Delta(h)}{dh} = 0. \tag{16}$$

3. Numerical Results

In this section, we discuss the implementation of our proposed algorithm and investigate its accuracy by applying the homotopy analysis transform method. The simplicity and accuracy of the proposed method are illustrated through the following numerical examples by computing the absolute error:

$$E_i(x) = \rceil\, u_i(x) - \widetilde{u_{im}}(x) \,\rceil, \quad 1 \le i \le n, \tag{17}$$

where $u_i(x)$ is the exact solution of the integral equations and $\widetilde{u_{im}}(x)$ is the approximate solution of the integral equations.

To demonstrate the effectiveness of the HATM algorithm, we have discussed above, several examples for some integral equations in this section. Here all the results are calculated by using the symbolic calculus software Mathematica 7.

Example 1. Consider the following system of Abel's integral equations [31]:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) + \int_{\mathbf{0}}^{\mathbf{x}} \frac{\mathbf{v}(\mathbf{t})}{\sqrt{\mathbf{x}-\mathbf{t}}}\mathbf{dt} &= x + \frac{\boldsymbol{\pi}}{2}x, \quad 0 \le x \le 1, \\ \mathbf{v}(\mathbf{x}) + \frac{\mathbf{1}}{\mathbf{2}}\int_{\mathbf{0}}^{\mathbf{x}} \frac{\mathbf{u}(\mathbf{t})+\mathbf{v}(\mathbf{t})}{\sqrt{\mathbf{x}-\mathbf{t}}}\mathbf{dt} &= \sqrt{\mathbf{x}} + \frac{2}{3}x^{3/2} + \frac{\boldsymbol{\pi}}{4}x, \end{aligned} \tag{18}$$

with the initial condition

$$\begin{aligned} \mathbf{u}(\mathbf{x},\mathbf{0}) &= \mathbf{x} + \frac{\boldsymbol{\pi}}{\mathbf{2}}\mathbf{x}, \\ \mathbf{v}(\mathbf{x},\mathbf{0}) &= \sqrt{\mathbf{x}} + \frac{2}{3}x^{3/2} + \frac{\boldsymbol{\pi}}{4}x \end{aligned} \tag{19}$$

and the exact solution

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{x}, \\ \mathbf{v}(\mathbf{x}) &= \sqrt{\mathbf{x}}, \end{aligned} \tag{20}$$

where

$$\mathscr{L}[\phi(x;q)] = L[\phi(x;q)], \tag{21}$$

with the property that

$$\mathscr{L}[c] = 0, \quad c \text{ is constants}, \tag{22}$$

which implies that

$$\mathscr{L}^{-1}(\bullet) = \int_0^t (\bullet)\,dt. \tag{23}$$

Taking Laplace transform of (18) of both sides subject to the initial condition, we get

$$\begin{aligned} &\mathbf{L}[\mathbf{u}(\mathbf{x})] - \mathbf{L}\left[\mathbf{x} + \frac{\boldsymbol{\pi}}{\mathbf{2}}\mathbf{x}\right] + \sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\mathbf{v}(\mathbf{x})]) = \mathbf{0}, \\ &\mathbf{L}[\mathbf{v}(\mathbf{x})] - \mathbf{L}\left[\sqrt{\mathbf{x}} + \frac{\mathbf{2}}{\mathbf{3}}\mathbf{x}^{\mathbf{3/2}} + \frac{\boldsymbol{\pi}}{\mathbf{4}}\mathbf{x}\right] \\ &\quad + \frac{1}{2}\sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\mathbf{u}(\mathbf{x})] + \mathbf{L}[\mathbf{v}(\mathbf{x})]) = \mathbf{0}. \end{aligned} \tag{24}$$

We now define the nonlinear operator as

$$\begin{aligned} &\mathbf{N}[\boldsymbol{\phi}_1(\mathbf{x};\mathbf{q}), \boldsymbol{\phi}_2(\mathbf{x};\mathbf{q})] \\ &\quad = \mathbf{L}[\boldsymbol{\phi}_1(\mathbf{x};\mathbf{q})] - \mathbf{L}\left[\mathbf{x} + \frac{\boldsymbol{\pi}}{\mathbf{2}}\mathbf{x}\right] + \sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}\mathbf{L}[\boldsymbol{\phi}_2(\mathbf{x};\mathbf{q})] = \mathbf{0}, \\ &\mathbf{N}[\boldsymbol{\phi}_1(\mathbf{x};\mathbf{q}), \boldsymbol{\phi}_2(\mathbf{x};\mathbf{q})] \\ &\quad = \mathbf{L}[\boldsymbol{\phi}_2(\mathbf{x};\mathbf{q})] - \mathbf{L}\left[\sqrt{\mathbf{x}} + \frac{\mathbf{2}}{\mathbf{3}}\mathbf{x}^{\mathbf{3/2}} + \frac{\boldsymbol{\pi}}{\mathbf{4}}\mathbf{x}\right] \\ &\qquad + \frac{\mathbf{1}}{\mathbf{2}}\sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}\mathbf{L}[\boldsymbol{\phi}_1(\mathbf{x};\mathbf{q}) + \boldsymbol{\phi}_2(\mathbf{x};\mathbf{q})] = \mathbf{0}, \end{aligned} \tag{25}$$

