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Research Article

On a Nonlinear Degenerate Evolution Equation with Nonlinear Boundary Damping

A. T. Lourêdo,[1] G. Siracusa,[2] and C. A. Silva Filho[3]

[1] *DM-UEPB, Campina Grande, PB, Brazil*
[2] *DMA-São Cristóvão, SE, Brazil*
[3] *DCET-UESC, Ilhéus, BA, Brazil*

Correspondence should be addressed to A. T. Lourêdo; aldolouredo@gmail.com

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This paper deals essentially with a nonlinear degenerate evolution equation of the form $Ku'' - \Delta u + \sum_{j=1}^{n} b_j(\partial u'/\partial x_j) + |u|^\sigma u = 0$ supplemented with nonlinear boundary conditions of Neumann type given by $\partial u/\partial \nu + h(\cdot, u') = 0$. Under suitable conditions the existence and uniqueness of solutions are shown and that the boundary damping produces a uniform global stability of the corresponding solutions.

1. Introduction

Let Ω be a smooth bounded open set of $\mathbb{R}^n$, with $n \geq 1$, and its boundary $\partial\Omega = \Gamma$ of class C^2. Assume that Γ is constituted by two disjoint closed parts Γ_0 and Γ_1 both with positive Lebesgue measure.

The main goal of this paper is to prove the existence and uniqueness as well as the uniform decay rates for the energy of the following nonlinear initial boundary value problem:

$$\begin{gathered} Ku'' - \Delta u + \sum_{j=1}^{n} b_j \frac{\partial u'}{\partial x_j} + |u|^\sigma u = 0 \quad \text{in } \Omega \times (0, \infty), \\ u = 0 \quad \text{on } \Gamma_0 \times (0, \infty), \\ \frac{\partial u}{\partial \nu} + h(\cdot, u') = 0 \quad \text{on } \Gamma_1 \times (0, \infty), \\ u(0, x) = u^0(x), \quad u'(0, x) = u^1(x) \quad \text{in } \Omega, \end{gathered} \tag{P}$$

where $K = K(x, t)$, $b_j = b_j(x, t)$, and $h = h(x, s)$ are real functions, $\nu = \nu(x)$ denotes the unit outward normal at $x \in \Gamma_1$, and $\sigma > 1$ is a constant.

The parabolic-hyperbolic equation $Ku'' - \Delta(u - u') + f(u) = 0$ when $n = 1$ or $n = 2$; this equation governs the motion of a nonlinear Kelvin solid. That is, a bar for $n = 1$ and a plate for $n = 2$, subject to no nonlinear elastic constraints. The function K represents the mass density of the solid.

The existence of solutions of the linear problem associated with (P) ($K = 1, b_j = 0$, and without the function $f(s) = |s|^\sigma s$ and with $h(x, s) = \delta(x)s$) was established by Komornik and Zuazua in [1], via semigroup theory and by Milla Miranda and Medeiros in [2], applying the Galerkin's method, with a special basis. The advantage of this second method is to define the Sobolev space where $\partial u/\partial \nu$ is lying. In the same context, applying this second method for a wave equation with a nonlinear term, Araruna and Maciel [3], derive similar results. In Cavalcanti et al. [4] the existence of solution and an exponential decay rate is established supposing $f = 0$ and h being a particular function considered in our work; see also Cavalcanti et al. [5].

For the wave equation with $K = 1$ and $b = 0$ there is a vast literature on this problem. We cite the papers Cavalcanti et al. [6], Lasiecka and Tataru [7], and references contained therein for the reader.

Following the ideas delivered in Milla Miranda and Medeiros [2], but bringing more technical difficulties, Milla Miranda and San Gil Jutuca [8] applied the Galerkin's method with a special basis to show the existence of solutions for Kirchhoff's equation with a linear dissipation on the boundary. Applying a similar approach, Lourêdo and Miranda [9] obtained the existence of solutions for a coupled system of Kirchhoff's equations with nonlinear boundary dissipation. For other models, but in the same context, we cite to the reader Lourêdo and Miranda [10], and Lourêdo et al. [11].

Park and Kang studied the existence, uniqueness, and uniform decay for the nonlinear degenerate equation with memory condition on the boundary in [12]. For the asymptotic behavior they also used the Nakao's method. de Lima Santos and Junior [13] studied the equation with a boundary condition with memory for Kirchhoff plates equations. An abstract formulation with the coefficient K satisfying the same conditions as in our paper was studied by Pereira in [14] and was established the existence, uniqueness, and asymptotic behavior for the solutions associated with a nonlinear beam equation.

In this paper we are interested in showing the global existence of solutions for Problem (P) under very general conditions to be fixed in the next section.

In our approach, we apply the Galerkin's method for a *perturbed problem* and a special basis; an appropriate Strauss' Lipschitz-continuous approximation h_l of h; the compactness method; and results on trace mapping of nonsmooth functions. Finally, the uniform stabilization of solutions is accomplished by using the Nakao's method.

2. Notations and Main Results

In order to establish the main results of this paper we assume the following assumptions on the objects of problem (P):

$(H1)$

$(A1)$ $b_j \in W^{1,\infty}(0,\infty; C^1(\overline{\Omega}))$ and there exists a positive constant $b_0 > 0$ such that

$(A2)$ $\operatorname{div} b(x,t) \leq -b_0 \ \forall x \in \Gamma_1,\ t \in [0,\infty)$, where $b(x,t) = (b_1(x,t), \dots, b_n(x,t))$;

$(H2)$

$$\frac{1}{n} < \sigma \leq \frac{2}{n-2} \quad \text{if } n \geq 3,$$
$$\sigma > \frac{1}{n} \quad \text{if } n = 1, 2. \tag{1}$$

$(H3)$ $h \in C^0(\mathbb{R}; L^\infty(\Gamma_1))$ with $h : \Gamma_1 \times \mathbb{R} \to \mathbb{R}$ strongly monotone; that is,

$$[h(x,s) - h(x,r)][s-r] \geq h_0 (s-r)^2 \quad \text{a.e. } x \in \Gamma_1,\ \forall s, r \in \mathbb{R}, \tag{2}$$

where $2h_0 > \|b\|_{\overline{\Omega}\times(0,\infty)}$. We use the notation $d_0 = 2h_0 - \|b\|_\infty$;

$(H4)$ $K \in C^1([0,T]; L^\infty(\Omega))$ with $K(x,t) \geq 0$, $\forall t \geq 0$, a.e. $x \in \Omega$ and there exists $\gamma > 0$ such that $K(x,0) \geq \gamma > 0$, a.e. $x \in \Omega$;

$(H5)$ $|\partial K(x,t)/\partial t|_{\mathbb{R}} \leq \delta + C(\delta)K(x,t)$, for all $\delta > 0$;

$(H6)$ $(u^0, u^1) \in D(-\Delta) \times H_0^1(\Omega)$.

The scalar product and norm of $L^2(\Omega)$ are denoted, respectively, by $(\cdot,\cdot)$ and $|\cdot|$. By V we represent the Hilbert space $V = \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_0\}$ which is equipped with the scalar product and norm

$$((u,v)) = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right),$$
$$\|u\|^2 = \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^2. \tag{3}$$

The operator $-\Delta$ is defined by the triplet $\{V, L^2(\Omega), ((\cdot,\cdot))\}$. Then its domain is given by

$$D(-\Delta) = \left\{ u \in V \cap H^2(\Omega);\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1 \right\}. \tag{4}$$

From spectral theory it follows that $D(-\Delta)$ is dense in V; see [15]. Moreover, it will be denoted

$$(u,v)_{L^2(\Gamma_1)} = \int_{\Gamma_1} u(x)\, v(x)\, d\Gamma,$$
$$|u|^2_{L^2(\Gamma_1)} = \int_{\Gamma_1} u^2(x)\, d\Gamma, \tag{5}$$
$$\|u\|_\infty = \operatorname*{ess\,sup}_{t\geq 0} \|u(t)\|_{L^\infty(\Omega)}.$$

Theorem 1. *Assume hypotheses (H1)–(H6); there exists at least a function $u : \Omega \times (0,\infty) \to \mathbb{R}$ in the class*

$$u \in L^\infty_{\text{loc}}(0,\infty; V),$$
$$u' \in L^\infty_{\text{loc}}(0,\infty; V),$$
$$u'' \in L^\infty_{\text{loc}}\left(0,\infty; L^2(\Omega)\right), \tag{6}$$
$$\Delta u \in L^\infty_{\text{loc}}\left(0,\infty; L^2(\Omega)\right),$$
$$\frac{\partial u}{\partial \nu} \in L^1_{\text{loc}}\left(0,\infty; L^1(\Gamma_1)\right),$$

satisfying

$$Ku'' - \Delta u + \sum_{j=1}^{n} b_j \frac{\partial u'}{\partial x_j} + |u|^{\sigma} u = 0 \quad \text{in } L^{\infty}_{\text{loc}}\left(0, \infty; L^2(\Omega)\right),$$
$$\frac{\partial u}{\partial \nu} + h\left(\cdot, u'\right) = 0 \quad \text{in } L^{1}_{\text{loc}}\left(0, \infty; L^1(\Gamma_1)\right),$$
$$u(0) = u^0, \quad u'(0) = u^1 \quad \text{in } \Omega. \tag{7}$$

In addition, if

($H7$) $|h(x,s)| \le h_1|s|$ $\forall s \in \mathbb{R}$ and a.e. x in Γ_1 (h_1 a positive constant), holds true, we have the following result.

