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*Research Article*

# VanderLaan Circulant Type Matrices

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Circulant matrices have become a satisfactory tools in control methods for modern complex systems. In the paper, VanderLaan circulant type matrices are presented, which include VanderLaan circulant, left circulant, and $g$-circulant matrices. The nonsingularity of these special matrices is discussed by the surprising properties of VanderLaan numbers. The exact determinants of VanderLaan circulant type matrices are given by structuring transformation matrices, determinants of well-known tridiagonal matrices, and tridiagonal-like matrices. The explicit inverse matrices of these special matrices are obtained by structuring transformation matrices, inverses of known tridiagonal matrices, and quasi-tridiagonal matrices. Three kinds of norms and lower bound for the spread of VanderLaan circulant and left circulant matrix are given separately. And we gain the spectral norm of VanderLaan $g$-circulant matrix.

## 1. Introduction

It is well known that the circulant matrices are one of the most important research tools in control methods for modern complex systems. There are many results on circulant systems. The concept of "chart" was used to investigate structural controllability and fixed models in [1] while more attention was paid to properties and controller design for a special class of circulant systems, called symmetrically circulant systems [2–5].

Systems with block symmetric circulant structure are a kind of complex systems and constitute an important class of large-scale systems where there may be a clear advantage of using multivariable control and where it is actually used in practice. Typical application examples can be found in the control of paper machine [6–8] and control of power systems with parallel structure [7]. Other examples, including multizone crystal growth furnaces and dyes for plastic films, were also listed in [7, 8] and the references therein. For those symmetric circulant composite systems, due to the high dimensionality of the overall system and information structural constraints, the control problem is more complex and few results on regional pole assignment are available in the literature. In [9], the authors considered the problem of placing the poles of uncertain symmetric circulant composite systems in a specified disk, which is also presented as the problem of quadratic D stabilisation. Lee et al. [10] presented linear quadratic (LQ) repetitive control (RC) methods for processes represented by a conventional FIR model and a circulant FIR model. The latter, which represents a FIR system under the assumption of a cyclic steady state, is named its input-output map. The map is represented by a circulant matrix. Using the complete frequency resolving property of a circulant matrix, a special tuning method for the LQ weights is proposed. Lee and Won considered properties of pulse response circulant matrix and applied that to MIMO control and identification in [11].

Furthermore, circulant type matrices have been put on the firm basis with the work in [12–18] and so on. These special matrices have significant applications in various disciplines.

The VanderLaan sequences are defined by the following recurrence relation [19]:

$$V_n = V_{n-2} + V_{n-3}, \tag{1}$$

where

$$V_0 = 0, \quad V_1 = 1, \quad V_2 = 0. \tag{2}$$

For the convenience of readers, we gave the first few values of the sequences as follows:

$$\begin{array}{c|cccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ \hline V_n & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & \cdots \end{array}. \quad (3)$$

The characteristic equation of VanderLaan numbers is $x^3 - x - 1 = 0$, and its roots are denoted by $r_1, r_2, r_3$; then

$$\begin{gathered} r_1 + r_2 + r_3 = 0, \\ r_1 r_2 + r_1 r_3 + r_2 r_3 = -1, \\ r_1 r_2 r_3 = 1. \end{gathered} \quad (4)$$

The Binet form for VanderLaan sequence is

$$V_n = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n, \quad (5)$$

where

$$\begin{aligned} c_1 &= \frac{r_2 + r_3}{(r_2 - r_1)(r_1 - r_3)}, \\ c_2 &= \frac{r_1 + r_3}{(r_2 - r_1)(r_3 - r_2)}, \\ c_3 &= \frac{r_1 + r_2}{(r_2 - r_3)(r_3 - r_1)}. \end{aligned} \quad (6)$$

Some authors have presented the explicit determinants and inverse of circulant type matrices involving famous numbers in recent years. For example, in [14], the nonsingularity of circulant type matrices with the sum and product of Fibonacci and Lucas numbers is discussed. And the exact determinants and inverses of these matrices are given. Determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers are given in [20]. Jiang et al. [15] studied circulant type matrices with the $k$-Fibonacci and $k$-Lucas numbers and obtained the explicit determinants and inverse matrices. Lin [21] proposed the determinants of the Fibonacci-Lucas quasi-cyclic matrices. In [22], Jiang et al. discussed the nonsingularity of the skew circulant type matrices and presented explicit determinants and inverse matrices of these special matrices. Furthermore, four kinds of norms and bounds for the spread of these matrices are given separately. In [23], Shen et al. discussed circulant matrices with Fibonacci and Lucas numbers and gave their explicit determinants and inverses. Authors [24] discussed the nonsingularity of the circulant matrix and presented the explicit determinant and inverse matrices. Moreover, the nonsingularity of the left circulant and $g$-circulant matrices is also studied. The explicit determinants and inverse matrices of the left circulant and $g$-circulant matrices are obtained by utilizing the relationship between left circulant and $g$-circulant matrices and circulant matrix, respectively.

This paper is aimed at getting the more beautiful results for the determinants and inverses of circulant type matrices via some surprising properties of VanderLaan numbers.

A VanderLaan circulant matrix is an $n \times n$ matrix of the following form:

$$\operatorname{Circ}(V_{r+1}, V_{r+2}, \ldots, V_{r+n}) = \begin{bmatrix} V_{r+1} & V_{r+2} & \cdots & V_{r+n} \\ V_{r+n} & V_{r+1} & \cdots & V_{r+n-1} \\ \vdots & \vdots & & \vdots \\ V_{r+2} & V_{r+3} & \cdots & V_{r+1} \end{bmatrix}. \quad (7)$$

A VanderLaan left circulant matrix is an $n \times n$ matrix of the following form:

$$L\operatorname{Circ}(V_{r+1}, V_{r+2}, \ldots, V_{r+n}) = \begin{bmatrix} V_{r+1} & V_{r+2} & \cdots & V_{r+n} \\ V_{r+2} & V_{r+3} & \cdots & V_{r+1} \\ \vdots & \vdots & & \vdots \\ V_{r+n} & V_{r+1} & \cdots & V_{r+n-1} \end{bmatrix}. \quad (8)$$

A VanderLaan $g$-circulant matrix is an $n \times n$ matrix with the following form:

$$A_{g,n} = \begin{pmatrix} V_{r+1} & V_{r+2} & \cdots & V_{r+n} \\ V_{r+n-g+1} & V_{r+n-g+2} & \cdots & V_{r+n-g} \\ V_{r+n-2g+1} & V_{r+n-2g+2} & \cdots & V_{r+n-2g} \\ \vdots & \vdots & \ddots & \vdots \\ V_{r+g+1} & V_{r+g+2} & \cdots & V_{r+g} \end{pmatrix}, \quad (9)$$

where $g$ is a nonnegative integer and each of the subscripts is understood to be reduced modulo $n$.