and then the mth order deformation equation is given by

$$\begin{aligned} \mathbf{L}[\mathbf{u}_m(\mathbf{x}) - \chi_m\mathbf{u}_{m-1}(\mathbf{x})] &= \hbar_1\mathbf{H}_1(\mathbf{x})\mathbf{R}_{1m}(\vec{u}_{m-1}), \\ \mathbf{L}[\mathbf{v}_m(\mathbf{x}) - \chi_m\mathbf{v}_{m-1}(\mathbf{x})] &= \hbar_2\mathbf{H}_2(\mathbf{x})\mathbf{R}_{2m}(\vec{v}_{m-1}). \end{aligned} \tag{26}$$

Taking inverse Laplace transform of (26), we get

$$\begin{aligned} \mathbf{u}_m(\mathbf{x}) &= \chi_m\mathbf{u}_{m-1} + \hbar_1\mathbf{L}^{-1}[\mathbf{H_1}(\mathbf{x})\mathbf{R_{1m}}(\vec{u}_{\mathbf{m-1}})], \\ \mathbf{v}_m(\mathbf{x}) &= \chi_m\mathbf{v}_{m-1} + \hbar_2\mathbf{L}^{-1}[\mathbf{H_1}(\mathbf{x})\mathbf{R_{2m}}(\vec{v}_{\mathbf{m-1}})], \end{aligned} \tag{27}$$

where

$$\begin{aligned} \mathbf{R}_{1m}(\vec{u}_{m-1}) &= \mathbf{L}[\mathbf{u_{m-1}}] - \mathbf{L}\left[\mathbf{x} + \frac{\boldsymbol{\pi}}{\mathbf{2}}\mathbf{x}\right](1-\chi_m) \\ &\quad + \sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\mathbf{v}_{m-1}]), \\ \mathbf{R}_{2m}(\vec{v}_{m-1}) &= \mathbf{L}[\mathbf{v_{m-1}}] - \mathbf{L}\left[\sqrt{\mathbf{x}} + \frac{\mathbf{2}}{\mathbf{3}}\mathbf{x}^{\mathbf{3/2}} + \frac{\boldsymbol{\pi}}{\mathbf{4}}\mathbf{x}\right](1-\chi_m) \\ &\quad + \frac{\mathbf{1}}{\mathbf{2}}\sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\mathbf{u_{m-1}}] + \mathbf{L}[\mathbf{v}_{m-1}]), \end{aligned} \tag{28}$$

with the assumption $\mathbf{H}_1(\mathbf{x}) = \mathbf{H}_2(\mathbf{x}) = \mathbf{1}$.

Let us take the initial approximation as

$$\begin{aligned} \mathbf{u}_0(x) &= \mathbf{x} + \frac{\boldsymbol{\pi}}{\mathbf{2}}\mathbf{x}, \\ \mathbf{v}_0(x) &= \sqrt{\mathbf{x}} + \frac{\mathbf{2}}{\mathbf{3}}\mathbf{x}^{\mathbf{3/2}} + \frac{\boldsymbol{\pi}}{\mathbf{4}}\mathbf{x}. \end{aligned} \tag{29}$$

The other components are given by

$$\mathbf{u}_1(x) = \frac{h\boldsymbol{\pi}\mathbf{x}}{2} + \frac{1}{3}h\boldsymbol{\pi}\mathbf{x}^{3/2} + \frac{1}{4}h\boldsymbol{\pi}\mathbf{x}^2,$$

$$\mathbf{v}_1(x) = \frac{h\boldsymbol{\pi}\mathbf{x}}{4} + \frac{1}{6}h(4+3\boldsymbol{\pi})\mathbf{x}^{3/2} + \frac{1}{8}h\boldsymbol{\pi}\mathbf{x}^2,$$

$$\mathbf{u}_2(x) = \frac{1}{2}h(1+h)\boldsymbol{\pi}\mathbf{x} + \frac{1}{3}h(1+2h)\boldsymbol{\pi}\mathbf{x}^{3/2} + \frac{1}{16}h\boldsymbol{\pi}(\mathbf{4}+h(8+3\boldsymbol{\pi}))\mathbf{x}^2 + \frac{2}{15}h^2\boldsymbol{\pi}\mathbf{x}^{5/2},$$

