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Research Article

The Hadamard Product of a Nonsingular General H-Matrix and Its Inverse Transpose Is Diagonally Dominant

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We study the combined matrix of a nonsingular H-matrix. These matrices can belong to two different H-matrices classes: the most common, invertible class, and one particular class named mixed class. Different results regarding diagonal dominance of the inverse matrix and the combined matrix of a nonsingular H-matrix belonging to the referred classes are obtained. We conclude that the combined matrix of a nonsingular H-matrix is always diagonally dominant and then it is an H-matrix. In particular, the combined matrix in the invertible class remains in the same class.

1. Introduction

The Hadamard product of a nonsingular general H-matrix and its inverse transpose, that is, the combined matrix, has been studied in several works such as [1–3]. A complete study of the combined matrix showing its linear application can be seen in [4]. In the last decade, new properties of the combined matrix have been presented in [5, 6]. It is in this last reference where the name of combined matrix appears for the first time.

It is well known that the sum by row or by column of the entries of the combined matrix of a nonsingular matrix A, $C(A)$, is exactly equal to 1. Then, if $C(A) \geq O$, the combined matrix is doubly stochastic. In [7, 8], the authors studied conditions under which the combined matrix of some classes of matrices is nonnegative. In particular, the authors have studied the nonnegativity of the combined matrix of totally positive (nonnegative) and totally negative (nonpositive) matrices and of sign regular matrices.

The combined matrix has different applications. In a process control problem, if A represents the relation among inputs and outputs, the combined matrix of A represents the relative gain array of the process. This interpretation was given in [9] and was applied in chemistry, for instance, in [10]. In mathematics, the combined matrix of A is used in [11] to compute the eigenvalues of A.

Results involving the Hadamard product of H-matrices can be found in [12, 13]. The result where the combined matrix of a nonsingular M-matrix is also a nonsingular M-matrix was obtained by Fiedler in [1]. The same statement can be deduced from [14, 15] as Fiedler and Markham indicated in [3]. In this work we extend this result to nonsingular H-matrices. Firstly, we recall nonsingular H-matrices properties and their relations with diagonal dominance. In Section 3.1, it is proven that the combined matrix of an H-matrix of the invertible class is also an H-matrix of this class. Moreover, in Section 3.2, we obtain properties on diagonal dominance of the inverse matrix and the combined matrix of a nonsingular H-matrix belonging to the mixed class. So, we conclude that the combined matrix of a nonsingular H-matrix of the mixed class is also an H-matrix.

2. Notations and Definitions

In this paper, we work with square matrices of size $n \times n$. Matrices we are considering are real or complex. We will point out when we assume real matrices.

Let the set of indices $N = \{1, 2, \ldots, n\}$. Given a matrix $A = [a_{ij}]$ of order n, the symbol $A(i \mid j)$ denotes the principal submatrix of order $n-1$ that results after deleting row i and column j of A, whereas the symbol A_{ij} denotes its determinant, $A_{ij} = \det A(i \mid j)$.

In order to introduce the concept of combined matrix, we recall that $A \circ B$ represents the *Hadamard product* or entrywise product of the matrices A and B:

$$(A \circ B)_{ij} = a_{ij} b_{ij}. \tag{1}$$

Definition 1. The *combined matrix* of the matrix $A \in \mathbb{C}^{n\times n}$ is defined as

$$C(A) = A \circ \left(A^{-1}\right)^{T}. \tag{2}$$

As a consequence, the (i, j) entry of $C(A)$ is given by

$$c_{ij} = \frac{a_{ij}(-1)^{i+j} \det A(i \mid j)}{\det A} = \frac{a_{ij}(-1)^{i+j} A_{ij}}{\det A} \tag{3}$$

according to the previous notation.

A class of matrices that allows us to obtain interesting properties of their combined matrices is the class of irreducible matrices. We recall that a matrix A is *reducible* if there exists a permutation matrix P such that PAP^T is block triangular. That is,

$$PAP^{T} = \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix}, \tag{4}$$

where the diagonal blocks Q_{11} and Q_{22} are square matrices. A matrix is *irreducible* when it is not reducible.

Applying repeatedly the triangular decomposition to the resulting diagonal blocks, provided they are reducible, we obtain the *Frobenius normal form* of a reducible matrix [16]. In other words, we obtain the block triangular decomposition:

$$PAP^{T} = \begin{bmatrix} R_{11} & 0 & \cdots & 0 \\ R_{21} & R_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{bmatrix}, \tag{5}$$

where each diagonal block R_{ii} is either an irreducible square block or a null block of size 1×1.