**Lemma 1.** *Define the $n \times n$ matrix by*

$$A_n = \begin{pmatrix} \kappa_1 & \kappa_2 & \cdots & 0 & 0 \\ \kappa_3 & \kappa_1 & \cdots & 0 & 0 \\ 0 & \kappa_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & \kappa_1 & \kappa_2 \\ 0 & 0 & \cdots & \kappa_3 & \kappa_1 \end{pmatrix}; \quad (10)$$

*then determinant of the matrix $A_n$ is*

$$\det A_n = \begin{cases} \left( \left( \dfrac{\kappa_1 + \sqrt{\kappa_1^2 - 4\kappa_2\kappa_3}}{2} \right)^{n+1} - \left( \dfrac{\kappa_1 - \sqrt{\kappa_1^2 - 4\kappa_2\kappa_3}}{2} \right)^{n+1} \right) \times \left( \sqrt{\kappa_1^2 - 4\kappa_2\kappa_3} \right)^{-1}, & \kappa_1^2 \neq 4\kappa_2\kappa_3, \\ (n+1)\left( \dfrac{\kappa_1}{2} \right)^n, & \kappa_1^2 = 4\kappa_2\kappa_3. \end{cases} \quad (11)$$

*Proof.* A calculation using the expansion of the last column for determinant of matrix $A_n$ shows that $\det A_n = \kappa_1 \cdot \det A_{n-1} - \kappa_2\kappa_3 \cdot \det A_{n-2}$; let $x + y = \kappa_1$, $xy = \kappa_2\kappa_3$; then let $x, y$ be the roots of the equation $x^2 - \kappa_1 x + \kappa_2\kappa_3 = 0$.

We have

$$\det A_n = y^n + xy^{n-1} + \cdots + x^{n-1}y + x^n = \begin{cases} \dfrac{x^{n+1} - y^{n+1}}{x - y}, & x \neq y, \\ (n+1)\,x^n, & x = y, \end{cases} \tag{12}$$

where

$$x = \frac{\kappa_1 + \sqrt{\kappa_1^2 - 4\kappa_2\kappa_3}}{2}, \qquad y = \frac{\kappa_1 - \sqrt{\kappa_1^2 - 4\kappa_2\kappa_3}}{2}. \tag{13}$$

We obtain

$$\det A_n = \begin{cases} \left(\left(\dfrac{\kappa_1 + \sqrt{\kappa_1^2 - 4\kappa_2\kappa_3}}{2}\right)^{n+1} - \left(\dfrac{\kappa_1 - \sqrt{\kappa_1^2 - 4\kappa_2\kappa_3}}{2}\right)^{n+1}\right) \times \left(\sqrt{\kappa_1^2 - 4\kappa_2\kappa_3}\right)^{-1}, & \kappa_1^2 \neq 4\kappa_2\kappa_3, \\ (n+1)\left(\dfrac{\kappa_1}{2}\right)^n, & \kappa_1^2 = 4\kappa_2\kappa_3. \end{cases} \tag{14}$$

□

**Lemma 2.** *Let*

$$B_n = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ \kappa_1 & \kappa_2 & \cdots & 0 & 0 \\ \kappa_3 & \kappa_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & \kappa_2 & 0 \\ 0 & 0 & \cdots & \kappa_1 & \kappa_2 \end{pmatrix} \tag{15}$$

*be a $n \times n$ matrix; then*

$$\det B_n = \sum_{i=1}^{n} (-1)^{1+i} \kappa_2^{n-i} a_i \cdot \det A_{i-1}, \tag{16}$$

*where, for $n \geq 3$,*

$$\det A_{i-1} = \begin{cases} \left(\left(\dfrac{\kappa_1 + \sqrt{\kappa_1^2 - 4\kappa_2\kappa_3}}{2}\right)^{i} - \left(\dfrac{\kappa_1 - \sqrt{\kappa_1^2 - 4\kappa_2\kappa_3}}{2}\right)^{i}\right) \times \left(\sqrt{\kappa_1^2 - 4\kappa_2\kappa_3}\right)^{-1}, & \kappa_1^2 \neq 4\kappa_2\kappa_3, \\ i\left(\dfrac{\kappa_1}{2}\right)^{i-1}, & \kappa_1^2 = 4\kappa_2\kappa_3, \end{cases} \tag{17}$$

*and* $\det A_0 = 1$.

*Proof.* Expanding the last column for the determinant of matrix $B_n$ and according to Lemma 1, we obtain

$$\begin{aligned} \det B_n &= \kappa_2 \cdot \det B_{n-1} + (-1)^{n+1} a_n \cdot \det A_{n-1} \\ &= \kappa_2^{n-1} a_1 + (-1)^3 \kappa_2^{n-2} a_2 \cdot \det A_1 \\ &\quad + \cdots + (-1)^{n+1} a_n \cdot \det A_{n-1} \\ &= \sum_{i=1}^{n} (-1)^{1+i} \kappa_2^{n-i} a_i \cdot \det A_{i-1}. \end{aligned} \tag{18}$$

This completes the proof. □

**Lemma 3.** *Let $\Phi = \left(\begin{smallmatrix} a & B \\ C & A \end{smallmatrix}\right)$ be a partitioned matrix; then*

$$\Phi^{-1} = \begin{pmatrix} \dfrac{1}{\ell} & -\dfrac{1}{\ell}BA^{-1} \\ -\dfrac{1}{\ell}A^{-1}C & A^{-1} + \dfrac{1}{\ell}A^{-1}CBA^{-1} \end{pmatrix}, \tag{19}$$