$$\mathbf{v}_2(x) = \frac{1}{4}h(1+h)\boldsymbol{\pi}\mathbf{x} + \frac{1}{6}h(4+3\boldsymbol{\pi}+h(4+6\boldsymbol{\pi}))\mathbf{x}^{3/2} + \frac{1}{32}h\boldsymbol{\pi}(4+h(8+5\boldsymbol{\pi}))\mathbf{x}^2 + \frac{1}{5}h^2\boldsymbol{\pi}\mathbf{x}^{5/2},$$

$$\vdots$$

$$\vdots \tag{30}$$

Proceeding in this manner, the rest of the components $y_n(x)$ for $n \geq 5$ can be completely obtained and the series solutions are thus entirely determined. The solution of the problem is given as

$$\mathbf{y}(\mathbf{x}) = \mathbf{y}_0(x) + \sum_{\mathbf{m}=\mathbf{1}}^{\infty}\mathbf{y}_m(\mathbf{x}); \tag{31}$$

however, mostly, the results given by the Laplace decomposition method and homotopy analysis transform method converge to the corresponding numerical solutions in a rather small region. But, different from those two methods, the homotopy analysis transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary h; if we select $h = -1$, then

$$\begin{aligned}\mathbf{u}(\mathbf{x}) &= \mathbf{u}_0(x) + \sum_{\mathbf{m}=\mathbf{1}}^{\infty}\mathbf{u}_m(\mathbf{x}) \\ &= \sum_{\mathbf{i}=\mathbf{0}}^{n}\mathbf{y}_i(\mathbf{x}) + \mathbf{O}\left(\mathbf{x}^{3/2+n/2}\right) \\ &\longrightarrow x \quad \text{as } n \longrightarrow \infty,\ h=-1, \\ \mathbf{v}(\mathbf{x}) &= \mathbf{v}_0(x) + \sum_{\mathbf{m}=\mathbf{1}}^{\infty}\mathbf{v}_m(\mathbf{x}) \\ &= \sum_{\mathbf{i}=\mathbf{0}}^{n}\mathbf{v}_i(\mathbf{x}) + \mathbf{O}\left(\mathbf{x}^{3/2+n/2}\right) \\ &\longrightarrow \sqrt{\mathbf{x}} \quad \text{as } n \longrightarrow \infty,\ h=-1.\end{aligned} \tag{32}$$

The above result is in complete agreement with [31].

The graphical comparison between the exact solution and the approximate solution which obtained by the HATM at $h_{\text{optimal}} = -0.98$. It can be seen that the solution obtained by the present method nearly identical to the exact solution. The above result is in complete agreement with [31].

Example 2. In this example, we considered the following system of Abel integral equations of second kind as

$$\mathbf{u}(\mathbf{x}) + \frac{1}{4}\int_{\mathbf{0}}^{\mathbf{x}}\frac{\mathbf{v}(\mathbf{t})-\mathbf{u}(\mathbf{t})}{\sqrt{\mathbf{x}-\mathbf{t}}}\mathbf{dt} = \sqrt{x} + \frac{3\pi}{32}x^2 - \frac{\pi}{8}x, \quad 0 \leq x \leq 1, \tag{33}$$

$$\mathbf{v}(\mathbf{x}) + \mathbf{2}\int_{\mathbf{0}}^{\mathbf{x}}\frac{\mathbf{u}(\mathbf{t})}{\sqrt{\mathbf{x}-\mathbf{t}}}\mathbf{dt} = x^{3/2} + \boldsymbol{\pi}x,$$

with the initial condition

$$\mathbf{u}_0(x) = \sqrt{\mathbf{x}} + \frac{\mathbf{3}\boldsymbol{\pi}}{\mathbf{32}}\mathbf{x}^{\mathbf{2}} - \frac{\boldsymbol{\pi}}{\mathbf{8}}\mathbf{x}, \qquad \mathbf{v}_0(x) = \mathbf{x}^{\mathbf{3/2}} + \boldsymbol{\pi}\mathbf{x}, \tag{34}$$

with the exact solution

$$\mathbf{u}(\mathbf{x}) = \sqrt{\mathbf{x}}, \qquad \mathbf{v}(\mathbf{x}) = \mathbf{x}^{3/2}, \tag{35}$$

where

$$\mathscr{L}[\phi(x;q)] = L[\phi(x;q)], \tag{36}$$

with the property that

$$\mathscr{L}[c] = 0, \quad c \text{ is constants}, \tag{37}$$

which implies that

$$\mathscr{L}^{-1}(\bullet) = \int_0^t(\bullet)\,dt. \tag{38}$$

Taking Laplace transform of (33) of both sides subject to the initial condition, we get