Corollary 2. *Under the hypothesis of Theorem 1 and (H7), the solution u of Problem (P) (obtained in Theorem 1) is unique and has the following regularity:*

$$u \in L^{\infty}_{\text{loc}}(0, \infty; V) \cap L^{2}_{\text{loc}}\left(0, \infty; H^{3/2}(\Omega)\right),$$
$$u' \in L^{\infty}_{\text{loc}}(0, \infty; V),$$
$$u'' \in L^{\infty}_{\text{loc}}\left(0, \infty; L^2(\Omega)\right),$$
$$Ku'' - \Delta u + \sum_{j=1}^{n} b_j \frac{\partial u'}{\partial x_j} + |u|^{\sigma} u = 0 \quad \text{in } L^{\infty}_{\text{loc}}\left(0, \infty; L^2(\Omega)\right),$$
$$\frac{\partial u}{\partial \nu} + h\left(\cdot, u'\right) = 0 \quad \text{in } L^{2}_{\text{loc}}\left(0, \infty; L^2(\Gamma_1)\right),$$
$$u(0) = u^0, \quad u'(0) = u^1 \quad \text{in } \Omega. \tag{8}$$

Remark 3. If we replace the function $f(s) = |s|^{\sigma}s$ in Theorem 1 by a continuous function g such that

$$[g(s) - g(r)](s - r) \ge g_0 (s - r)^2, \quad \forall s, r \in \mathbb{R}, \ (g_0 \text{ a positive constant}), \tag{9}$$

and further

$$\left|g(s)\right| \le g_1 |s|, \quad \forall s \in \mathbb{R}, \tag{10}$$

Theorem 1 remains valid. Indeed, from (10), we obtain

$$\left|g_l(s)\right| \le g_1 |s|, \quad \forall s \in \mathbb{R}, \tag{11}$$

where (g_l) is the Strauss' approximations (see [16]) of the function g.

Remark 4. Analogously if $|h(x,s)| \le h_1|s|$ $\forall s \in \mathbb{R}$ and a.e. x in Γ_1 (h_1 a positive constant), we obtain $|h_l(x,s)| \le (3/2)h_1|s|$ for all $s \in \mathbb{R}$ (see [10]).

For use later, note that

$$|w|^2 \le \frac{1}{\lambda_1} \|w\|^2,$$
$$\|w\|_{L^2(\Gamma_1)} \le k_1 \|w\|,$$
$$\forall w \in V, \quad (k_1 \text{ positive constants}); \tag{12}$$

where λ_1 is the first eigenvalue of the spectral problem $((u, v)) = \lambda(u, v)$ for all $v \in V$ (see [15]).

In order to establish the uniform decay rate for energy, we assume

($H8$) $\|K'\|_{\infty} < b_0/2$;

($H9$) $\|K\|_{\infty} = \text{ess sup}_{t\ge 0}\|K(t)\|_{L^{\infty}(\Omega)}$;

($H10$) $\|\operatorname{div} b\|_{\infty} \le \delta_0$;

($H11$) $\|b\|_{\infty} < (21/32)\lambda_1, b_0 > (32(1/\lambda_1)/M)\|K\|_{\infty}, \|K\|_{\infty} > \|b\|_{\infty}/2 + 8(1/\lambda_1)/M$, where $M = \min\{1, 1 - (9/32 + (1/\lambda_1)\|b\|_{\infty})\}$.

Remark 5. There are functions that satisfy hypothesis ($H5$) and ($H8$). In fact, the function $K(x,t) = (\alpha(x)/\|\alpha\|_{\infty})\beta(t)$ with $\alpha \in C^1(\overline{\Omega})$, $\alpha(x) \ge 0$, and $\beta(t) = (b_0/4)e^{-t}$ satisfies such hypothesis, since

$$\left|K'(x,t)\right| \le \frac{\alpha(x)}{\|\alpha\|_{\infty}} \frac{b_0}{4} e^{-t} \le \frac{b_0}{4} < \frac{b_0}{2}, \tag{13}$$

and so

$$\left|K'(x,t)\right| \le \delta + K(x,t), \quad \forall \delta \ge 0. \tag{14}$$

Theorem 6. *Under the hypothesis of Theorem 1 and (H7)–(H11), with $h_1(s)$ satisfying (9) and (10), the energy*

$$E(t) = \frac{1}{2}\left\{K^{1/2}(t)\left|u'(t)\right|^2 + \|u(t)\|^2 + \frac{2}{\sigma+2}\int_{\Omega} |u(t)|^{\sigma+2} dx\right\} \quad \text{for } t \ge 1, \tag{15}$$

associated with the solution, u, obtained in Corollary 2 is uniformly stable; that is, there exists a positive constant such that

$$E(t) \le C \exp(-\varpi t), \quad \forall t \ge 1, \tag{16}$$

where C and ϖ are positive constants.

For use later, we observe that hypothesis in $(H3)_1$ on σ implies $q^* = 2n/(n-2) \ge 2\sigma + 2$ and $q^* \ge \sigma n$. Thus, the Sobolev's embedding gives

$$V \hookrightarrow L^{q^*}(\Omega) \hookrightarrow L^{2\sigma+2}(\Omega) \quad \text{for } n \ge 3,$$
$$V \hookrightarrow L^{2\sigma+2}(\Omega) \quad \text{for } n = 1, 2, \tag{17}$$

$$V \hookrightarrow L^{q^*}(\Omega) \hookrightarrow L^{\sigma n}(\Omega) \quad \text{for } n \ge 3,$$
$$V \hookrightarrow L^{\sigma n}(\Omega) \quad \text{for } n = 1, 2. \tag{18}$$

Here, $X \hookrightarrow Y$ indicates that the subspace X is continuously embedded in the space Y.

Next, following the ideas contained in Strauss [16], we approximate the function h by Lipschitz-continuous ones h_l.

3. Proof of Theorem 1

For our purposes we need the following previous results, whose proof can be seen in [10].

Lemma 7. *Let h be a function satisfying hypothesis (H3). Then there exists a sequence (h_l) of functions in $C^0(\mathbb{R}; L^\infty(\Gamma_1))$ such that*

(i) $h_l(x,0) = 0$ *a.e. x in Γ_1;*

(ii) $[h_l(x,s) - h_l(x,r)](s-r) \geq h_0(s-r)^2$, $\forall s, r \in \mathbb{R}$, *and a.e. x in Γ_1; (h_0 positive constant);*

(iii) *for any l there exists a function c_l in $L^\infty(\Gamma_1)$ satisfying*

$$\left|h_l(x,s) - h_l(x,r)\right| \leq c_l |s-r|, \quad \forall s, r \in \mathbb{R}, \ \text{a.e. } x \text{ in } \Gamma_1; \tag{19}$$

(iv) *(h_l) converges to g uniformly on bounded sets on $\mathbb{R}$ and a.e. x in Γ_1.*

Lemma 8. *Let $T > 0$ be a real number. Consider the sequence (w_l) of vectors in $L^2(0,T; H^{-1/2}(\Gamma_1)) \cap L^1(0,T; L^1(\Gamma_1))$ and vectors $w \in L^2(0,T; H^{-1/2}(\Gamma_1))$ and $\chi \in L^1(0,T; L^1(\Gamma_1))$ such that*

(i) *$w_l \to w$ weak in $L^2(0,T; H^{-1/2}(\Gamma_1))$;*

(ii) *$w_l \to \chi$ in $L^1(0,T; L^1(\Gamma_1))$.*

Then, $w = \chi$.

Lemma 9. *Let $p : \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz-continuous function with $p(0) = 0$ and let*

$$\gamma_0 : V \longrightarrow H^{1/2}(\Gamma_1) \tag{20}$$

be the continuous trace of order zero. Consider $u \in L^2(0,T;V)$ then $p(u) \in L^2(0,T;V)$, $p(\gamma_0 u) \in L^2(0,T; H^{1/2}(\Gamma_1))$, and $\gamma_0 p(u(t)) = p(\gamma_0 u(t))$ a.e. $t \in (0,T)$.

Proof. We see that

$$\begin{aligned} p &: V \longrightarrow V, \\ p &: H^{1/2}(\Gamma_1) \longrightarrow H^{1/2}(\Gamma_1) \end{aligned} \tag{21}$$

are continuous maps (see Brezis and Cazenave [17] and Marcus and Mizel [18]). Let $v \in V$. Consider a sequence (v_k) of functions of $C^1(\overline{\Omega})$ such that

$$v_k \longrightarrow v \quad \text{in } V. \tag{22}$$

Then by (21) and (22), we have $p(v_k) \to p(v)$ in V, and by (20)

$$\gamma_0 p(v_k) \longrightarrow \gamma_0 p(v) \quad \text{in } H^{1/2}(\Gamma_1). \tag{23}$$

Also by (20) and (22), we deduce

$$p(\gamma_0 v_k) \longrightarrow p(\gamma_0 v) \quad \text{in } H^{1/2}(\Gamma_1). \tag{24}$$

As $\gamma_0 p(v_k) = p(\gamma_0 v_k)$, it follows from (23) and (24) that $\gamma_0 p(v) = p(\gamma_0 v)$. This implies

$$\gamma_0 p(u(t)) = p(\gamma_0 u(t)) \quad \text{a.e. in } (0,T). \tag{25}$$

Now, we consider the set $\mathcal{O} = \{s \in \mathbb{R} : p \text{ is not differentiable in } s\}$. Then

$$\frac{\partial}{\partial x_i} p(u(x,t)) = \begin{cases} p'(u(x,t)) \dfrac{\partial u}{\partial x_i}(x,t), & u(x,t) \notin \mathcal{O}, \\ 0, & u(x,t) \in \mathcal{O}, \end{cases} \tag{26}$$

with $i = 1, 2, \ldots, n$ (see Brezis and Cazenave, loc. cit). As $p' \in L^\infty(\mathbb{R})$ then

$$p(u) \in L^2(0,T;V). \tag{27}$$

From this and since $\gamma_0 : L^2(0,T;V) \to L^2(0,T; H^{1/2}(\Gamma_1))$ then $\gamma_0 p(u) \in L^2(0,T; H^{1/2}(\Gamma_1))$. This and (25) furnish

$$p(\gamma_0 u) \in L^2\left(0,T; H^{1/2}(\Gamma_1)\right). \tag{28}$$

From (25) to (28) we have the results of this Lemma. □

Proof of Theorem 1. We will use the Faedo-Galerkin's method with a special basis of $V \cap H^2(\Omega)$. Thus, let us consider the Strauss' approximation (h_l) of h given by Lemma 7. Let us consider (u_l^1) a sequence of vectors in $\mathscr{D}(\Omega)$ such that

$$u_l^1 \longrightarrow u^1 \quad \text{in } H_0^1(\Omega). \tag{29}$$

Note that $h_l(x, u_l^1) = 0$ and $\partial u^0/\partial \nu = 0$ on Γ_1 since $u^0 \in D(-\Delta)$. Thus,

$$\frac{\partial u^0}{\partial \nu} + h_l\left(\cdot, u_l^1\right) = 0 \quad \text{on } \Gamma_1, \forall l. \tag{30}$$

Now, we fix l and construct the basis $\{w_1^l, w_2^l, \ldots\}$ of $V \cap H^2(\Omega)$ such that u^0, u_l^1 belong to the subspace $[w_1^l, w_2^l]$ spanned by w_1^l and w_2^l. Let $V_m^l = [w_1^l, w_2^l, \ldots, w_m^l]$ be the subspace of $V \cap H^2(\Omega)$ spanned by $w_1^l, \ldots, w_m^l$. With this basis we determine the approximate solutions $u_{\varepsilon lm}(t)$ of Problem (AP), where $0 < \varepsilon < 1$ fixed.