*where $\ell = a - BA^{-1}C$, $B$ is a row vector, and $C$ is a column vector.*

*Proof.* From direct calculation by matrix multiplication, we get

$$\Phi\Phi^{-1} = I_n, \qquad \Phi^{-1}\Phi = I_n, \tag{20}$$

where

$$\Phi = \begin{pmatrix} a & B \\ C & A \end{pmatrix}, \quad \Phi^{-1} = \begin{pmatrix} \dfrac{1}{\ell} & -\dfrac{1}{\ell}BA^{-1} \\ -\dfrac{1}{\ell}A^{-1}C & A^{-1} + \dfrac{1}{\ell}A^{-1}CBA^{-1} \end{pmatrix}. \tag{21}$$

□

**Lemma 4.** *Let the matrix $\mathcal{G} = [h_{i,j}]_{i,j=1}^{n-3}$ be of the form*

$$h_{i,j} = \begin{cases} c, & i = j, \\ b, & i = j+1, \\ d, & i = j+2, \\ 0, & \text{otherwise}; \end{cases} \tag{22}$$

*then the inverse $\mathcal{G}^{-1} = [h'_{i,j}]_{i,j=1}^{n-3}$ of the matrix $\mathcal{G}$ is equal to*

$$h'_{i,j} = \begin{cases} \dfrac{1}{c}\left(\dfrac{\beta^{i-j+1} - \alpha^{i-j+1}}{\beta - \alpha}\right), & i \geq j, \\ 0, & i < j, \end{cases} \tag{23}$$

*where*

$$
\begin{aligned}
c &= V_{r+1} - V_{r+n+1}, \\
b &= V_{r+2} - V_{r+n} - V_{r+n-1}, \\
d &= V_r - V_{r+n}, \\
\alpha &= \frac{-b + \sqrt{b^2 - 4cd}}{2c}, \\
\beta &= \frac{-b - \sqrt{b^2 - 4cd}}{2c}.
\end{aligned} \tag{24}
$$

*Proof.* Let $c'_{i,j} = \sum_{k=1}^{n-3} h_{i,k} h'_{k,j}$; if $i = j$, we have $c'_{i,i} = h_{i,i} h'_{i,i} = 1$; distinctly, $c'_{i,j} = 0$ for $i < j$. And, for $i \geq j + 1$, we have

$$
\begin{aligned}
c'_{i,j} &= h_{i,i-2} h'_{i-2,j} + h_{i,i-1} h'_{i-1,j} + h_{i,i} h'_{i,j} \\
&= \frac{d}{c} \left( \frac{\beta^{i-j-1} - \alpha^{i-j-1}}{\beta - \alpha} \right) + \frac{\beta^{i-j+1} - \alpha^{i-j+1}}{\beta - \alpha} \\
&\quad + \frac{b}{c} \left( \frac{\beta^{i-j} - \alpha^{i-j}}{\beta - \alpha} \right) \\
&= 0.
\end{aligned} \tag{25}
$$

We obtain $\mathscr{G}\mathscr{G}^{-1} = I_{n-3}$, where $I_{n-3}$ is $(n-3) \times (n-3)$ identity matrix. Evidenced by the same token, $\mathscr{G}^{-1}\mathscr{G} = I_{n-3}$. The proof is completed. □

We first introduce the following notations:

$$
\begin{aligned}
\nu &= e + \sum_{i=2}^{n-1} \delta_{i+1} \Delta^{n-i}, \\
\Delta &= \frac{-b \pm \sqrt{b^2 - 4cd}}{2c}, \\
e &= V_{r+1} - \frac{V_{r+2} V_{r+n}}{V_{r+1}}, \\
\delta_{i+1} &= V_{r+i+1} - \frac{V_{r+2} V_{r+i}}{V_{r+1}} \quad (i = 2, 3, \ldots, n-1), \\
\kappa &= \sum_{i=2}^{n-2} (-1)^{i+1} c^{n-i-2} \epsilon_{n-i} \phi_{i-1} + \epsilon_1 c^{n-3}, \\
\epsilon_1 &= V_{r+1} - \frac{V_{r+3} V_{r+n-1}}{V_{r+1}}, \\
\epsilon_i &= V_{r+i+2} - \frac{V_{r+3} V_{r+i}}{V_{r+1}} \quad (i = 2, 3, \ldots, n-2), \\
\phi_{i-1} &= \begin{cases} \left( \left( \dfrac{b + \sqrt{b^2 - 4cd}}{2} \right)^i - \left( \dfrac{b - \sqrt{b^2 - 4cd}}{2} \right)^i \right) \times \left( \sqrt{b^2 - 4cd} \right)^{-1}, & b^2 \neq 4cd, \\ i \left( \dfrac{b}{2} \right)^{i-1}, & b^2 = 4cd, \end{cases} \\
\theta &= f + \sum_{i=2}^{n-2} \epsilon_i \Delta^{n-i}, \\
f &= V_{r+2} - \frac{V_{r+3} V_{r+n}}{V_{r+1}} + \epsilon_1 \Delta, \\
\tau &= \sum_{i=1}^{n-2} (-1)^{i+1} c^{n-i-2} \delta_{n-i+1} \phi_{i-1}, \\
p &= V_{r+2} - V_{r+n+2}.
\end{aligned} \tag{26}
$$

## 2. Determinant and Inverse of VanderLaan Circulant Matrix

In this segment, let $\mathscr{V}_{r,n}$ be a VanderLaan circulant matrix. To begin with, we give the determinant of the matrix $\mathscr{V}_{r,n}$. And then we draw a conclusion that $\mathscr{V}_{r,n}$ is an invertible matrix for $n \neq 2k\pi / \arctan(\sqrt{4cd - p^2} / \pm p)$; finally we find the inverse of the matrix $\mathscr{V}_{r,n}$.