$$\mathbf{L}[\mathbf{u}(\mathbf{x})] - \mathbf{L}\left[\sqrt{\mathbf{x}} + \frac{\mathbf{3}\boldsymbol{\pi}}{\mathbf{32}}\mathbf{x}^{\mathbf{2}} - \frac{\boldsymbol{\pi}}{\mathbf{8}}\mathbf{x}\right] + \frac{1}{4}\sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\mathbf{v}(\mathbf{x})] - \mathbf{L}[\mathbf{y}(\mathbf{x})]) = \mathbf{0}, \tag{39}$$

$$\mathbf{L}[\mathbf{v}(\mathbf{x})] - \mathbf{L}\left[\mathbf{x}^{\mathbf{3/2}} + \boldsymbol{\pi}\mathbf{x}\right] + 2\sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\mathbf{u}(\mathbf{x})]) = \mathbf{0}.$$

We now define the nonlinear operator as

$$\begin{aligned}&\mathbf{N}[\boldsymbol{\phi}_1(\mathbf{x};\mathbf{q}), \boldsymbol{\phi}_2(\mathbf{x};\mathbf{q})] \\ &\quad = \mathbf{L}[\boldsymbol{\phi}_1(\mathbf{x};\mathbf{q})] - \mathbf{L}\left[\sqrt{\mathbf{x}} + \frac{\mathbf{3}\boldsymbol{\pi}}{\mathbf{32}}\mathbf{x}^{\mathbf{2}} - \frac{\boldsymbol{\pi}}{\mathbf{8}}\mathbf{x}\right] + \frac{\mathbf{1}}{\mathbf{4}}\sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}(\mathbf{L}[\boldsymbol{\phi}_2(\mathbf{x};\mathbf{q})] - \mathbf{L}[\boldsymbol{\phi}_1(\mathbf{x};\mathbf{q})]) = \mathbf{0}, \\ &\mathbf{N}[\boldsymbol{\phi}_1(\mathbf{x};\mathbf{q}), \boldsymbol{\phi}_2(\mathbf{x};\mathbf{q})] \\ &\quad = \mathbf{L}[\boldsymbol{\phi}_2(\mathbf{x};\mathbf{q})] - \mathbf{L}\left[\mathbf{x}^{\mathbf{3/2}} + \boldsymbol{\pi}\mathbf{x}\right] + \mathbf{2}\sqrt{\frac{\boldsymbol{\pi}}{\mathbf{s}}}\mathbf{L}[\boldsymbol{\phi}_1(\mathbf{x};\mathbf{q})] = \mathbf{0},\end{aligned} \tag{40}$$

and then the mth order deformation equation is given by

$$\mathbf{L}\left[\mathbf{u}_m(\mathbf{x})-\chi_m\mathbf{u}_{m-1}(\mathbf{x})\right]=\hbar_1\mathbf{H}_1(\mathbf{x})\mathbf{R}_{1m}(\vec{u}_{m-1}),$$
$$\mathbf{L}\left[\mathbf{v}_m(\mathbf{x})-\chi_m\mathbf{v}_{m-1}(\mathbf{x})\right]=\hbar_2\mathbf{H}_2(\mathbf{x})\mathbf{R}_{2m}(\vec{v}_{m-1}). \quad (41)$$

Taking inverse Laplace transform of (41), we get

$$\mathbf{u}_m(\mathbf{x})=\chi_m\mathbf{u}_{m-1}+\hbar_1\mathbf{L}^{-1}\left[\mathbf{H_1}(\mathbf{x})\mathbf{R_{1m}}(\vec{u}_{\mathbf{m-1}})\right],$$
$$\mathbf{v}_m(\mathbf{x})=\chi_m\mathbf{v}_{m-1}+\hbar_2\mathbf{L}^{-1}\left[\mathbf{H_1}(\mathbf{x})\mathbf{R_{2m}}(\vec{v}_{\mathbf{m-1}})\right], \quad (42)$$

where

$$\mathbf{R}_{1m}(\vec{u}_{m-1})=\mathbf{L}\left[\mathbf{u_{m-1}}\right]-\mathbf{L}\left[\sqrt{\mathbf{x}}+\frac{3\pi}{32}\mathbf{x}^2-\frac{\pi}{8}\mathbf{x}\right](1-\chi_m)+\sqrt{\frac{\pi}{\mathbf{s}}}\left(\mathbf{L}\left[\mathbf{v}_{m-1}\right]\right),$$
$$\mathbf{R}_{2m}(\vec{v}_{m-1})=\mathbf{L}\left[\mathbf{v_{m-1}}\right]-\mathbf{L}\left[\mathbf{x}^{3/2}+\pi\mathbf{x}\right](1-\chi_m)+\frac{1}{2}\sqrt{\frac{\pi}{\mathbf{s}}}\left(\mathbf{L}\left[\mathbf{u_{m-1}}\right]+\mathbf{L}\left[\mathbf{v}_{m-1}\right]\right), \quad (43)$$

with the assumption $\mathbf{H}_1(\mathbf{x})=\mathbf{H_2}(\mathbf{x})=\mathbf{1}$.