Approximated Perturbed Problem. This consists to find the functions $u_{\varepsilon lm}(t) = \sum_{j=1}^{m} g_{\varepsilon jlm}(t) w_j^l$, solutions of the problem

$$\begin{gathered}\left((K(t)+\varepsilon) u_{\varepsilon lm}''(t), w_j^l\right) + \left(\left(u_{\varepsilon lm}(t), w_j^l\right)\right) \\ + \left(\sum_{j=1}^{n} b_j(t) \frac{\partial u_{\varepsilon lm}'(t)}{\partial x_j}, w_j^l\right) \\ + \left(\left|u_{\varepsilon lm}(t)\right|^{\sigma} u_{\varepsilon lm}(t), w_j^l\right) \\ + \left(h_l\left(\cdot, u_{\varepsilon lm}'(t)\right), w_j^l\right)_{L^2(\Gamma_1)} = 0, \quad j = 1, 2, \ldots, m, \\ u_{\varepsilon lm}(0) = u^0, \\ u_{\varepsilon lm}'(0) = u_l^1.\end{gathered} \tag{AP}$$

The above finite-dimensional system has a solution, $u_{\varepsilon lm}(t)$, defined on $[0, t_{\varepsilon lm}[$. The following estimate allows us to extend this solution to the whole interval $[0, \infty)$. □

3.1. Estimates

3.1.1. First Estimate. Considering $w = 2u_{\varepsilon lm}'(t)$ in $(AP)_1$, integrating from 0 to t $(0 \le t < t_{\varepsilon lm})$, using the fact that $h_l(x,s)s \ge h_0 s^2$ (see Part (ii) of Lemma 7), assumptions $(H4)$ and $(H5)$, and since $0 < \varepsilon < 1$, we obtain

$$\begin{aligned}&\left(K(t), u_{\varepsilon lm}'^2(t)\right) + \varepsilon\left|u_{\varepsilon lm}'(t)\right|^2 + \left\|u_{\varepsilon lm}(t)\right\|^2 \\ &\quad + \frac{2}{\sigma+2}\int_\Omega \left|u_{\varepsilon lm}(t)\right|^{\sigma+2} dx \\ &\quad + 2h_0 \int_0^t \int_{\Gamma_1} \left(u_{\varepsilon lm}'(x,s)\right)^2 d\Gamma\, ds \\ &\quad + 2\int_0^t \sum_{j=1}^{n} \int_\Omega b_j(s) \frac{\partial u_{\varepsilon lm}'(s)}{\partial x_j} u_{\varepsilon lm}'(s)\, dx\, ds \\ &\le \left|u_l^1\right|^2 + \left\|u^0\right\|^2 + |K(0)| \left|u_l^1\right|^2 + \frac{2}{\sigma+2}\int_\Omega \left|u^0\right|^{\sigma+2} dx \\ &\quad + \int_0^t \left[\delta\left|u_{\varepsilon lm}'(s)\right|^2 + C(\delta)\left(K(s), u_{\varepsilon lm}'^2(s)\right)\right] ds.\end{aligned} \tag{31}$$

Note that

$$\begin{aligned}\sum_{j=1}^{n}\int_\Omega b_j(t) \frac{\partial u_{\varepsilon lm}'}{\partial x_j}(t)\, u_{\varepsilon lm}'(t)\, dx = &-\frac{1}{2}\int_\Omega \operatorname{div}(b) \left|u_{\varepsilon lm}'\right|^2 dx \\ &+ \frac{1}{2}\int_{\Gamma_1} b\cdot\nu \left|u_{\varepsilon lm}'\right|^2 d\Gamma.\end{aligned} \tag{32}$$

In fact, by the Gauss's formula we have

$$\begin{aligned}\int_\Omega b_j u_{\varepsilon lm}' \frac{\partial u_{\varepsilon lm}'}{\partial x_j} dx = &-\frac{1}{2}\int_\Omega \frac{\partial b_j}{\partial x_j}\left|u_{\varepsilon lm}'\right|^2 dx \\ &+ \frac{1}{2}\int_{\Gamma_1} b_j \nu_j \left|u_{\varepsilon lm}'\right|^2 d\Gamma,\end{aligned} \tag{33}$$

where ν_j is the jth entry of the normal vector ν. Hence, by $(A2)$ we obtain

$$\begin{aligned}&\int_0^t \sum_{j=1}^{n}\int_\Omega b_j u_{\varepsilon lm}' \frac{\partial u_{\varepsilon lm}'(s)}{\partial x_j} dx\, ds \\ &\quad \ge \frac{b_0}{2}\int_0^t \int_\Omega \left|u_{\varepsilon lm}'(s)\right|^2 dx\, ds \\ &\qquad + \frac{1}{2}\int_0^t \int_{\Gamma_1} b(s)\cdot\nu \left|u_{\varepsilon lm}'(t)\right|^2 d\Gamma\, ds.\end{aligned} \tag{34}$$

Using the hypothesis $\operatorname{div} b(t) \le -b_0$, choosing $\delta = b_0/4 > 0$, and plugging (34) in (31), we find

$$\begin{aligned}&\left(K(t), u_{\varepsilon lm}'^2(t)\right) + \varepsilon\left|u_{\varepsilon lm}'(t)\right|^2 + \left\|u_{\varepsilon lm}(t)\right\|^2 \\ &\quad + \frac{b_0}{2}\int_0^t \left|u_{\varepsilon lm}'(s)\right|^2 ds + \frac{2}{\sigma+2}\int_\Omega \left|u_{\varepsilon lm}(t)\right|^{\sigma+2} dx \\ &\quad + d_0 \int_0^t \int_{\Gamma_1} \left(u_{\varepsilon lm}'(x,s)\right)^2 d\Gamma\, ds \\ &\le c_0\left|u^1\right|^2 + \left\|u^0\right\|^2 + c_1\left\|u^0\right\|^{\sigma+2} \\ &\quad + C(\delta)\int_0^t \left(K(s), u_{\varepsilon lm}'^2(s)\right) ds,\end{aligned} \tag{35}$$

where $d_0 = 2h_0 - \|b\|_\infty/2 > 0$ for all $l \ge l_0$, $\forall m$ and $0 < \varepsilon < 1$. Moreover, $\|v\|_{L^{\sigma+2}(\Omega)} \le c_1\|v\|$ for all $v \in V$. Therefore by the Gronwall's inequality and (35)

$$\begin{gathered}\left(K^{1/2} u_{\varepsilon lm}'\right) \text{ is bounded in } L^\infty_{\text{loc}}\left(0,\infty; L^2(\Omega)\right), \\ \left(u_{\varepsilon lm}\right) \text{ is bounded in } L^\infty_{\text{loc}}\left(0,\infty; V\right), \\ \left(u_{\varepsilon lm}'\right) \text{ is bounded in } L^2_{\text{loc}}\left(0,\infty; L^2(\Gamma_1)\right).\end{gathered} \tag{36}$$

3.1.2. Second Estimate. Differentiating with respect to t the approximate equation $(AP)_1$ and taking $w = 2u_{\varepsilon lm}''(t)$, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(K(t) u_{\varepsilon lm}''(t), u_{\varepsilon lm}''(t)\right) + \frac{d}{dt}\varepsilon\left|u_{\varepsilon lm}''(t)\right|^2 \\ &\quad + \frac{d}{dt}\left\|u_{\varepsilon lm}'(t)\right\|^2 + \left(K'(t) u_{\varepsilon lm}''(t), u_{\varepsilon lm}''(t)\right) \\ &\quad + 2\int_{\Gamma_1} h_l'\left(\cdot, u_{\varepsilon lm}'(t)\right)\left|u_{\varepsilon lm}''(t)\right|^2 d\Gamma\end{aligned}$$

$$+2\sum_{i=1}^{n}\left(b_i\frac{\partial u''_{\varepsilon lm}(t)}{\partial x_i},u''_{\varepsilon lm}(t)\right)$$
$$+2(\sigma+1)\left(\left|u_{\varepsilon lm}(t)\right|^{\sigma}u'_{\varepsilon lm}(t),u''_{\varepsilon lm}(t)\right)$$
$$+2\sum_{i=1}^{n}\left(b'_i(t)\frac{\partial u'_{\varepsilon lm}(t)}{\partial x_i}(t),u''_{\varepsilon lm}(t)(t)\right)=0. \tag{37}$$