**Theorem 5.** *Let $\mathscr{V}_{r,n} = Circ(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan circulant matrix, so its determinant is*

$$
\det \mathscr{V}_{r,n} = V_{r+1} \left( \nu\kappa - \theta\tau \right), \tag{27}
$$

*where $V_{r+n}$ is the $(r+n)$th VanderLaan number.*

*Proof.* Apparently, $\det \mathscr{V}_{0,4} = 3$ meets formula (27). If $n > 4$, let

$$
\Gamma_1 = \begin{pmatrix}
1 & & & & & & \\
-\dfrac{V_{r+2}}{V_{r+1}} & & & & & & 1 \\
\dfrac{-V_{r+3}}{V_{r+1}} & & & 0 & & & 0 \\
-1 & & & & \iddots & & -1 \\
0 & & & & \iddots & & -1 \\
\vdots & & \iddots & \iddots & \iddots & \iddots & \vdots \\
0 & 1 & 0 & -1 & -1 & \cdots & 0
\end{pmatrix},
$$

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \Delta^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \Delta^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & \Delta & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{28}$$

be two $n \times n$ matrices; we have

$$\Gamma_1 \mathscr{V}_{r,n} \Pi_1 = \begin{pmatrix} V_{r+1} & \mu & V_{r+n-1} & \cdots & V_{r+2} \\ 0 & \nu & \delta_n & \cdots & \delta_3 \\ 0 & \theta & \epsilon_1 & \cdots & \epsilon_2 \\ 0 & 0 & b & \cdots & 0 \\ 0 & 0 & d & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \tag{29}$$

where

$$\mu = \sum_{i=2}^{n} V_{r+i} \Delta^{n-i}; \tag{30}$$

we obtain

$$\det \Gamma_1 \det \mathscr{V}_{r,n} \det \Pi_1 = V_{r+1} \cdot (\nu\kappa - \theta\tau). \tag{31}$$

Let

$$\mathscr{B}_{n-2} = \begin{pmatrix} \epsilon_1 & \epsilon_{n-2} & \epsilon_{n-3} & \cdots & \epsilon_2 \\ b & c & 0 & \cdots & 0 \\ d & b & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & c \end{pmatrix},$$
$$\mathscr{C}_{n-2} = \begin{pmatrix} \delta_n & \delta_{n-1} & \delta_{n-2} & \cdots & \delta_3 \\ b & c & 0 & \cdots & 0 \\ d & b & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & c \end{pmatrix} \tag{32}$$

be two $(n-2) \times (n-2)$ matrices, and

$$\kappa = \det \mathscr{B}_{n-2}, \qquad \tau = \det \mathscr{C}_{n-2}. \tag{33}$$

By Lemmas 1 and 2, the following equations hold:

$$\tau = \sum_{i=1}^{n-2} (-1)^{i+1} c^{n-i-2} \delta_{n-i+1} \phi_{i-1},$$
$$\kappa = \sum_{i=2}^{n-2} (-1)^{i+1} c^{n-i-2} \epsilon_{n-i} \phi_{i-1} + \epsilon_1 c^{n-3}; \tag{34}$$

while

$$\det \Gamma_1 = \det \Pi_1 = (-1)^{(n-1)(n-2)/2}, \tag{35}$$

we have

$$\det \mathscr{V}_{r,n} = V_{r+1} (\nu\kappa - \theta\tau), \tag{36}$$

which completes the proof. □

**Theorem 6.** *Let $\mathscr{V}_{r,n} = Circ(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan circulant matrix; in the case of $n \neq 2k\pi/\arctan(\sqrt{4cd-p^2}/\pm p)$, $\mathscr{V}_{r,n}$ is a nonsingular matrix.*

*Proof.* According to Theorem 5, in case of $n \leq 4$, we have $\det \mathscr{V}_{r,n} \neq 0$. When $n > 4$, since $V_n = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n$, where

$$c_1 = \frac{r_2 + r_3}{(r_2 - r_1)(r_1 - r_3)},$$
$$c_2 = \frac{r_1 + r_3}{(r_2 - r_1)(r_3 - r_2)}, \tag{37}$$
$$c_3 = \frac{r_1 + r_2}{(r_2 - r_3)(r_3 - r_1)},$$

we get

$$\begin{aligned} f(\omega^k) &= \sum_{j=1}^{n} V_{r+j} (\omega^k)^{j-1} \\ &= \sum_{j=1}^{n} \left(c_1 r_1^{r+j} + c_2 r_2^{r+j} + c_3 r_3^{r+j}\right)(\omega^k)^{j-1} \\ &= \frac{d\omega^{2k} + (V_{r+2} - V_{r+n+2})\omega^k + c}{\sigma}, \end{aligned} \tag{38}$$

where

$$\sigma = (1 - r_1\omega^k)(1 - r_2\omega^k)(1 - r_3\omega^k), \quad (k = 1, \ldots, n-1). \tag{39}$$

If there exists $\omega^l$ $(l = 1, 2, \ldots, n-1)$ showing that $f(\omega^l) = 0$, here we get $d\omega^{2k} + p\omega^k + c = 0$ for $(1-r_1\omega^k)(1-r_2\omega^k)(1-r_3\omega^k) \neq 0$. If $p^2 - 4cd \geq 0$, thus, $\omega^l = (-p \pm \sqrt{p^2 - 4cd})/2$ is a real number. While

$$\omega^l = \exp\left(\frac{2l\pi i}{n}\right) = \cos\left(\frac{2l\pi}{n}\right) + i \sin\left(\frac{2l\pi}{n}\right), \tag{40}$$

$\sin(2l\pi/n) = 0$, we obtain $\omega^l = -1$ for $0 < (2l\pi/n) < 2\pi$. But $x = -1$ is not the root of the equation $dx^2 + px + c = 0$. We have $f(\omega^k) \neq 0$ for any $\omega^k$ $(k = 1, 2, \ldots, n-1)$, while $f(1) = \sum_{j=1}^{n} V_{r+j} = ((V_r + V_{r+4} - V_{r+n} - V_{r+n+4})/((1-r_1)(1-r_2)(1-r_3))) \neq 0$. If $p^2 - 4cd < 0$, we get that $\omega^k$ is an imaginary number, if and only if

$$\cos\left(\frac{2k\pi}{n}\right) = \frac{-p}{2d}, \qquad \sin\left(\frac{2l\pi}{n}\right) = \frac{\pm\sqrt{4cd - p^2}}{2d}. \tag{41}$$

We obtain $n = 2k\pi/\arctan(\sqrt{4cd-p^2}/\pm p)$, such that $f(\omega^k) = 0$. If $(1-r_1\omega^k)(1-r_2\omega^k)(1-r_3\omega^k) = 1 - \omega^{2k} - \omega^{3k} = 0$, we have $\omega^k = 1/r_1, 1/r_2$, or $1/r_3$; obviously, $\omega^k \neq 0$ and $\omega^k \neq \pm 1$. In the same way, we know that $1/r_i \neq 0$ and $1/r_i \neq \pm 1$. So, $f(1/r_i) = \sum_{j=1}^{n} V_{r+j}(1/r_i)^{j-1} \neq 0$, $(i = 1, 2, 3)$. By Lemma 1 in [14], the proof is obtained. □