Let us take the initial approximation as

$$\mathbf{u}_0(x)=\sqrt{\mathbf{x}}+\frac{3\pi}{32}\mathbf{x}^2-\frac{\pi}{8}\mathbf{x},$$
$$\mathbf{v}_0(x)=\mathbf{x}^{3/2}+\pi\mathbf{x}. \quad (44)$$

The other components are given by

$$\mathbf{u}_1(x)=-\frac{1}{8}h\pi\mathbf{x}+\frac{3}{8}h\pi\mathbf{x}^{3/2}+\frac{3}{32}h\pi\mathbf{x}^2-\frac{1}{40}h\pi\mathbf{x}^{5/2},$$
$$\mathbf{v}_1(x)=h\pi\mathbf{x}-\frac{1}{3}h\pi\mathbf{x}^{3/2}+\frac{1}{5}h\pi\mathbf{x}^{5/2},$$
$$\mathbf{u}_2(x)=-\frac{1}{8}h(1+h)\pi\mathbf{x}+\frac{3}{8}h(1+2h)\pi\mathbf{x}^{3/2}+\frac{1}{256}h(24+h(24-17\pi))\pi\mathbf{x}^2-\frac{1}{40}h(1+2h)\pi\mathbf{x}^{5/2}+\frac{9}{512}h^2\pi^2x^3,$$
$$\mathbf{v}_2(x)=h(1+h)\pi\mathbf{x}-\frac{1}{3}h(1+2h)\pi\mathbf{x}^{3/2}+\frac{9}{32}h^2\pi^2x^2+\frac{1}{5}h(1+2h)\pi\mathbf{x}^{5/2}-\frac{1}{64}h^2\pi^2x^3,$$
$$\vdots$$
$$\vdots \quad (45)$$

Table 1: The values of h.

The values of h-curve derived from Figures 2 and 3		
$u(x)$	$-1.15 \le h$	≤ -0.4
$v(x)$	$-1.11 \le h$	≤ -0.4

Table 2: The values of h.

The values of h-curve derived from Figures 5 and 6		
$u(x)$	$-1.2 \le h$	≤ -0.8
$v(x)$	$-1.1 \le h$	≤ -0.94

Hence the solution of (41) is given as

$$\mathbf{u}(\mathbf{x})=\mathbf{u}_0(x)+\sum_{\mathbf{m=1}}^{\infty}\mathbf{u}_m(\mathbf{x})=\sum_{\mathbf{i=0}}^{n}\mathbf{y}_i(\mathbf{x})+\mathbf{O}\left(\mathbf{x}^{2+n/2}\right)\longrightarrow\sqrt{\mathbf{x}}\quad\text{as } n\longrightarrow\infty,\ h=-1,$$
$$\mathbf{v}(\mathbf{x})=\mathbf{v}_0(x)+\sum_{\mathbf{m=1}}^{\infty}\mathbf{v}_m(\mathbf{x})=\sum_{\mathbf{i=0}}^{n}\mathbf{v}_i(\mathbf{x})+\mathbf{O}\left(\mathbf{x}^{2+n/2}\right)\longrightarrow\mathbf{x}^{3/2}\quad\text{as } n\longrightarrow\infty,\ h=-1. \quad (46)$$

The homotopy analysis transform method provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter h; if we select $h=-1$, then the above result is in complete agreement with [31].

In general, by means of the so-called h-curve, it is straightforward to choose an appropriate range for h which ensures the convergence of the solution series. To study the influence of h on the convergence of solution, the h-curves of $u(0.1)$ and $v(0.1)$ are sketched, as shown in Figures 5 and 6 (see Table 1). For better presentation, these valid regions have been listed in Table 2. The absolute error $E(x)=u_{\text{exact}}-u_{\text{OHATM}}$ is exhibited in Figure 4 and also from Figures 1 to 6 we show the graphical comparison between the exact solution and the approximate solution obtained by the OHATM. It can be seen that the solution obtained by the present method nearly identical to the exact solution. The above result is in complete agreement with [31].

4. Conclusions

The main aim of this work is to provide the system of Abel integral equations of the second kind which has been studied by the optimal homotopy analysis transform method OHATM. The OHATM is more suitable than other analytic methods. OHATM is coupling of homotopy analysis and Laplace transform method. The new modification is a powerful tool to search for solutions of Abel's integral equation.