The properties that we use in the study of the combined matrix of H-matrices are listed below (see [11]).

Theorem 2. *Let A be a nonsingular $n \times n$ matrix and let $C(A) = [c_{ij}]$ be its combined matrix. Then*

(1) *the sum by row and by column satisfies*

$$\begin{aligned} &\sum_{j} c_{ij} = 1, \quad \forall i \in N, \\ &\sum_{i} c_{ij} = 1, \quad \forall j \in N; \end{aligned} \tag{6}$$

(2) *if D is a nonsingular diagonal matrix,*

$$C(DA) = C(AD) = C(A); \tag{7}$$

(3) *if P and Q are permutation matrices,*

$$C(PAQ) = PC(A)Q; \tag{8}$$

(4) *if A is a reducible matrix, then $C(A)$ is a block diagonal matrix. More precisely, if $PAP^T = [R_{ij}]$ is the Frobenius normal form of A, then*

$$C(A) = P^{T} \begin{bmatrix} C(R_{11}) & 0 & \cdots & 0 \\ 0 & C(R_{22}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C(R_{kk}) \end{bmatrix} P. \tag{9}$$

Regarding the combined matrix of a reducible matrix, it is enough to study the combined matrix of each irreducible diagonal block.

Next we are going to recall basic definitions related to H-matrices.

Definition 3. The matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is said to be *diagonally dominant* (DD) if

$$\left|a_{ii}\right| \geq \sum_{j\neq i} \left|a_{ij}\right|, \quad i = 1, 2, \ldots, n, \tag{10}$$

and A is called *strictly diagonal dominant* (SDD) if inequalities (10) are strict.

We are going to work with *general H-matrices*. Traditionally, H-matrices were considered only in the case where their comparison matrices are nonsingular M-matrices. Nevertheless, as we can see in [17], there are nonsingular H-matrices with singular comparison matrix. For example,

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{11}$$

is a nonsingular H-matrix, but its comparison matrix is a singular M-matrix. For this reason, we have to consider singular M-matrices in order to obtain all possible cases of nonsingular H-matrices.

The *comparison matrix* of a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is defined as

$$\mathscr{M}(A) = \left[m_{ij}\right] = 2\left|\operatorname{diag}(A)\right| - |A|; \tag{12}$$

therefore,

$$m_{ij} = \begin{cases} -\left|a_{ij}\right|, & \text{if } i \neq j, \\ \left|a_{ij}\right|, & \text{if } i = j, \end{cases} \quad i, j = 1, 2, \ldots, n. \tag{13}$$

Definition 4. A real matrix A is an *M-matrix* if $A = \mathscr{M}(A)$ and

$$\mathscr{M}(A) = sI - B, \quad \text{with } B \geq 0,\ s \geq \rho(B), \tag{14}$$

where $\rho(B)$ represents the spectral radius of B.

Definition 5. A matrix $A \in \mathbb{C}^{n\times n}$ is called an *H-matrix* if it satisfies condition (14); that is, its comparison matrix is an M-matrix.

When the last inequality in (14) is strict, it follows that $\mathscr{M}(A)$ is a nonsingular M-matrix. Otherwise, $\mathscr{M}(A)$ is singular.

Analyzing the properties of nonsingular and singular H-matrices, a partition of the set of H-matrices in three classes is given in [17]:

(1) A belongs to the *invertible class* $\mathscr{H}_I$ if and only if $\mathscr{M}(A)$ is a nonsingular M-matrix.

(2) A belongs to the *mixed class* $\mathscr{H}_M$ if and only if $\mathscr{M}(A)$ is a singular M-matrix and its diagonal entries are not null.

(3) A belongs to the *singular class* $\mathscr{H}_S$ if and only if $\mathscr{M}(A)$ is an M-matrix with at least one null diagonal entry.

The conclusions related to the singularity of each class are the following: all H-matrices in $\mathscr{H}_I$ are nonsingular, in $\mathscr{H}_M$ there are nonsingular and singular H-matrices, and all H-matrices in $\mathscr{H}_S$ are singular and reducible. Moreover, if A is irreducible, then all singular H-matrices in $\mathscr{H}_M$ are diagonally equivalent to the singular matrix $\mathscr{M}(A)$ [18].

We study the combined matrix of nonsingular H-matrices. More precisely, we study the combined matrix of nonsingular H-matrices of both the invertible and the mixed classes.

We now recall the generalized diagonal dominance definitions.