**Theorem 7.** *Let $\mathscr{V}_{r,n} = Circ(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan circulant matrix. If $n \neq 2k\pi/\arctan(\sqrt{4cd - p^2}/\pm p)$, so its inverse is*

$$\mathscr{V}_{r,n}^{-1} = Circ\left(\frac{1}{\nu} - \frac{\theta}{\nu}y_3' - y_4' - y_5', -\frac{V_{r+2}}{\nu V_{r+1}} + \left(\frac{\theta V_{r+2}}{\nu V_{r+1}} - \frac{V_{r+3}}{V_{r+1}}\right)y_3' - y_4', y_n', \ldots, y_3' - y_5' - y_6'\right), \tag{42}$$

*where*

$$\begin{aligned} y_1' &= 0, \\ y_2' &= \frac{1}{\nu}, \\ y_3' &= \frac{b_3}{\xi} - \frac{c\sum_{i=2}^{n-2} h_{i1}b_{i+2}}{\xi}, \\ y_4' &= -\frac{b_3\sum_{i=1}^{n-3} h_{i1}\rho_{i+3}}{\xi} + \sum_{i=1}^{n-3} h_{i1}b_{i+3} - \frac{c\sum_{i=1}^{n-3} h_{i1}\rho_{i+3}\sum_{i=2}^{n-2} h_{i1}b_{i+2}}{\xi}, \\ &\vdots \\ y_k' &= -\frac{b_3\sum_{i=1}^{n-k+1} h_{i1}\rho_{i+k-1}}{\xi} + \sum_{i=1}^{n-k+1} h_{i1}b_{i+k-1} - \frac{c\sum_{i=1}^{n-k+1} h_{i1}\rho_{i+k-1}\sum_{i=2}^{n-2} h_{i1}b_{i+2}}{\xi}, \quad (k \geq 4), \\ \xi &= \rho_3 - b\sum_{i=1}^{n-3} h_{i1}\rho_{i+3} - d\sum_{i=1}^{n-4} h_{i1}\rho_{i+4}, \\ \rho_3 &= V_{r+1} - \frac{V_{r+3}V_{r+n-1}}{V_{r+1}} - \frac{\theta}{\nu}\delta_n, \\ \rho_i &= V_{r+n-i+4} - \frac{\theta}{\nu}\delta_{n-i+3} - \frac{V_{r+3}V_{r+n+2-i}}{V_{r+1}} \quad (i = 4, \ldots, n), \\ b_j &= \frac{V_{r+2}V_{r+n+2-j} - V_{r+1}V_{r+n+3-j}}{\nu V_{r+1}} \quad (j = 3, \ldots, n). \end{aligned} \tag{43}$$

*Proof.* Let

$$\Gamma_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -\frac{\theta}{\nu} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}; \tag{44}$$

thus

$$\Gamma V_n \Pi_1 = \begin{pmatrix} V_{r+1} & \mu & V_{r+n-1} & \cdots & V_{r+2} \\ 0 & \nu & \delta_n & \cdots & \delta_3 \\ 0 & 0 & \rho_3 & \cdots & \rho_n \\ 0 & 0 & b & \cdots & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}. \tag{45}$$

Also we have $\Gamma = \Gamma_2\Gamma_1$, according to Lemma 3; letting

$$\Psi = \begin{pmatrix} \rho_3 & V \\ U & \mathscr{G} \end{pmatrix}, \tag{46}$$

be a $(n-2)\times(n-2)$ partitioned matrix, we obtain

$$\Psi^{-1} = \begin{pmatrix} \frac{1}{\xi} & -\frac{V\mathscr{G}^{-1}}{\xi} \\ -\frac{\mathscr{G}^{-1}U}{\xi} & \mathscr{G}^{-1} + \frac{UV\mathscr{G}^{-1}}{\xi} \end{pmatrix}, \tag{47}$$

where

$$\begin{aligned} \xi &= \rho_3 - V\mathscr{G}^{-1}U, \\ U &= (b, d, 0, \ldots, 0)^T, \\ V &= (\rho_4, \rho_5, \ldots, \rho_n). \end{aligned} \tag{48}$$

Let

$$\Pi_2 = \begin{pmatrix} 1 & -\frac{\mu}{V_{r+1}} & \varrho_3 & \cdots & \varrho_n \\ 0 & 1 & b_3 & \cdots & b_n \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{49}$$

where

$$\varrho_i = \frac{\mu b_i - V_{r+n-i+2}}{V_{r+1}} \quad (i = 3, \ldots, n). \tag{50}$$

So

$$\Gamma\mathscr{V}_{r,n}\Pi_1\Pi_2 = \mathscr{D} \oplus \Psi, \tag{51}$$

where $\mathscr{D} = \operatorname{diag}(V_{r+1}, \nu)$ is a diagonal matrix and $\mathscr{D} \oplus \Psi$ is the direct sum of $\mathscr{D}$ and $\Psi$. If we denote $\Pi = \Pi_1\Pi_2$, then we obtain

$$\mathscr{V}_{r,n}^{-1} = \Pi\left(\mathscr{D}^{-1} \oplus \Psi^{-1}\right)\Gamma,$$

$$\Gamma = \begin{pmatrix} 1 & & & & & & 0 & 0 \\ -\dfrac{V_{r+2}}{V_{r+1}} & & 0 & & & & 0 & 1 \\ \chi & & & & & \iddots & 1 & -\dfrac{\theta}{\nu} \\ -1 & & & & \iddots & \iddots & 0 & -1 \\ 0 & & & & & \iddots & -1 & -1 \\ \vdots & & \iddots & \iddots & \iddots & \iddots & \iddots & 0 \\ 0 & 0 & 1 & 0 & -1 & \iddots & 0 & \vdots \\ 0 & 1 & 0 & -1 & -1 & 0 & \cdots & 0 \end{pmatrix}, \tag{52}$$

where

$$\chi = \frac{\theta V_{r+2}}{\nu V_{r+1}} - \frac{V_{r+3}}{V_{r+1}}. \tag{53}$$