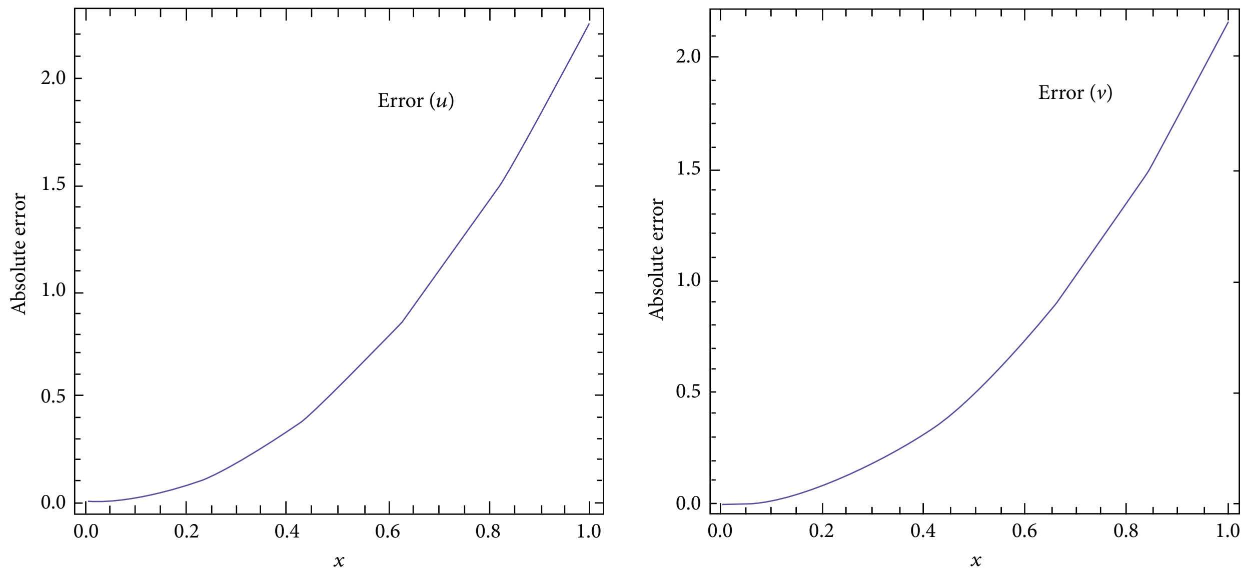

FIGURE 1: The absolute error between the exact solution and the approximate solution of Abel integral equation (18) at $h_{\text{optimal}} = -0.98$.

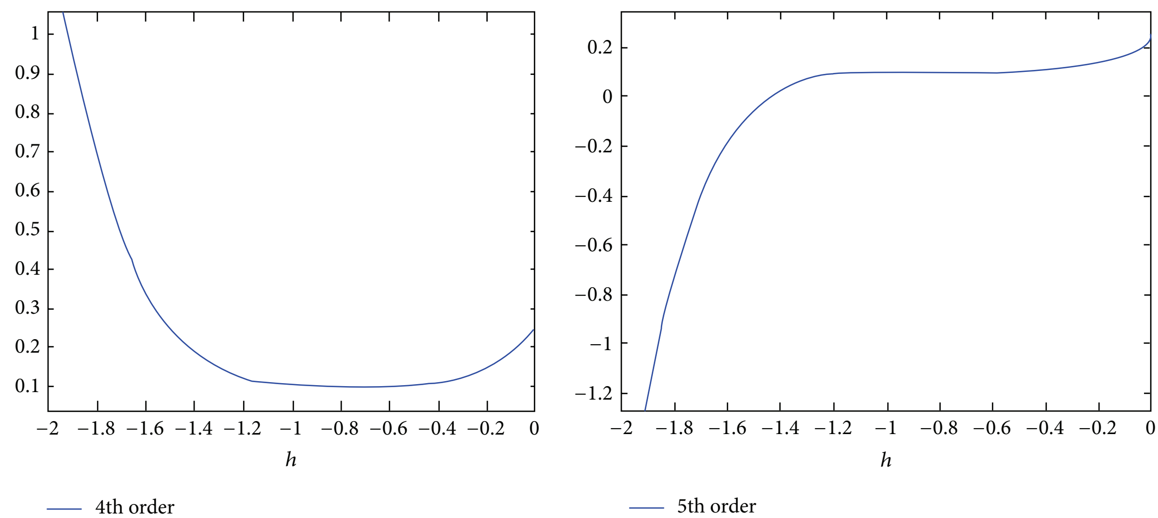

FIGURE 2: The HATM approximate solutions $u(x)$ with $x = 0.1$ lead to the h-curves.