Note that

$$\sum_{j=1}^{n}\left(b_j(t)\frac{\partial u''_{\varepsilon lm}(t)}{\partial x_j}(t),u''_{\varepsilon lm}(t)\right)$$
$$=-\frac{1}{2}\int_{\Omega}\operatorname{div}(b)\left|u''_{\varepsilon lm}(t)\right|^2dx+\frac{1}{2}\int_{\Gamma_1}b(t)\cdot\nu\left|u''_{\varepsilon lm}(t)\right|^2d\Gamma \tag{38}$$

and that

$$\left|\sum_{j=1}^{n}\left(b'_j(t)\frac{\partial u'_{\varepsilon lm}(t)}{\partial x_j},u''_{\varepsilon lm}(t)\right)\right|$$
$$\leq n\max_{1\leq j\leq n}\left\|b'_j\right\|_{\infty}\left|\frac{\partial u'_{\varepsilon lm}}{\partial x_j}(t)\right|\left|u''_{\varepsilon lm}(t)\right|$$
$$\leq n\max_{1\leq j\leq n}\left\|b'_j\right\|_{\infty}\left|\nabla u'_{\varepsilon lm}(t)\right|\left|u''_{\varepsilon lm}(t)\right|$$
$$\leq\frac{n^{1/2}\max_{1\leq j\leq n}\left\|b'_j\right\|_{\infty}}{2}C_{\rho}\left|\nabla u'_{\varepsilon lm}(t)\right|^2+\rho\left|u''_{\varepsilon lm}(t)\right|^2, \tag{39}$$

where the last inequality becomes from Young's inequality. Therefore, if

$$K_2:=C_{\rho}\frac{n^{1/2}\max_{1\leq j\leq n}\left\|b_j\right\|_{\infty}}{2}, \tag{40}$$

then $|\sum_{j=1}^{n}(b'_j(t)(\partial u'_{\varepsilon lm}(t)/\partial x_j)(t),u''_{\varepsilon lm}(t))|\leq K_2\|u'_{\varepsilon lm}(t)\|^2+\rho|u''_{\varepsilon lm}(t)|^2$. Combining this inequality and (38) with (37), after that using that $h'_l(\cdot,s)\geq h_0$, $\operatorname{div}b(t)\leq -b_0$ and integrating from 0 to t, we obtain

$$\left(K(t),u''^2_{\varepsilon lm}(t)\right)+\varepsilon\left|u''_{\varepsilon lm}(t)\right|^2+\left\|u'_{\varepsilon lm}(t)\right\|^2$$
$$+\left(\frac{b_0}{2}-\rho\right)\int_0^t\left|u''_{\varepsilon lm}(s)\right|^2ds$$
$$+\left(2h_0-\frac{\|b\|_{\infty}}{2}\right)\int_0^t\int_{\Gamma_1}\left|u''_{\varepsilon lm}(s)\right|^2d\Gamma\,ds$$
$$\leq\left|u''_{\varepsilon lm}(0)\right|^2+\left\|u_l^1\right\|^2+\left|K^{1/2}(0)\right|\left|u''_{\varepsilon lm}(0)\right|$$
$$-2(\sigma+1)\int_0^t\left(\left|u_{\varepsilon lm}(s)\right|^{\sigma}u'_{\varepsilon lm}(s),u''_{\varepsilon lm}(s)\right)ds$$
$$+\delta\int_0^t\left|u''_{\varepsilon lm}(s)\right|^2ds+K_2\int_0^t\left\|u'_{\varepsilon lm}(s)\right\|^2$$
$$+\int_0^tC(\delta)\left(K(s),u''^2_{\varepsilon lm}(s)\right)ds. \tag{41}$$

We also have used the above hypotheses $(H4)$ and $(H5)$. Now, by Hölder's inequality for $1/q_1^*+1/n+1/2=1$, embedding (18) and first estimate (36), we find

$$\left|\left(\left|u_{\varepsilon lm}(t)\right|^{\sigma}u'_{\varepsilon lm}(t),u''_{\varepsilon lm}(t)\right)\right|$$
$$\leq\left(\int_{\Omega}\left|u_{\varepsilon lm}(t)\right|^{\sigma n}\right)^{1/n}\left(\int_{\Omega}\left|u'_{\varepsilon lm}(t)\right|^{q_1^*}\right)^{1/q_1^*}$$
$$\cdot\left(\int_{\Omega}\left|u''_{\varepsilon lm}(t)\right|^2\right)^{1/2} \tag{42}$$
$$\leq C\left\|u_{\varepsilon lm}(t)\right\|^{\sigma}\left\|u'_{\varepsilon lm}(t)\right\|\left|u''_{\varepsilon lm}(t)\right|$$
$$\leq C\left\|u'_{\varepsilon lm}(t)\right\|\left|u''_{\varepsilon lm}(t)\right|$$
$$\leq C_{\eta}\left\|u'_{lm}(t)\right\|^2+\eta\left|u''_{lm}(t)\right|^2,$$

where the constant C is independent of l, m, ε and $t\geq 0$, and C_{η} is a positive constant that depends on η. Substituting this inequality in (41) yields

$$\left(K(t)u''_{\varepsilon lm}(t),u''_{\varepsilon lm}(t)\right)+\varepsilon\left|u''_{\varepsilon lm}(t)\right|^2$$
$$+\left\|u'_{\varepsilon lm}(t)\right\|^2\left(\frac{b_0}{2}-(\rho+\eta)\right)\int_0^t\left|u''_{\varepsilon lm}(s)\right|^2ds$$
$$+\left(2h_0-\frac{\|b\|_{\infty}}{2}\right)\int_0^t\int_{\Gamma_1}\left|u''_{\varepsilon lm}(s)\right|^2d\Gamma\,ds$$
$$\leq\left|u''_{\varepsilon lm}(0)\right|^2+\left\|u_l^1\right\|^2+\sup_{x\in\Omega}K(x,0)\left|u''_{\varepsilon lm}(0)\right|^2 \tag{43}$$
$$+K_2\int_0^t\left\|u'_{\varepsilon lm}(s)\right\|^2+\delta\int_0^t\left|u''_{\varepsilon lm}(s)\right|^2ds$$
$$+\int_0^tC(\delta)\left(K(s)u''_{\varepsilon lm}(s),u''_{\varepsilon lm}(s)\right)ds.$$

Choosing $\delta=b_0/4$ and ρ,η small such that $b_1=(b_0/2-(\rho+\eta))>0$, we find

$$\left(K(t),u''^2_{\varepsilon lm}(t)\right)+\varepsilon\left|u''_{\varepsilon lm}(t)\right|^2+\left\|u'_{\varepsilon lm}(t)\right\|^2$$
$$+b_1\int_0^t\left|u''_{\varepsilon lm}(s)\right|^2ds+d_0\int_0^t\int_{\Gamma_1}\left|u''_{\varepsilon lm}(s)\right|^2d\Gamma\,ds$$
$$\leq\left|u''_{\varepsilon lm}(0)\right|^2+\left\|u_l^1\right\|^2+\sup_{x\in\Omega}K(x,0)\left|u''_{\varepsilon lm}(0)\right|^2$$
$$+K_2\int_0^t\left\|u'_{\varepsilon lm}(s)\right\|^2+C(\delta)\left(K(s)u''_{\varepsilon lm}(s),u''_{\varepsilon lm}(s)\right)ds. \tag{44}$$

Now we need to derive an estimate for $(u''_{\varepsilon lm}(0))$. Thus, taking $t = 0$ in approximate Problem $(AP)_1$ and choosing $v = u''_{\varepsilon lm}(0)$, one has

$$\begin{aligned}&\left((K(0)+\varepsilon)u''_{\varepsilon lm}(0), u''_{\varepsilon lm}(0)\right)+\left(\left(u^0, u''_{\varepsilon lm}(0)\right)\right)\\&\quad+\int_{\Gamma_1} h_l\left(\cdot, u_l^1\right)u''_{\varepsilon lm}(0)\,d\Gamma+\sum_{i=1}^{n}\left(b_j(0)\frac{\partial u_l^1}{\partial x_i}, u''_{\varepsilon lm}(0)\right)\\&\quad+\left(\left|u^0\right|^{\sigma}u^0, u''_{\varepsilon lm}(0)\right)=0.\end{aligned} \tag{45}$$

Applying Green's formula

$$\begin{aligned}&\left((K(0)+\varepsilon)u''_{\varepsilon lm}(0), u''_{\varepsilon lm}(0)\right)-\left(\Delta u^0, u''_{\varepsilon lm}(0)\right)\\&\quad+\int_{\Gamma_1}\left[\frac{\partial u^0}{\partial \nu}+h_l\left(\cdot, u_l^1\right)\right]u''_{\varepsilon lm}(0)\,d\Gamma\\&\quad+\sum_{i=1}^{n}\left(b_j(0)\frac{\partial u_l^1}{\partial x_i}, u''_{\varepsilon lm}(0)\right)+\left(\left|u^0\right|^{\sigma}u^0, u''_{\varepsilon lm}(0)\right)=0.\end{aligned} \tag{46}$$

Thanks to (30), the integrals on Γ_1 in (46) are null. Using convergence (29), embedding (17), and the hypothesis $(H4)$ give

$$\left|u''_{\varepsilon lm}(0)\right|\le C,\quad \forall l, m, \varepsilon. \tag{47}$$

From this boundedness and (46), we get

$$\begin{aligned}&\left(K(t)u''_{\varepsilon lm}(t), u''_{\varepsilon lm}(t)\right)+\varepsilon\left|u''_{\varepsilon lm}(t)\right|^2+\left\|u'_{\varepsilon lm}(t)\right\|^2\\&\quad+b_1\int_0^t\left|u''_{\varepsilon lm}(s)\right|^2ds+d_0\int_0^t\int_{\Gamma_1}\left|u''_{\varepsilon lm}(s)\right|^2d\Gamma\,ds\\&\le C_1+K_2\int_0^t\left\|u'_{\varepsilon lm}(s)\right\|^2\\&\quad+\int_0^t C(\delta)\left(K(s)u''_{\varepsilon lm}(t), u''_{\varepsilon lm}(s)\right)ds,\end{aligned} \tag{48}$$

where C_1 is constant independent of l, m, and ε. Applying Gronwall's inequality in (48) and using estimate (47), we have

$$\begin{aligned}&\left(K^{1/2}u''_{\varepsilon lm}\right)\text{ is bounded in } L^{\infty}_{\text{loc}}\left(0,\infty; L^2(\Omega)\right);\\&\left(u''_{\varepsilon lm}\right)\text{ is bounded in } L^{2}_{\text{loc}}\left(0,\infty; L^2(\Omega)\right);\\&\left(u'_{\varepsilon lm}\right)\text{ is bounded in } L^{\infty}_{\text{loc}}\left(0,\infty; V\right);\\&\left(u''_{\varepsilon lm}\right)\text{ is bounded in } L^{2}_{\text{loc}}\left(0,\infty; L^2(\Gamma_1)\right).\end{aligned} \tag{49}$$