Since the last row elements of the matrix $\Pi$ are $0, 1, (V_{r+2}V_{r+n-1} - V_{r+1}V_{r+n})/\nu V_{r+1}, \dots, (V_{r+2}^2 - V_{r+1}V_{r+3})/\nu V_{r+1}$, then for $\Pi(\mathscr{D}_1^{-1} \oplus \Psi^{-1})$ its last row elements are given by the following equations:

$$\begin{aligned} y_1' &= 0, \\ y_2' &= \frac{1}{\nu}, \\ y_3' &= \frac{b_3}{\xi} - \frac{c\sum_{i=2}^{n-2} h_{i1}b_{i+2}}{\xi}, \\ y_4' &= -\frac{b_3\sum_{i=1}^{n-3} h_{i1}\rho_{i+3}}{\xi} + \sum_{i=1}^{n-3} h_{i1}b_{i+3} \\ &\quad - \frac{c\sum_{i=1}^{n-3} h_{i1}\rho_{i+3}\sum_{i=2}^{n-2} h_{i1}b_{i+2}}{\xi}, \\ &\;\vdots \\ y_k' &= -\frac{b_3\sum_{i=1}^{n-k+1} h_{i1}\rho_{i+k-1}}{\xi} + \sum_{i=1}^{n-k+1} h_{i1}b_{i+k-1} \\ &\quad - \frac{c\sum_{i=1}^{n-k+1} h_{i1}\rho_{i+k-1}\sum_{i=2}^{n-2} h_{i1}b_{i+2}}{\xi}, \quad (k \geq 4). \end{aligned} \tag{54}$$

By Lemma 4, if $\mathscr{V}_{r,n}^{-1} = \operatorname{Circ}(y_1, y_2, \dots, y_n)$, we have its last row elements as follows:

$$\begin{aligned} y_2 &= -\frac{V_{r+2}}{\nu V_{r+1}} + \left(\frac{\theta V_{r+2}}{\nu V_{r+1}} - \frac{V_{r+3}}{V_{r+1}}\right)y_3' - y_4', \\ y_3 &= y_n', \\ y_4 &= y_{n-1}', \\ y_5 &= y_{n-2}' - y_n', \\ &\;\vdots \\ y_k &= y_{n-k+3}' - y_{n-k+5}' - y_{n-k+6}' \quad (5 < k \leq n), \\ y_1 &= \frac{1}{\nu} - \frac{\theta}{\nu}y_3' - y_4' - y_5'. \end{aligned} \tag{55}$$

Proof is completed. □

## 3. Norms and Bound of Spread for VanderLaan Circulant Matrix

In this section, we gain three kinds of norms and lower bound for the spread of VanderLaan circulant matrix separately.

**Theorem 8.** *Let $\mathscr{V}_{r,n} = \operatorname{Circ}(V_{r+1}, V_{r+2}, \dots, V_{r+n})$ be a VanderLaan circulant matrix. One has its two kinds of norms:*

$$\|\mathscr{V}_{r,n}\|_1 = \|\mathscr{V}_{r,n}\|_\infty = V_{r+n+5} - V_{r+5}. \tag{56}$$

*Proof.* By definition of norms in [25] and (4), we obtain

$$\begin{aligned} \|\mathscr{V}_{r,n}\|_1 &= \|\mathscr{V}_{r,n}\|_\infty \\ &= \sum_{j=1}^{n} V_{r+j} \\ &= \frac{V_r + V_{r+1} + V_{r+2} - V_{r+n} - V_{r+n+1} - V_{r+n+2}}{(1 - r_1)(1 - r_2)(1 - r_3)} \\ &= \frac{V_{r+5} - V_{r+n+5}}{(1 - r_1)(1 - r_2)(1 - r_3)} \\ &= \frac{V_{r+5} - V_{r+n+5}}{1 - (r_1 + r_2 + r_3) + (r_1r_2 + r_2r_3 + r_1r_3) - r_1r_2r_3} \\ &= V_{r+n+5} - V_{r+5}. \end{aligned} \tag{57}$$

Theorem 8 is proved. □

**Theorem 9.** *Let $\mathscr{V}_{r,n} = \operatorname{Circ}(V_{r+1}, V_{r+2}, \dots, V_{r+n})$ be a VanderLaan circulant matrix. So one has*

$$\|\mathscr{V}_{r,n}\|_2 = \sum_{j=1}^{n} V_{r+j} = V_{r+n+5} - V_{r+5}. \tag{58}$$

*Proof.* According to Theorem 2 in [26] and (4), we get

$$\begin{aligned}\|\mathscr{V}_{r,n}\|_2 &= \sum_{j=1}^{n} V_{r+j} \\ &= \frac{V_r + V_{r+1} + V_{r+2} - V_{r+n} - V_{r+n+1} - V_{r+n+2}}{(1-r_1)(1-r_2)(1-r_3)} \\ &= \frac{V_{r+5} - V_{r+n+5}}{1-(r_1+r_2+r_3)+(r_1r_2+r_2r_3+r_1r_3)-r_1r_2r_3} \\ &= V_{r+n+5} - V_{r+5}.\end{aligned} \tag{59}$$

The proofs are completed. □

**Theorem 10.** *Let $\mathscr{V}_{r,n} = Circ(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan circulant matrix; the lower bound for the spread of $\mathscr{V}_{r,n}$ is*

$$s(\mathscr{V}_{r,n}) \geq \frac{n(V_{r+n+5} - V_{r+6})}{n-1}. \tag{60}$$

*Proof.* By (19) in [27] and (4), we obtain

$$s(\mathscr{V}_{r,n}) \geq \frac{n}{n-1}\sum_{j=2}^{n} V_{r+j}. \tag{61}$$

Since

$$\begin{aligned}\sum_{j=2}^{n} V_{r+j} &= \frac{c_1 r_1^{r+2}(1-r_1^{n-1})}{(1-r_1)} + \frac{c_2 r_2^{r+2}(1-r_2^{n-1})}{(1-r_2)} \\ &\quad + \frac{c_3 r_3^{r+2}(1-r_3^{n-1})}{(1-r_3)} \\ &= \frac{V_{r+1} + V_{r+2} + V_{r+3} - V_{r+n} - V_{r+n+1} - V_{r+n+2}}{(1-r_1)(1-r_2)(1-r_3)} \\ &= \frac{V_{r+6} - V_{r+n+5}}{1-(r_1+r_2+r_3)+(r_1r_2+r_2r_3+r_1r_3)-r_1r_2r_3} \\ &= V_{r+n+5} - V_{r+6},\end{aligned} \tag{62}$$