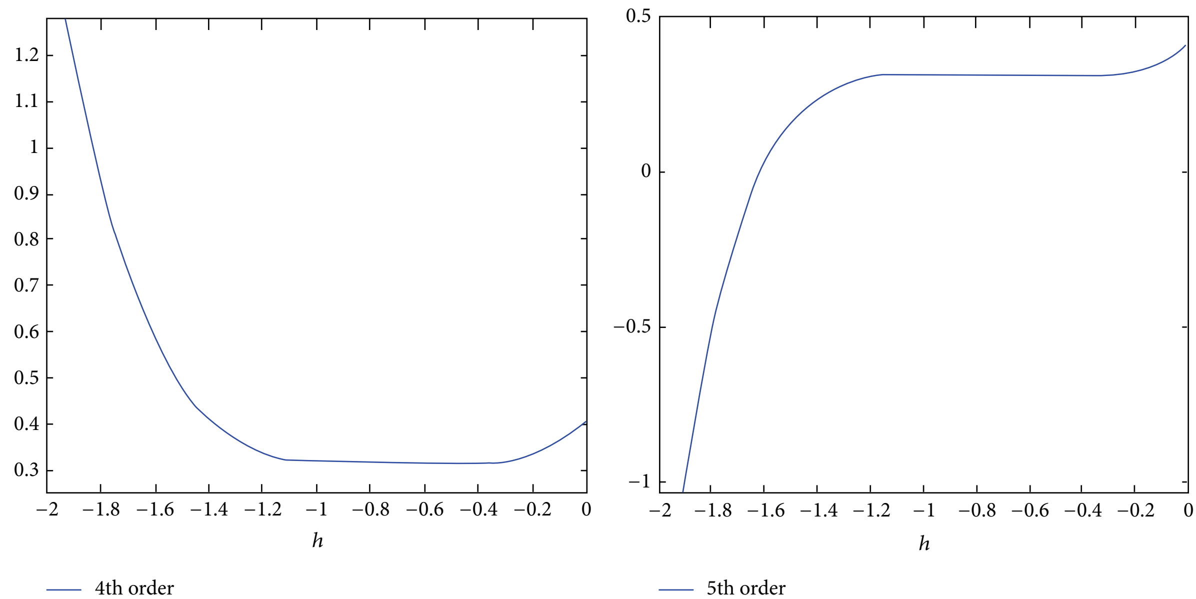

FIGURE 3: The HATM approximate solutions $v(x)$ with $x = 0.1$ lead to the h-curves.

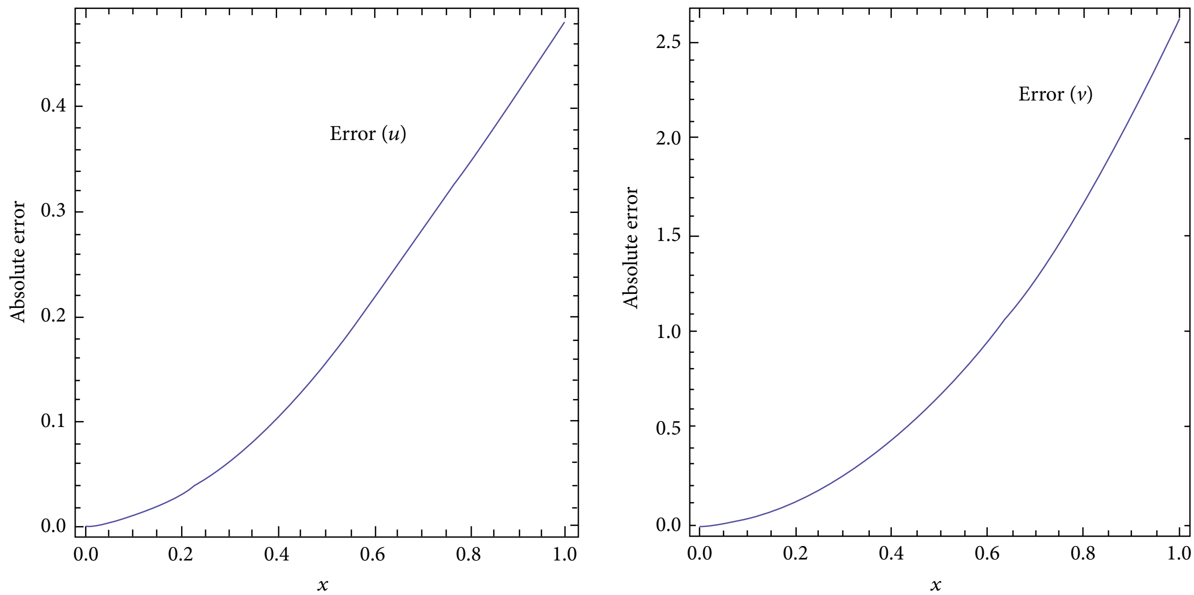

Figure 4: The absolute error between the exact solution and the approximate solution of Abel integral equation (33) at $h_{\text{optimal}} = -0.99$.