From estimates (36), (49), induction, and diagonal process, we obtain a subsequence of $(u_{\varepsilon lm})$, which is still denoted by $(u_{\varepsilon lm})$, and a function $u_{\varepsilon l}:\Omega\times]0,\infty[\to\mathbb{R}$, such that

$$\begin{aligned}u_{\varepsilon lm}&\longrightarrow u_{\varepsilon l}\text{ weak star in } L^{\infty}_{\text{loc}}(0,\infty;V);\\u'_{\varepsilon lm}&\longrightarrow u'_{\varepsilon l}\text{ weak star in } L^{\infty}_{\text{loc}}(0,\infty;V);\\Ku''_{\varepsilon lm}&\longrightarrow Ku''_{\varepsilon l}\text{ weak star in } L^{\infty}_{\text{loc}}\left(0,\infty;L^2(\Omega)\right);\\u'_{\varepsilon lm}&\longrightarrow u'_{\varepsilon l}\text{ weak in } L^{2}_{\text{loc}}\left(0,\infty;L^2(\Gamma_1)\right);\\u''_{\varepsilon lm}&\longrightarrow u''_{\varepsilon l}\text{ weak in } L^{2}_{\text{loc}}\left(0,\infty;L^2(\Gamma_1)\right).\end{aligned} \tag{50}$$

From the convergence $(49)_2$ we obtain

$$\begin{aligned}&\sum_{i=1}^{n}b_j\cdot\frac{\partial u'_{\varepsilon lm}}{\partial x_j}\\&\quad\longrightarrow\sum_{i=1}^{n}b_j\cdot\frac{\partial u'_{\varepsilon l}}{\partial x_j}\text{ weak star in } L^{\infty}_{\text{loc}}\left(0,\infty;L^2(\Omega)\right).\end{aligned} \tag{51}$$

3.1.3. Analysis of the Nonlinear Terms. By estimates $(36)_1$, $(36)_2$, compactness method (cf. Lions [19] or Simon [20]), embedding (17), induction, and diagonal process, we obtain a subsequence of $(u_{\varepsilon lm})$, which also is denoted by $(u_{\varepsilon lm})$, such that

$$\begin{aligned}&\left|u_{\varepsilon lm}\right|^{\sigma}u_{lm}\\&\quad\longrightarrow\left|u_{\varepsilon l}\right|^{\sigma}u_{\varepsilon l}\text{ weak star in } L^{\infty}_{\text{loc}}\left(0,\infty;L^2(\Omega)\right).\end{aligned} \tag{52}$$

From $(48)_1$, we have that $(u'_{\varepsilon lm})$ is bounded in $L^{\infty}_{\text{loc}}(0,\infty; H^{1/2}(\Gamma_1))$. Thus, estimate $(49)_3$ and the compactness embedding of $H^{1/2}(\Gamma_1)$ in $L^2(\Gamma_1)$, give $u'_{\varepsilon lm}\to u'_{\varepsilon l}$ in $L^2_{\text{loc}}(0,\infty; L^2(\Gamma_1))$. From this, property (iii) and Lemma 7 yield

$$h_l\left(\cdot, u'_{\varepsilon lm}\right)\longrightarrow h_l\left(\cdot, u'_{\varepsilon l}\right)\quad\text{in } L^2_{\text{loc}}\left(0,\infty;L^2(\Gamma_1)\right). \tag{53}$$

3.1.4. Passage to the Limit as m. Convergences (49), (51), and (53) permit us to pass to the limits in approximate equations (AP), as $m\to\infty$. Thus, this fact and the density of V_m^l in V, give

$$\begin{aligned}&\int_0^T\left((K+\varepsilon)u''_{\varepsilon l}(t), v\right)\psi(t)\,dt+\int_0^T\left(\left(u_{\varepsilon l}(t), v\right)\right)\psi(t)\,dt\\&\quad+\int_0^T\int_{\Gamma_1}h_l\left(\cdot, u'_{\varepsilon l}(t)\right)v\psi(t)\,d\Gamma\,dt\\&\quad+\sum_{i=1}^{n}\int_0^T\left(b_i\frac{\partial u'_{\varepsilon l}}{\partial x_i}(t), v\right)\psi(t)\,dt\\&\quad+\int_0^T\left(\left(\left|u_{\varepsilon l}(t)\right|^{\sigma}u_{\varepsilon}(t)\right), v\right)\psi(t)=0,\\&\qquad\forall\psi\in\mathscr{D}(0,T),\quad\forall v\in V.\end{aligned} \tag{54}$$

Now, if $\psi \in \mathscr{D}(0,T)$ and $v \in \mathscr{D}(\Omega)$ we obtain using the regularity of $u_{\varepsilon l}$ (given by (49)), that

$$(K+\varepsilon)\,u''_{\varepsilon l} - \Delta u_{\varepsilon l} + \sum_{i=1}^{n} b_i \frac{\partial u'_{\varepsilon l}}{\partial x_i} + \left|u_{\varepsilon l}\right|^{\sigma} u_{\varepsilon l} = 0 \quad \text{in } L^{\infty}_{\text{loc}}\left(0,\infty; L^2(\Omega)\right). \tag{55}$$

From $(49)_1$ and (55) we have $u_{\varepsilon l} \in L^{\infty}(0,T;V)$ and $\Delta u_{\varepsilon l} \in L^{\infty}(0,T;L^2(\Omega))$, respectively. Thus, $\partial u_{\varepsilon l}/\partial \nu \in L^{\infty}(0,T;H^{-1/2}(\Gamma_1))$. (compare to Lions [19] and Medeiros and Milla Miranda [21]). Multiplying both sides of (55) by $v\psi$, with $v \in V$ and $\psi \in \mathscr{D}(0,T)$, and integrating over $\Omega \times [0,T[$, then the preceding regularity, $\partial u_{\varepsilon l}/\partial \nu$, gives

$$\int_0^T \left((K+\varepsilon)\,u''_{\varepsilon l}, v\right)\psi\,dt + \int_0^T \left(\left(u_{\varepsilon l}, v\right)\right)\psi\,dt - \int_0^T \left\langle \frac{\partial u_{\varepsilon l}}{\partial \nu}, v\right\rangle \psi\,dt + \int_0^T \left(\left|u_{\varepsilon l}\right|^{\sigma} u_{\varepsilon l}, v\right)\psi\,dt + \sum_{i=1}^{n}\int_0^T \left(b_i\frac{\partial u'_{\varepsilon l}}{\partial x_i}, v\right)\psi\,dt = 0, \tag{56}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality paring between $H^{-1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$.

Comparing (54) and (56) and using the Lipschitz property of h_l, we obtain

$$\frac{\partial u_{\varepsilon l}}{\partial \nu} + h_l\left(\cdot, u'_{\varepsilon l}\right) = 0 \quad \text{in } L^{2}_{\text{loc}}\left(0,\infty; L^2(\Gamma_1)\right). \tag{57}$$

3.1.5. Passage to the Limit in $\varepsilon \to 0$ and $l \to \infty$. As estimates (36) and (49) are independent of l, m, and ε we obtain a subsequence of $(u_{\varepsilon l})$, which still denoted by $(u_{\varepsilon l})$, and a function u_l such that all convergences (49) and (52) are valid. These convergences will be denoted by $(49)_\varepsilon$, $(51)_\varepsilon$, and $(52)_\varepsilon$, respectively. These results imply that there exists a function u_l belonging to class (49) and it is a solution of equation

$$K u''_l - \Delta u_l + \sum_{i=1}^{n} b_i \frac{\partial u'_l}{\partial x_i} + \left|u_l\right|^{\sigma} u_l = 0 \quad \text{in } L^{\infty}_{\text{loc}}\left(0,\infty; L^2(\Omega)\right), \tag{58}$$

$$\frac{\partial u_l}{\partial \nu} + h_l\left(\cdot, u'_l\right) = 0 \quad \text{in } L^{2}_{\text{loc}}\left(0,\infty; L^2(\Gamma_1)\right). \tag{59}$$

Denoting these convergence in l by $(49)_l$, $(51)_l$, and $(52)_l$, respectively, then the convergence $(49)_l$ gives us $u_l \to u$ weak star in $L^{\infty}(0,\infty;V)$. From this and (55), $\Delta u_l \to \Delta u$ weak star in $L^{\infty}_{\text{loc}}(0,\infty;L^2(\Omega))$. Then

$$\frac{\partial u_l}{\partial \nu} \longrightarrow \frac{\partial u}{\partial \nu} \text{ weak star in } L^{\infty}_{\text{loc}}\left(0,\infty; H^{-1/2}(\Gamma_1)\right). \tag{60}$$

Moreover, convergence $(49)_l$ furnishes $u'_l \to u'$ in $L^2_{\text{loc}}(0,\infty;L^2(\Gamma_1))$. Now, we fix $T > 0$. The preceding convergence implies

$$u'_l(x,t) \longrightarrow u'(x,t) \quad \text{a.e in } \Sigma_1 = \Gamma_1 \times]0,T[\,. \tag{61}$$

Fixing $(x,t) \in \Sigma_1$, then by (61) the set $\{u'_l(x,t) : l \in \mathbb{N}\}$ is bounded. Part (iv) of Lemma 7 says that (h_l) converges to h uniformly in bounded sets of $\mathbb{R}$, a.e. x in Γ_1. These two results and (61) give

$$h_l\left(x, u'_l(x,t)\right) \longrightarrow h\left(x, u'(x,t)\right) \quad \text{a.e in } \Sigma_1 = \Gamma_1 \times]0,T[\,. \tag{62}$$

On the other hand, by (58) and (59), we obtain

$$\int_{\Gamma_1} h_l\left(\cdot, u'_l(t)\right) u'_l(t)\,d\Gamma = -\left(K(t)\,u''_l(t), u'_l(t)\right) - \frac{1}{2}\frac{d}{dt}\left\|u_l(t)\right\|^2 - \sum_{i=1}^{n}\left(b_i\frac{\partial u'_l(t)}{\partial x_i}, u'_l(t)\right) - \left(\left|u_l(t)\right|^{\sigma} u_l(t), u'_l(t)\right). \tag{63}$$

By familiar inequalities,

$$\left|\sum_{i=1}^{n}\left(b_i\frac{\partial u'_l(t)}{\partial x_i}, u'_l(t)\right)\right| \le C\left[\left\|u'_l(t)\right\|^2 + \left|u'_l(t)\right|^2\right], \tag{64}$$

and from embedding (17),

$$\left|\left(\left|u_l(t)\right|^{\sigma} u_l(t), u'_l(t)\right)\right| \le C\left[\left\|u_l(t)\right\|^{2(\sigma+2)} + \left|u'_l(t)\right|^2\right], \tag{65}$$

where $C > 0$ is a constant independent of l and $t \in [0,T]$.