Definition 6. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is said to be *generalized diagonally dominant* (GDD) if there exists a nonnegative diagonal and nonsingular matrix D of size n such that AD is diagonally dominant; that is,

$$\left|a_{ii}d_{ii}\right| \geq \sum_{j\neq i}\left|a_{ij}d_{ij}\right|, \quad i = 1, 2, \ldots, n, \tag{15}$$

and A is called *generalized strictly diagonal dominant* (GSDD) if the inequalities in (15) are strict.

We have the following well known results (see [17, 19]).

Theorem 7. *Let $A \in \mathbb{C}^{n\times n}$. Then,*

(1) *$A \in \mathscr{H}_I$ if and only if it is GSDD;*

(2) *if A is GDD, then it is an H-matrix;*

(3) *if $A \in \mathscr{H}_M$ and is irreducible, then it is GDD.*

3. Combined Matrices of H-Matrices

In this section, we are going to extend to nonsingular and real H-matrices the following theorem that we have already commented on in the previous section (see [1]). All matrices considered in this section are real.

Theorem 8. *The combined matrix of a nonsingular M-matrix is also a nonsingular M-matrix.*

In fact, we are going to prove the diagonal dominance of the combined matrix of a nonsingular H-matrix.

3.1. Combined Matrices of H-Matrices of the Invertible Class. Let us start with H-matrices of the invertible class.

Theorem 9. *Let $A \in \mathscr{H}_I$. Then its combined matrix is strictly diagonal dominant.*

Proof. Since $C(AD) = C(A)$, without loss of generality, we can suppose that A is SDD; therefore,

$$\left|a_{ii}\right| > \sum_{j\neq i}\left|a_{ij}\right|, \quad \forall i \in N. \tag{16}$$

In addition, according to Theorem 2.5.12 in [11], we know that $A^{-1} = [\alpha_{ij}]$ satisfies the strict inequalities

$$\left|\alpha_{ii}\right| > \left|\alpha_{ji}\right|, \quad \forall i, j \in N, \ j \neq i. \tag{17}$$

We should notice that, with this same notation, the entries of the inverse matrix are equal to

$$\alpha_{ij} = \frac{(-1)^{i+j} A_{ji}}{\det A} \tag{18}$$

and inequality (17) can be expressed in the form

$$\left|A_{ii}\right| > \left|A_{ij}\right|, \quad \forall i, j \in N, \ j \neq i. \tag{19}$$

Then, taking into account inequalities (16) and (19), it results that

$$\left|a_{ii}\right|\left|A_{ii}\right| > \sum_{j\neq i}\left|a_{ij}\right|\left|A_{ii}\right| > \sum_{j\neq i}\left|a_{ij}\right|\left|A_{ij}\right| \tag{20}$$

and consequently

$$\left|c_{ii}\right| > \sum_{j\neq i}\left|c_{ij}\right|, \tag{21}$$

where c_{ij} are the entries of $C(A)$ (3). This implies that the combined matrix $C(A)$ is SDD. □

Corollary 10. *If $A \in \mathscr{H}_I$, then $C(A) \in \mathscr{H}_I$.*

Proof. It is straightforward. □

It is well known that the $\mathscr{H}_I$ class has different subclasses: α_1- and α_2-matrices, S-SDD, and so forth (see [20, 21]). Since all strictly diagonally dominant matrices belong to these subclasses, we can establish the following general corollary.

Corollary 11. *Let $\mathcal{S}$ be a subclass of H-matrices of the invertible class that contains all SDD matrices; that is,*

$$\left\{A \in \mathbb{C}^{n\times n} : A \text{ is SDD}\right\} \subset \mathcal{S} \subset \mathscr{H}_I. \tag{22}$$

If $A \in \mathcal{S}$, then $C(A)$ belongs to the same subclass $\mathcal{S}$.

Proof. It is obvious. □

It is worth noticing that this result can be obtained combining Theorems 3.1 and 3.5 of [12], where the authors work with the W-class of matrices satisfying condition (17).

Note that we can prove Theorem 8 as a consequence of Theorem 9.

Corollary 12. *If A is a nonsingular M-matrix, then $C(A)$ is a nonsingular M-matrix.*

Proof. Let us recall first that $A \in \mathscr{H}_I$; then $C(A)$ is SDD. Since $A^{-1} \geq 0$, $C(A)$ has the signs pattern of A; consequently $C(A)$ is a nonsingular M-matrix. □

3.2. Combined Matrices of Nonsingular H-Matrices of the Mixed Class. We extend the previous results to nonsingular H-matrices of the mixed class. For this purpose, we need to extend two results of [11] on SDD matrices to nonsingular DD matrices.

We will denote by $\operatorname{sgn}(x)$ the sign of the real number x.