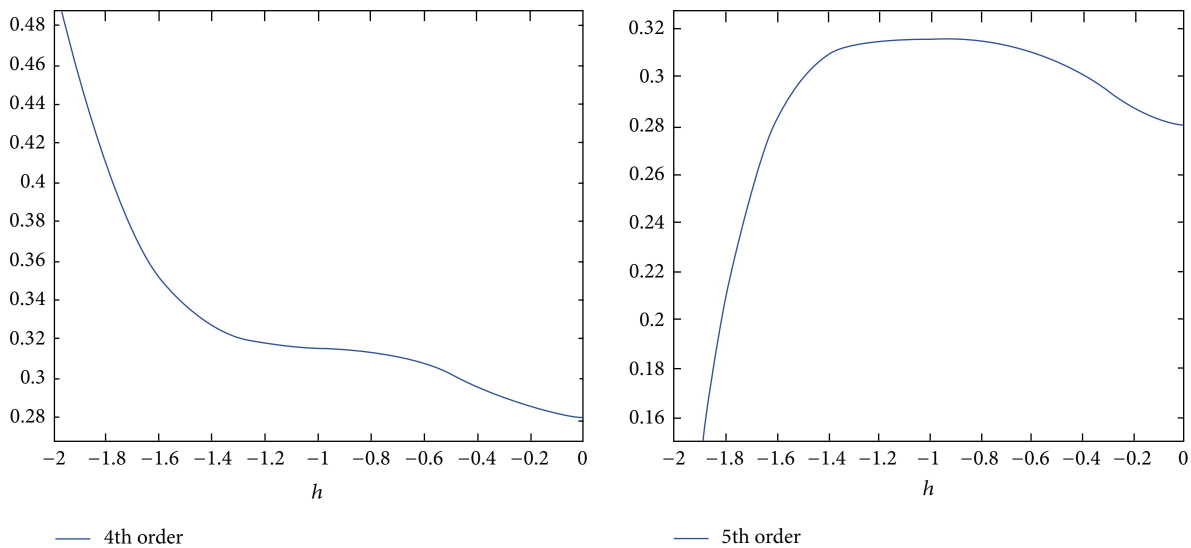

Figure 5: The curves are gotten from the HATM approximate solutions of $u(x)$ with $x = 0.1$.

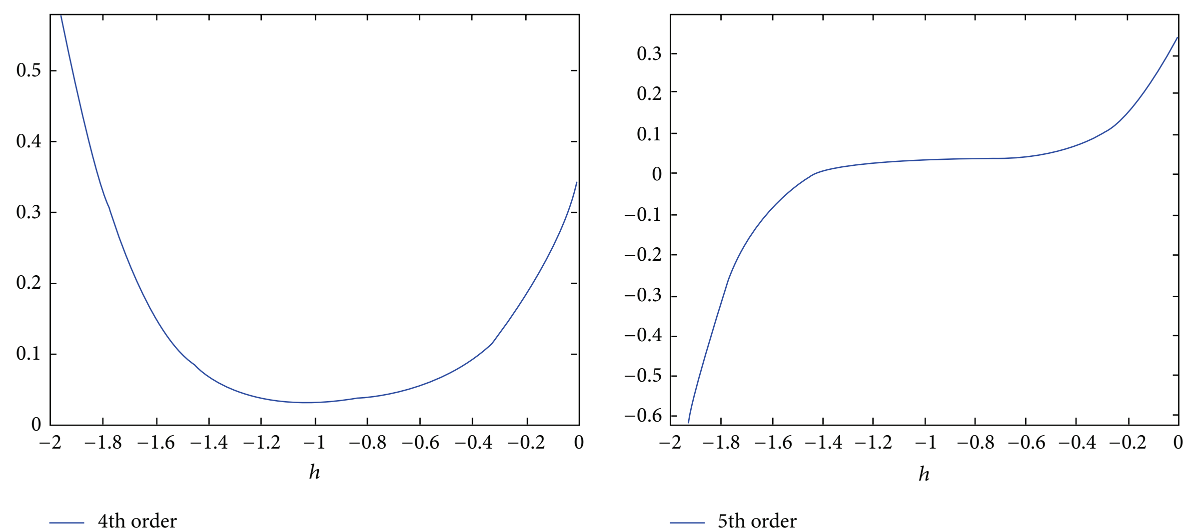

Figure 6: The h-curves are gotten from the HATM approximate solutions of $v(x)$ with $x = 0.1$.

An excellent agreement is achieved. The proposed method is employed without using linearization, discretization, or transformation. It may be concluded that the OHATM is very powerful and efficient in finding the analytical solutions for a wide class of differential and integral equations. The approximate solution of this system is calculated in this form of series whose components are computed by applying a recursive relation. Results indicate that the solution obtained by this method converges rapidly to an exact solution. The graphs plotted confirm the results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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