As

$$\left(K(t)\,u''_l(t), u'_l(t)\right) = \frac{1}{2}\frac{d}{dt}\left(K(t)\,u'_l(t), u'_l(t)\right) - \frac{1}{2}\left(K'(t)\,u'_l(t), u'_l(t)\right), \tag{66}$$

integrating the inequality above from 0 to T, and using the hypothesis $(H5)$ and $(H6)$, we find

$$\begin{aligned}
\int_0^T \left(K(t)\,u''_l(t), u'_l(t)\right)dt &= \frac{1}{2}\left[\left(K(T)\,u'_l(T), u'_l(T)\right) - \left(K(0)\,u^1_l, u^1_l\right)\right] - \frac{1}{2}\int_0^T \left(K'(t)\,u'_l(t), u'_l(t)\right)dt \\
&\le \|K\|_\infty\left[\left|u'_l(T)\right|^2 + \left|u^1_l\right|^2\right] + \int_0^T \left[\delta\left|u'_l(s)\right|^2 + C(\delta)\left(K(s), u'^2_l(s)\right)\right]ds \\
&\le \|K\|_\infty\left[\left|u'_l(T)\right|^2 + \left|u^1_l\right|^2\right] + C\left(\delta, \|K\|_\infty\right)\int_0^T \left|u'_l(t)\right|^2 dt.
\end{aligned} \tag{67}$$

Note that $u_l \in C^0([0,T];V)$, $u_l' \in C^0([0,T];L^2(\Omega))$ and that $(u_l(T))$ and $(u_l'(T))$ are bounded in V and in $L^2(\Omega)$, respectively. Taking into account the preceding considerations, estimates $(49)_l$ and convergence $(36)_l$ in (63), we obtain

$$\int_0^T \int_{\Gamma_1} h_l\left(\cdot, u_l'(t)\right) u_l'(t)\, d\Gamma\, dt \le C, \qquad \forall l \ge l_0, \quad \forall t \in [0,T], \tag{68}$$

where $C > 0$ is a constant independent of $l \ge l_0$ and $t \in [0,T]$. Note that $h_l(\cdot, u_l'(t))u_l'(t) \ge 0$.

From (62), (68), Strauss' approximations, Lemma 9 and from a diagonal process, we get

$$h_l\left(\cdot, u_l'\right) \longrightarrow h\left(\cdot, u'\right) \quad \text{in } L^1_{\text{loc}}\left(0,\infty; L^1(\Gamma_1)\right). \tag{69}$$

Convergences (60) and (69) imply $\partial u_l/\partial\nu \to \partial u/\partial\nu$ in $\mathscr{D}'(\Gamma_1\times]0,\infty[)$ and $h_l(\cdot,u_l') \to h(\cdot,u')$ in $\mathscr{D}'(\Gamma_1\times]0,\infty[)$. Now we take the limit in (59). Moreover the last two convergences and the regularity of $h(\cdot,u')$ imply $\partial u/\partial\nu + h(\cdot,u') = 0$ in $L^1_{\text{loc}}(0,\infty;L^1(\Gamma_1))$, which shows that u satisfies

$$Ku'' - \Delta u + \sum_{i=1}^n b_i \frac{\partial u'}{\partial x_i} + |u|^\sigma u = 0 \qquad \text{in } L^\infty_{\text{loc}}\left(0,\infty; L^2(\Omega)\right). \tag{70}$$

Hence, the result is done as in (58).

The verification of the initial conditions follows by convergence $(49)_l$.

Remark 10. If $|h(x,s)| \le h_1|s|$ a.e. $x \in \Gamma$, then the sequence h_l, which converges to h, satisfies $|h_l(x,s)| \le (3/2)h_1|s|$ (see Lourêdo and Miranda [10]). In these conditions, the solution is unique and $\partial u/\partial\nu + h(\cdot,u') = 0$ in $L^2_{\text{loc}}(0,\infty;L^2(\Gamma_1))$. Consequently, $u \in L^\infty(0,\infty, V \cap H^{3/2}(\Omega))$.

The proof of Corollary 2 follows from Remark 10 and from regularity of elliptic problems (see Lions and Magenes [22]).

4. Asymptotic Behavior

In this section, by applying Nakao's method (see [23]), we will prove the uniform stabilization of the energy associated with the solution of the Problem (P).

Proof of Theorem 6. First, we prove the inequality (15) for the approximate energy $E_l(t)$ given by

$$E_l(t) = \frac{1}{2}\left\{K^{1/2}(t)\left|u_l'(t)\right|^2 + \left\|u_l(t)\right\|^2 + \frac{2}{\sigma+2}\int_\Omega \left|u_l(t)\right|^{\sigma+2} dx\right\} \quad \text{for } t \ge 1, \tag{71}$$

and Theorem 6 will follow by taking the lim inf in l.

Taking the scalar product of $L^2(\Omega)$ in both sides of

$$Ku_l'' - \Delta u_l + \sum_{j=1}^n b_j \frac{\partial u_l'}{\partial x_j} + |u_l|^\sigma u_l = 0 \qquad \text{in } L^\infty_{\text{loc}}\left(0,\infty; L^2(\Omega)\right) \tag{72}$$

with u_l', we find

$$\begin{aligned} \frac{d}{dt}E_l(t) = {}& \frac{1}{2}\left(\frac{\partial K(t)}{\partial t}u_l'(t), u_l'(t)\right) - \left(h_l\left(\cdot, u_l'(t)\right), u_l'(t)\right)_{\Gamma_1} \\ & + \frac{1}{2}\int_\Omega \operatorname{div} b(t)\left|u_l'(t)\right|^2 dx - \frac{1}{2}\int_{\Gamma_1} b(t)\cdot\nu\left|u_l(t)\right|^2 dx. \end{aligned} \tag{73}$$

Using (73) and the hypotheses $(A2)$, $(H3)$, $(H8)$, and $(H2)$, we obtain

$$\frac{d}{dt}E_l(t) \le -\frac{b_0}{4}\left|u_l'(t)\right|^2 + \left(\frac{\|b\|_\infty}{2} - h_0\right)\left|u_l'(t)\right|^2_{\Gamma_1} \le 0. \tag{74}$$

Note that $E_l(t)$ is decreasing. From (73), the hypotheses $(H1)$, $(H2)$, and Remark 10, we have that

$$\frac{d}{dt}E_l(t) \le \frac{b_0}{2}\left|u_l'(t)\right|^2 + d_1\left|u_l'(t)\right|^2_{\Gamma_1} + \frac{\delta_0}{2}\left|u_l'(t)\right|^2 + \frac{\|b\|_\infty}{2}\left|u_l'(t)\right|^2_{\Gamma_1}. \tag{75}$$

Thus,

$$\frac{d}{dt}E_l(t) \le N_1\left|u_l'(t)\right|^2 + N_2\left|u_l'(t)\right|^2_{\Gamma_1}, \tag{76}$$

where $N_1 = (b_0/2 + \delta_0/2) > 0$ and $N_2 = d_1 + (\|b\|_\infty/2) > 0$. Integrating (76) from t to $t+1$, we obtain

$$E_l(t+1) - E_l(t) \le N_1\int_t^{t+1}\left|u_l'(s)\right|^2 ds + N_2\int_t^{t+1}\left|u_l'(s)\right|^2_{\Gamma_1} ds. \tag{77}$$

Furthermore, taking the scalar product in both sides of (72) with u_l, we find

$$\left(K(t)u_l''(t), u_l(t)\right) + \left\|u_l(t)\right\|^2 + \left(b\cdot\nabla u_l'(t), u_l(t)\right) + \left(h_l\left(\cdot, u_l'(t)\right), u_l(t)\right)_{\Gamma_1} + \left(\left|u_l(t)\right|^\sigma u_l(t), u_l(t)\right) = 0. \tag{78}$$

As

$$\left(K(t)u_l''(t), u_l(t)\right) = \frac{d}{dt}\left(K(t)u_l'(t), u_l(t)\right) - \left(K'(t)u_l'(t), u_l(t)\right) - \left(K(t)u_l'(t), u_l'(t)\right), \tag{79}$$

then

$$\begin{aligned}\|u_l(t)\|^2 &+ |u_l(t)|^{\sigma+2}_{L^{\sigma+2}(\Omega)} \\ &= -\left(b\cdot\nabla u_l'(t), u_l(t)\right) - \left(h_l\left(\cdot, u_l'(t)\right), u_l(t)\right)_{\Gamma_1} \\ &\quad + \left(K'(t)u_l'(t), u_l(t)\right) + \left(K(t)u_l'(t), u_l'(t)\right) \\ &\quad - \frac{d}{dt}\left(K(t)u_l'(t), u_l(t)\right).\end{aligned} \tag{80}$$