we get

$$s(\mathscr{V}_{r,n}) \geq \frac{n(V_{r+n+5} - V_{r+6})}{n-1}, \tag{63}$$

which completes the proof. □

## 4. Determinant, Inverse, and Norms and Spread of VanderLaan Left Circulant Matrix

In this part, let $\mathscr{U}_{r,n}$ be a VanderLaan left circulant matrix. By using the obtained conclusions, we give a determinant formula for the matrix $\mathscr{U}_{r,n}$. Afterwards, we prove that $\mathscr{U}_{r,n}$ is an nonsingular matrix for $n \neq 2k\pi/\arctan(\sqrt{4cd-p^2}/\pm p)$. The inverse of the matrix $\mathscr{U}_{r,n}$ is also presented. Finally, three kinds of norms and lower bound for the spread of VanderLaan left circulant matrix are given.

According to Lemma 2 in [15] and Theorems 5, 6, and 7, we can obtain the following theorems.

**Theorem 11.** *Let $\mathscr{U}_{r,n} = LCirc(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan left circulant matrix; then one has*

$$\det \mathscr{U}_{r,n} = (-1)^{(n-1)(n-2)/2}\left[V_{r+1}(\nu\kappa - \theta\tau)\right], \tag{64}$$

*where $V_{r+n}$ is the $(r+n)$th VanderLaan number.*

**Theorem 12.** *Let $\mathscr{U}_{r,n} = LCirc(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan left circulant matrix; if $n \neq 2k\pi/\arctan(\sqrt{4cd-p^2}/\pm p)$, then $\mathscr{U}_{r,n}$ is a nonsingular matrix.*

**Theorem 13.** *Let $\mathscr{U}_{r,n} = LCirc(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan left circulant matrix; in case of $n \neq 2k\pi/\arctan(\sqrt{4cd-p^2}/\pm p)$, its inverse is*

$$\begin{aligned}\mathscr{U}_{r,n}^{-1} = LCirc\Bigg(&\frac{1}{\nu} - \frac{\theta}{\nu}y_3' - y_4' - y_5', y_3' - y_5' - y_6', \ldots, y_n', \\ &-\frac{V_{r+2}}{\nu V_{r+1}} + \left(\frac{\theta V_{r+2}}{\nu V_{r+1}} - \frac{V_{r+3}}{V_{r+1}}\right) y_3' - y_4'\Bigg),\end{aligned} \tag{65}$$

*where*

$$\begin{aligned}y_1' &= 0, \\ y_2' &= \frac{1}{\nu}, \\ y_3' &= \frac{b_3}{\xi} - \frac{c\sum_{i=2}^{n-2} h_{i1} b_{i+2}}{\xi}, \\ y_4' &= -\frac{b_3 \sum_{i=1}^{n-3} h_{i1}\rho_{i+3}}{\xi} \\ &\quad + \sum_{i=1}^{n-3} h_{i1} b_{i+3} \\ &\quad - \frac{c\sum_{i=1}^{n-3} h_{i1}\rho_{i+3} \sum_{i=2}^{n-2} h_{i1} b_{i+2}}{\xi}, \\ &\vdots \\ y_k' &= -\frac{b_3 \sum_{i=1}^{n-k+1} h_{i1}\rho_{i+k-1}}{\xi} + \sum_{i=1}^{n-k+1} h_{i1} b_{i+k-1} \\ &\quad - \frac{c\sum_{i=1}^{n-k+1} h_{i1}\rho_{i+k-1} \sum_{i=2}^{n-2} h_{i1} b_{i+2}}{\xi}, \quad (k \geq 4).\end{aligned} \tag{66}$$

**Theorem 14.** *Let $\mathscr{U}_{r,n} = LCirc(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan left circulant matrix, so one can get two kinds of norms of $\mathscr{U}_{r,n}$:*

$$\|\mathscr{U}_{r,n}\|_1 = \|\mathscr{U}_{r,n}\|_\infty = V_{r+n+5} - V_{r+5}. \tag{67}$$

*Proof.* By definition of norms in [25] and (57), we have

$$\begin{aligned}\|\mathscr{U}_{r,n}\|_1 &= \|\mathscr{U}_{r,n}\|_\infty \\ &= \sum_{j=1}^{n} V_{r+j} \\ &= V_{r+n+5} - V_{r+5}.\end{aligned} \tag{68}$$

This completes the proof. □

**Theorem 15.** *Let $\mathscr{U}_{r,n} = LCirc(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan left circulant matrix; then one has the spectral norm of $\mathscr{U}_{r,n}$:*

$$\|\mathscr{U}_{r,n}\|_2 = V_{r+n+5} - V_{r+5}. \tag{69}$$

*Proof.* From Theorem 2 in [26] and (57), we can have

$$\|\mathscr{U}_{r,n}\|_2 = \sum_{j=1}^{n} V_{r+j} = V_{r+n+5} - V_{r+5}. \tag{70}$$

The proofs are completed. □

**Theorem 16.** *Let $\mathscr{U}_{r,n} = LCirc(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan left circulant matrix; the lower bound for the spread of $\mathscr{U}_{r,n}$ is*

$$s(\mathscr{U}_{r,n}) \geq \begin{cases} V_{r+n+5} - V_{r+5}, & n \text{ is odd}, \\ t, & n \text{ is even}, \end{cases} \tag{71}$$

*where*

$$t = \frac{n}{n-1}(V_{r+n+5} - V_{r+5}) - \frac{2}{n-1}(V_{r+n+2} - V_{r+2}). \tag{72}$$

*Proof.* By (19) in [27], we obtain

$$s(\mathscr{U}_{r,n}) \geq \frac{1}{n-1}\sum_{j\neq i} V_{r+j}. \tag{73}$$

When $n$ is odd

$$\begin{aligned}\frac{1}{n-1}\sum_{j\neq i} V_{r+j} &= \frac{1}{n-1}\left[n\sum_{j=1}^{n} V_{r+j} - \sum_{j=1}^{n} V_{r+j}\right] \\ &= \sum_{j=1}^{n} V_{r+j} \\ &= V_{r+n+5} - V_{r+5}.\end{aligned} \tag{74}$$