Now note that

$$\left(b\cdot\nabla u_l'(t), u_l(t)\right) = \int_\Omega \sum_{j=1}^n b_j(t)\frac{\partial u_l'}{\partial x_j}(t)\,u_l(t)\,dx \tag{81}$$

and by Gauss' formula, we have

$$\begin{aligned}\frac{\partial}{\partial x_j}\left(b_j(t)u_l(t)u_l'(t)\right) &= b_j(t)u_l(t)\frac{\partial u_l'}{\partial x_j}(t) \\ &\quad + \frac{\partial b_j(t)}{\partial x_j}u_l(t)u_l'(t) \\ &\quad + b_j(t)\frac{\partial u_l}{\partial x_j}(t)u_l'(t).\end{aligned} \tag{82}$$

Therefore,

$$\begin{aligned}\sum_{j=1}^n \int_\Omega b_j(t)u_l(t)\frac{\partial u_l'(t)}{\partial x_j}dx &= \int_\Gamma b(t)\cdot\nu u_l(t)u_l'(t)\,d\Gamma \\ &\quad - \int_\Omega (\operatorname{div} b(t))\,u_l(t)u_l'(t)\,dx \\ &\quad - \int_\Omega b(t)\cdot\nabla u_l(t)u_l'(t)\,dx.\end{aligned} \tag{83}$$

Hence,

$$\begin{aligned}\left(b\cdot\nabla u_l'(t), u_l(t)\right) &= \int_{\Gamma_1} b(t)\nu u_l(t)u_l'(t)\,d\Gamma \\ &\quad - \int_\Omega (\operatorname{div} b(t))\,u_l(t)u_l'(t)\,dx \\ &\quad - \int_\Omega b(t)\cdot\nabla u_l(t)u_l'(t)\,dx.\end{aligned} \tag{84}$$

Substituting (84) in (80), we obtain

$$\begin{aligned}\|u_l(t)\|^2 &+ |u_l(t)|^{\sigma+2}_{L^{\sigma+2}(\Omega)} \\ &= -\int_{\Gamma_1} b(t)\cdot\nu u_l(t)u_l'(t)\,d\Gamma + \int_\Omega (\operatorname{div} b(t))\,u_l(t)u_l'(t)\,dx \\ &\quad + \int_\Omega b(t)\cdot\nabla u_l(t)u_l'(t)\,dx - \left(h_l\left(\cdot,u_l'(t)\right), u_l(t)\right)_{L^2(\Gamma_1)} \\ &\quad + \left(K'(t)u_l'(t), u_l(t)\right) + \left(K(t)u_l'(t), u_l'(t)\right) \\ &\quad - \frac{d}{dt}\left(K(t)u_l'(t), u_l(t)\right).\end{aligned} \tag{85}$$

Integrating (85) from t_1 to t_2 yields

$$\begin{aligned}&\int_{t_1}^{t_2}\left(\|u_l(t)\|^2 + |u_l(t)|^{\sigma+2}_{L^{\sigma+2}(\Omega)}\right)dt \\ &\le \|b\|_\infty \int_{t_1}^{t_2} |u_l(t)|_{\Gamma_1}\left|u_l'(t)\right|_{\Gamma_1}dt + \delta_0\int_{t_1}^{t_2}|u_l(t)|\left|u_l'(t)\right|dt \\ &\quad + \|b\|_\infty\int_{t_1}^{t_2}|\nabla u_l(t)|\left|u_l'(t)\right|dt \\ &\quad + \int_{t_1}^{t_2}\left|h_l\left(\cdot,u_l'(t)\right)\right|_{\Gamma_1}|u_l(t)|_{\Gamma_1}dt \\ &\quad + \int_{t_1}^{t_2}\frac{b_0}{2}\left|u_l'(t)\right||u_l(t)|\,dt + \|K\|_\infty\int_{t_1}^{t_2}\left|u_l'(t)\right|^2dt \\ &\quad - \left[\left(K(t_2)u_l'(t_2), u_l(t_2)\right) - \left(K(t_1)u_l'(t_1), u_l(t_1)\right)\right].\end{aligned} \tag{86}$$

Using the embedding of V in $L^2(\Omega)$ and V in $L^2(\Gamma_1)$ (see (12)), it follows that

(i) $\|b\|_\infty \int_{t_1}^{t_2}|u_l(t)|_{\Gamma_1}|u_l'(t)|_{\Gamma_1}dt \le 32\|b\|_\infty^2 k_1^2\int_{t_1}^{t_2}|u_l'(t)|^2_{\Gamma_1}dt + (1/32)\int_{t_1}^{t_2}\|u_l(t)\|^2dt$;

(ii) $\delta_0\int_{t_1}^{t_2}|u_l(t)||u_l'(t)|dt \le 32(1/\lambda_1)\int_{t_1}^{t_2}|u_l'(t)|^2dt + (1/32)\int_{t_1}^{t_2}\|u_l(t)\|^2dt$;

(iii) $\|b\|_\infty\int_{t_1}^{t_2}|\nabla u_l(t)||u_l'(t)|dt \le (\|b\|_\infty^2/2)\int_{t_1}^{t_2}\|u_l(t)\|^2dt + (1/2)\int_{t_1}^{t_2}|u_l'(t)|^2dt$.

(iv) $\int_{t_1}^{t_2}|h_l(\cdot,u_l'(t))|_{\Gamma_1}|u_l(t)|_{\Gamma_1}dt \le (3/2)h_1k_1\int_{t_1}^{t_2}|u_l'(t)|_{\Gamma_1}\cdot|u_l(t)|_{\Gamma_1}dt \le ((3/2)h_1k_1)^2(\lambda_1^{1/2})^2 16\int_{t_1}^{t_2}|u_l'(t)|^2_{\Gamma_1}dt + (1/32)\int_{t_1}^{t_2}\|u_l(t)\|^2dt$;

(v) $\int_{t_1}^{t_2}(b_0/2)|u_l'(t)||u_l(t)|dt \le 32(b_0/2)^2(1/\lambda_1)\int_{t_1}^{t_2}|u_l'(t)|^2 + (1/32)\int_{t_1}^{t_2}\|u_l(t)\|^2dt$;

(vi) $|(K(t_2)u_l'(t_2), u_l(t_2)) - (K(t_1)u_l'(t_1), u_l(t_1)| \le \|K\|_\infty\cdot[|u_l'(t_2)||u_l(t_2)| + |u_l'(t_1)||u_l(t_1)|]$.

Note that

$$|u_l(t_2)| \le \left(\frac{1}{\sqrt{\lambda_1}}\right)\|u_l(t)\| \le \left(\frac{1}{\sqrt{\lambda_1}}\right)E_l^{1/2}(t_2), \tag{87}$$

since $E_l(t)$ is decreasing and $t \le t_1, t_2 \le t+1$. Analogously, we obtain

$$|u_l(t_1)| \le \left(\frac{1}{\sqrt{\lambda_1}}\right)E_l^{1/2}(t_1). \tag{88}$$

Using (87) and (88) in (vi) we find

$$\left|\left(K(t_2)u_l'(t_2),u_l(t_2)\right)-\left(K(t_1)u_l'(t_1),u_l(t_1)\right)\right| \le \|K\|_\infty\left(\frac{1}{\lambda_1^{1/2}}\right)\sup_{t_1\le s\le t_2}E_l^{1/2}(s)\left[\left|u_l'(t_2)\right|+\left|u_l'(t_1)\right|\right]. \quad (89)$$

Substituting (i)–(vi) and (89) in (86), we obtain

$$\begin{aligned}\int_{t_1}^{t_2}\left[\|u_l(t)\|^2+|u_l(t)|_{L^{\sigma+2}(\Omega)}^{\sigma+2}\right]dt &\le \left(32+\|b\|_\infty\frac{1}{\lambda_1}\right)\int_{t_1}^{t_2}\|u_l(t)\|^2dt\\ &+\left(32\left(\frac{1}{\lambda_1}\right)^2+8b_0^2\left(\frac{1}{\lambda_1}\right)^2\right)\int_{t_1}^{t_2}\left|u_l'(t)\right|^2dt\\ &+\left(32\|b\|_\infty^2k_1^2+18(h_1k_1)^2\lambda_1+\frac{1}{2}\right)\int_{t_1}^{t_2}\left|u_l'(t)\right|_{\Gamma_1}^2dt\\ &+\|K\|_\infty\left(\frac{1}{\sqrt{\lambda_1}}\right)\sup_{t_1\le s\le t_2}E_l^{1/2}(s)\left[\left|u_l'(t_2)\right|+\left|u_l'(t_1)\right|\right].\end{aligned} \quad (90)$$

It follows from (74) that

$$\begin{aligned}E_l(t+1)-E_l(t)&\le -N_3\int_t^{t+1}\left|u_l'(s)\right|^2ds\\ &-N_4\int_t^{t+1}\left|u_l'(s)\right|_{\Gamma_1}ds,\end{aligned} \quad (91)$$

where $N_3=b_0>0$ and $N_4=h_0-\|b\|_\infty/2>0$. Hence

$$\begin{aligned}D_l^2(t)&=E_l(t)-E_l(t+1)\\ &\ge N_3\int_t^{t+1}\left|u_l'(s)\right|^2ds+N_4\int_t^{t+1}\left|u_l'(t)\right|_{\Gamma_1}^2ds\\ &\ge d_1\left[\int_t^{t+1}\left|u_l'(s)\right|^2ds+\int_t^{t+1}\left|u_l'(t)\right|_{\Gamma_1}^2ds\right],\end{aligned} \quad (92)$$

where $d_1=\min\{N_3,N_4\}>0$.

From (92), we obtain

$$\begin{aligned}\frac{D_l^2(t)}{d_1}&\ge\int_t^{t+1}\left|u_l'(s)\right|^2ds,\\ \frac{D_l^2(t)}{d_1}&\ge\int_t^{t+1}\left|u_l'(s)\right|_{\Gamma_1}^2ds,\end{aligned} \quad (93)$$

and from (93), we get

$$\begin{aligned}\int_t^{t+1/4}\left|u_l'(s)\right|_{\Gamma_1}^2ds&\le\int_t^{t+1}\left|u_l'(s)\right|_{\Gamma_1}^2ds\le\frac{D_l^2(t)}{d_1},\\ \int_{t+3/4}^{t+1}\left|u_l'(s)\right|_{\Gamma_1}^2ds&\le\int_t^{t+1}\left|u_l'(s)\right|_{\Gamma_1}^2ds\le\frac{D_l^2(t)}{d_1}.\end{aligned} \quad (94)$$

By the Mean Value Theorem, there are $t_1\in(t,t+1/4)$ and $t_2\in(t+3/4,t+1)$, such that

$$\begin{aligned}\frac{1}{4}\left|u_l'(t_1)\right|^2&=\int_t^{t+1/4}\left|u_l'(s)\right|^2ds\le\frac{D_l^2(t)}{d_1},\\ \frac{1}{4}\left|u_l'(t_2)\right|^2&=\int_{t+3/4}^{t+1}\left|u_l'(s)\right|^2ds\le\frac{D_l^2(t)}{d_1}.\end{aligned} \quad (95)$$

From (95), we can write

$$\left|u_l'(t_1)\right|+\left|u_l'(t_2)\right|\le\frac{2D_l(t)}{\sqrt{d_1}}. \quad (96)$$

Now from (93) we have

$$\begin{aligned}N_5\int_{t_1}^{t_2}\left|u_l'(s)\right|_{\Gamma_1}^2ds&\le N_5\int_t^{t+1}\left|u_l'(s)\right|_{\Gamma_1}^2ds\\ &\le\frac{N_5}{d_1}D_l^2(t),\end{aligned} \quad (97)$$

where $N_5=32\|b\|_\infty+18(h_1k_1)^2$.

Analogously we obtain

$$N_6\int_{t_1}^{t_2}\left|u_l'(s)\right|^2ds\le N_6\int_t^{t+1}\left|u_l'(s)\right|^2ds\le\frac{N_6}{d_1}D_l^2(t), \quad (98)$$

where $N_6=32(1/\lambda_1)+8b_0^2(1/\lambda_1)$.

Substituting (96), (97), and (98) in (90) and considering

$$N_7=\frac{9}{32}+\left(\frac{1}{\lambda_1}\right)\|b\|_\infty, \quad (99)$$

we get

$$\begin{aligned}\int_{t_1}^{t_2}\left[\|u_l(t)\|^2+|u_l(t)|_{L^{\sigma+2}(\Omega)}^{\sigma+2}\right]dt &\le N_7\int_{t_1}^{t_2}\|u_l(t)\|^2dt+N_5\int_{t_1}^{t_2}\left|u_l'(t)\right|^2dt\\ &+N_6\int_{t_1}^{t_2}\left|u_l'(t)\right|_{\Gamma_1}^2dt\\ &+\|K\|_\infty\left(\frac{1}{\sqrt{\lambda_1}}\right)\sup_{t_1\le s\le t_2}E_l^{1/2}\left[\left|u_l'(t_2)\right|+\left|u_l'(t_1)\right|\right]\\ &\le N_7\int_{t_1}^{t_2}\|u_l(t)\|^2dt+\frac{N_5}{d_1}D_l^2(t)+\frac{N_6}{d_1}D_l^2(t)\\ &+\frac{2\|K\|_\infty\left(1/\sqrt{\lambda_1}\right)}{\sqrt{d_1}}\sup_{t_1\le s\le t_2}E_l^{1/2}D_l(t)\\ &\le N_7\int_{t_1}^{t_2}\|u_l(t)\|^2dt+\left(\frac{N_5}{d_1}+\frac{N_6}{d_1}\right)D_l^2(t)\\ &+\frac{2\|K\|_\infty\left(1/\sqrt{\lambda_1}\right)}{\sqrt{d_1}}D_l(t)E_l^{1/2}(t).\end{aligned} \quad (100)$$

As $D_l^2(t) = E_l(t) - E_l(t+1) \leq E_l(t)$ then $D_l(t) \leq E_l^{1/2}(t)$, and substituting this inequality in (100), we obtain

$$\begin{aligned}\int_{t_1}^{t_2} \left(\|u_l(t)\|^2 + |u_l(t)|_{L^{\sigma+2}(\Omega)}^{\sigma+2}\right) dt &\leq N_7 \int_{t_1}^{t_2} \|u_l(t)\|^2 dt + \left(\frac{N_5+N_6}{d_1}\right) D_l^2(t) \\ &+ \frac{2\|K\|_\infty \left(1/\sqrt{\lambda_1}\right)}{\sqrt{d_1}} D_l(t) E_l(t).\end{aligned} \tag{101}$$

Replacing $t = t_1$ and $t+1 = t_2$ in (93), we have

$$\begin{aligned}\int_{t_1}^{t_2} K(t) \left|u_l'(t)\right|^2 &\leq \|K\|_\infty \int_{t_1}^{t_2} \left|u_l'(t)\right|^2 dt \\ &\leq \frac{\|K\|_\infty}{d_1} D_l^2(t).\end{aligned} \tag{102}$$

Adding (101) and (102), we obtain

$$\begin{aligned}\int_{t_1}^{t_2} \left[K(t)\left|u_l'(t)\right|^2 + \|u_l(t)\|^2 + |u_l(t)|_{L^{\sigma+2}(\Omega)}^{\sigma+2}\right] dt &\leq N_7 \int_{t_1}^{t_2} \|u_l(t)\|^2 dt + \left(\frac{\|K\|_\infty + N_5 + N_6}{d_1}\right) D_l^2(t) \\ &+ \frac{2\left(1/\sqrt{\lambda_1}\right)\|K\|_\infty}{\sqrt{d_1}} E_l(t),\end{aligned} \tag{103}$$

and this implies

$$\begin{aligned}\int_{t_1}^{t_2} \left[K(t)\left|u_l'(t)\right|^2 + (1-N_7)\|u_l(t)\|^2 + |u_l(t)|_{L^{\sigma+2}(\Omega)}^{\sigma+2}\right] dt \\ \leq N_8 D_l^2(t) + N_9 E_l(t),\end{aligned} \tag{104}$$

where

$$\begin{aligned} N_8 &= \frac{\|K\|_\infty + N_5 + N_6}{d_1}, \\ N_9 &= \frac{2\left(1/\sqrt{\lambda_1}\right)\|K\|_\infty}{\sqrt{d_1}}.\end{aligned} \tag{105}$$

The hypothesis $(H11)$ yields $\|b\|_\infty < 21\lambda_1/32$ and as $1-N_7 > 0$ then

$$\begin{aligned}\int_{t_1}^{t_2} \left[\frac{1}{2}K(t)\left|u_l'(t)\right|^2 + \frac{1}{2}\|u_l(t)\|^2 + \frac{1}{\sigma+2}|u_l(t)|_{L^{\sigma+2}(\Omega)}^{\sigma+2}\right] dt \\ \leq N_{10} D_l^2(t) + N_{11} E_l(t).\end{aligned} \tag{106}$$

Since $1/(\sigma+2) < 1$, $N_{10} = N_8/M$ and $N_{11} = N_9/M$, where $M = \min\{1, 1-N_7\} > 0$.

From (106), we obtain

$$\int_{t_1}^{t_2} E_l(t)\, dt \leq N_{10} D_l^2(t) + N_{11} E_l(t). \tag{107}$$

Note that by hypothesis $(H11)$, $N_{11} < 1/4$, thus from (107), we find

$$\int_{t_1}^{t_2} E_l(t)\, dt \leq N_{10} D_l^2(t) + \frac{1}{4} E_l(t). \tag{108}$$

Again, by the Mean Value Theorem there exists $t^* \in (t_1, t_2)$, such that

$$\int_{t_1}^{t_2} E_l(s)\, ds = E_l(t^*)(t_2 - t_1) \geq \frac{1}{2} E_l(t^*). \tag{109}$$

Integrating (76) from t^* to t and using (77) and (93), it follows that

$$\begin{aligned} E_l(t) &\leq E_l(t^*) + N_1 \int_{t^*}^{t} \left|u_l'(s)\right|^2 ds \\ &+ N_2 \int_{t^*}^{t} \left|u_l'(s)\right|_{\Gamma_1}^2 ds \\ &\leq E_l(t^*) + \frac{N_1}{d_1} D_l^2(t) + \frac{N_2}{d_1} D_l^2(t).\end{aligned} \tag{110}$$

Substituting (109) in (110), we get

$$E_l(t) \leq 2\int_{t_1}^{t_2} E_l(s)\, ds + \left(\frac{N_1}{d_1} + \frac{N_2}{d_1}\right) D_l^2(t). \tag{111}$$

Now, substituting (108) in (111), we obtain

$$E_l(t) \leq \frac{1}{2} E_l(t) + \left(2N_{10} + \frac{N_1}{d_1} + \frac{N_2}{d_1}\right) D_l^2(t), \tag{112}$$

$$E_l(t) \leq 2\left(2N_{10} + \frac{N_1}{d_1} + \frac{N_2}{d_1}\right) D_l^2(t). \tag{113}$$

As $E_l(t)$ is decreasing, the inequality (113) provides that

$$\sup_{t \leq s \leq t+1} E_l(s) \leq N_{12}\left[E_l(t) - E_l(t+1)\right], \tag{114}$$

where

$$N_{12} = 2\left(2N_{10} + \frac{N_1}{d_1} + \frac{N_2}{d_1}\right) > 0. \tag{115}$$

Thus, it follows using (114) and from Nakao's Lemma, (see [23]), that

$$E_l(t) \leq c e^{-wt}, \quad t \geq 1. \tag{116}$$

Taking the lim inf as $l \to \infty$ in (116), we obtain

$$E(t) \leq c e^{-wt}, \quad \forall t \geq 1, \tag{117}$$

where c is positive constant. □

Conflict of Interests

The authors report that there is no conflict of interests in the publication of this paper.

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