If $n$ is even

$$\begin{aligned}\frac{1}{n-1}\sum_{j\neq i} V_{r+j} &= \frac{1}{n-1}\left[n\sum_{j=1}^{n} V_{r+j} - 2\sum_{j=1}^{n/2} V_{r+2j-1}\right] \\ &= \frac{n}{n-1}(V_{r+n+5} - V_{r+5}) - \frac{2}{n-1}\sum_{j=1}^{n/2} V_{r+2j-1} \\ &= \frac{n}{n-1}V_{r+n+5} - V_{r+5} - \frac{2}{n-1}(V_{r+n+2} - V_{r+2}) \\ &= t.\end{aligned} \tag{75}$$

We get

$$s(\mathscr{U}_{r,n}) \geq \begin{cases} V_{r+n+5} - V_{r+5}, & n \text{ is odd}, \\ t, & n \text{ is even}. \end{cases} \tag{76}$$

The proofs are completed. □

## 5. Determinant, Inverse, and Spectral Norm of VanderLaan $g$-Circulant Matrix

In this section, let $\mathscr{W}_{g,r,n}$ be a VanderLaan $g$-circulant matrix. We give a determinant formula for the matrix $\mathscr{W}_{g,r,n}$ by the means of the gained results. Afterwards, we get the inverse of the matrix $\mathscr{W}_{g,r,n}$ and obtain that $\mathscr{W}_{g,r,n}$ is an invertible matrix for $n \neq 2k\pi/\arctan(\sqrt{4cd - p^2}/\pm p)$. At last, we gain the spectral norm of VanderLaan $g$-circulant matrix.

From Lemmas 3 and 4 in [15] and Theorems 5, 6, and 7, the following results are deduced.

**Theorem 17.** *Let $\mathscr{W}_{g,r,n} = g\text{-}Circ(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan $g$-circulant matrix and $(n, g) = 1$; one obtains*

$$\det \mathscr{W}_{g,r,n} = V_{r+1}(\nu\kappa - \theta\tau)\det \mathbb{Q}_g, \tag{77}$$

*where $V_{r+n}$ is the $(r+n)$th VanderLaan number and the matrix $\mathbb{Q}_g$ is given Lemma 3 in [15].*

**Theorem 18.** *Let $\mathscr{W}_{g,r,n} = g\text{-}Circ(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan $g$-circulant matrix and $(n, g) = 1$; if $n \neq 2k\pi/\arctan(\sqrt{4cd - p^2}/\pm p)$, $\mathscr{W}_{g,r,n}$ is a nonsingular matrix.*

**Theorem 19.** *Let $\mathscr{W}_{g,r,n} = g\text{-}Circ(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan $g$-circulant matrix and $(n, g) = 1$; when $n \neq 2k\pi/\arctan(\sqrt{4cd - p^2}/\pm p)$, the inverse of matrix $\mathscr{W}_{g,r,n}$ is*

$$\begin{aligned}\mathscr{W}_{g,r,n}^{-1} = \Bigg[Circ\Bigg(&\frac{1}{\nu} - \frac{\theta}{\nu}y_3' - y_4' - y_5', -\frac{V_{r+2}}{\nu V_{r+1}} \\ &+ \left(\frac{\theta V_{r+2}}{\nu V_{r+1}} - \frac{V_{r+3}}{V_{r+1}}\right)y_3' - y_4', y_n', \ldots, y_3' \\ &- y_5' - y_6'\Bigg)\Bigg]\mathbb{Q}_g^T,\end{aligned} \tag{78}$$

*where*

$$y_1' = 0,$$

$$y_2' = \frac{1}{\nu},$$

$$y_3' = \frac{b_3}{\xi} - \frac{c\sum_{i=2}^{n-2} h_{i1}b_{i+2}}{\xi},$$

$$y_4' = -\frac{b_3 \sum_{i=1}^{n-3} h_{i1}\rho_{i+3}}{\xi} + \sum_{i=1}^{n-3} h_{i1} b_{i+3} - \frac{c \sum_{i=1}^{n-3} h_{i1}\rho_{i+3} \sum_{i=2}^{n-2} h_{i1} b_{i+2}}{\xi},$$
$$\vdots$$
$$y_k' = -\frac{b_3 \sum_{i=1}^{n-k+1} h_{i1}\rho_{i+k-1}}{\xi} + \sum_{i=1}^{n-k+1} h_{i1} b_{i+k-1} - \frac{c \sum_{i=1}^{n-k+1} h_{i1}\rho_{i+k-1} \sum_{i=2}^{n-2} h_{i1} b_{i+2}}{\xi}, \quad (k \geq 4). \tag{79}$$

**Theorem 20.** *Let $\mathscr{W}_{g,r,n} = g\text{-}Circ(V_{r+1}, V_{r+2}, \ldots, V_{r+n})$ be a VanderLaan g-circulant matrix and $(n, g) = 1$. One has its spectral norm*

$$\left\|\mathscr{W}_{g,r,n}\right\|_2 = V_{r+n+5} - V_{r+5}. \tag{80}$$

*Proof.* According to the definition of norms, Lemmas 3 and 4 in [15], and Theorem 9, if $(g, n) = 1$, we get $\mathscr{W}_{g,r,n}^T \mathscr{W}_{g,r,n} = \mathscr{V}_{r,n}^T \mathbb{Q}_g^T \mathbb{Q}_g \mathscr{V}_{r,n} = \mathscr{V}_{r,n}^T \mathscr{V}_{r,n}$; then $\|\mathscr{W}_{g,r,n}\|_2 = \|\mathscr{V}_{r,n}\|_2 = V_{r+n+5} - V_{r+5}$.

This completes the proofs. □

## 6. Conclusion

In this paper, we considered VanderLaan circulant type matrices. We discussed the nonsingularity of these special matrices and presented the exact determinants and inverse matrices of VanderLaan circulant type matrices. Three kinds of norms and lower bound for the spread of VanderLaan circulant and left circulant matrix are given separately. And we gain the spectral norm of VanderLaan $g$-circulant matrix.

Furthermore, based on circulant matrices technology we will consider solving the problem in [28–31].

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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