Lemma 13. *If A is diagonally dominant and nonsingular, then*

$$\operatorname{sgn}(\det A) = \operatorname{sgn}\left(\prod_i a_{ii}\right). \tag{23}$$

Proof. Since A is DD and nonsingular, then $a_{ii} \neq 0$ for all $i \in N$.

Let us suppose first that $a_{ii} > 0$ for all i. Given $\epsilon \in [0, 1[$ we build the matrix

$$A_\epsilon = D + \epsilon (A - D), \tag{24}$$

where $D = \operatorname{diag}(A)$.

Since $A_\epsilon = \epsilon A + D(1-\epsilon)$ is SDD because it is the sum of a DD matrix and a SDD matrix and $\operatorname{diag}(A_\epsilon) = \operatorname{diag}(A) = D$, then (see [11, page 125]) $\operatorname{sgn}(\det A_\epsilon) = \operatorname{sgn}(a_{11}a_{22}\cdots a_{nn})$ and, therefore, $\det A_\epsilon > 0$. Since the determinant is a multilinear function, it is continuous, and since $\lim_{\epsilon \to 1^-} A_\epsilon = A$, then $\det A \geq 0$. Therefore,

$$\operatorname{sgn}(\det A) = \operatorname{sgn}(a_{11}a_{22}\cdots a_{nn}) > 0 \tag{25}$$

since A is nonsingular.

Now let us suppose that there are some negative diagonal entries. We build the diagonal matrix

$$\overline{D} = \operatorname{sgn}(\operatorname{diag}(D)) = \operatorname{diag}(\operatorname{sgn}(a_{ii})). \tag{26}$$

Then, matrix $B = A\overline{D}$ has its diagonal entries positive and is DD. Therefore, applying the first part of this proof to B we have $\det B > 0$.

Since $\det B = \det A \det \overline{D} = \det A \prod_i \operatorname{sgn}(a_{ii})$, we conclude that

$$\operatorname{sgn}(\det A) = \operatorname{sgn}\left(\prod_i \operatorname{sgn} a_{ii}\right) = \operatorname{sgn}\left(\prod_i a_{ii}\right). \tag{27}$$

□

We should notice that if A is singular and DD, then the sign of its determinant may not match with the sign of the diagonal entries product. This means that it can occur that $\det A = 0$, but $\prod_i a_{ii} \neq 0$. Besides, if $\prod_i a_{ii} = 0$, then A would have at least one null row.

Lemma 13 can be extended to GDD matrices.

Lemma 14. *If matrix A is generalized diagonally dominant and nonsingular, then*

$$\operatorname{sgn}(\det A) = \operatorname{sgn}\left(\prod_i a_{ii}\right). \tag{28}$$

Proof. It is enough to observe that if $D = \operatorname{diag}(d_i)$ is a diagonal, nonsingular, and nonnegative matrix that transforms AD in a diagonally dominant matrix, then $\det(AD) = \det A \prod_i d_i$ and the diagonal entries of AD are equal to $a_{ii}d_i$. □

The following result is essential for achieving our goals.

Theorem 15. *Suppose that A is diagonally dominant with nonzero diagonal entries. Then,*

$$\left|A_{ii}\right| \geq \left|A_{ij}\right|, \quad \forall i, j \in N. \tag{29}$$

Proof. Let us start supposing that $a_{ii} > 0$ for all $i \in N$. In order to prove inequalities (29), let us consider, without loss of generality, that $i = 1$ and $j = 2$. We are going to prove that $A_{11} \pm A_{12} \geq 0$. Then

$$\begin{aligned} A_{11} \pm A_{12} &= \det \begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \\ &\quad + \det \begin{bmatrix} \pm a_{21} & a_{23} & \cdots & a_{2n} \\ \pm a_{31} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ \pm a_{n1} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \\ &= \det \begin{bmatrix} \pm a_{21} + a_{22} & a_{23} & \cdots & a_{2n} \\ \pm a_{31} + a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ \pm a_{n2} + a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = \det B. \end{aligned} \tag{30}$$

Let us prove that this auxiliary matrix B is DD. In its first row, since A is DD, we know that

$$a_{22} = \left|a_{22}\right| \geq \sum_{k \neq 2} \left|a_{2k}\right|; \tag{31}$$

in particular, $a_{22} \geq |a_{21}|$; therefore,

$$a_{22} \pm a_{21} = \left|a_{22} \pm a_{21}\right| \geq a_{22} - \left|a_{21}\right| \geq \sum_{k \neq 1,2} \left|a_{2k}\right|. \tag{32}$$

For the remaining rows, $j \neq 1$, it is true that

$$\begin{aligned}&\left|a_{j2} \pm a_{j1}\right| + \sum_{k\neq 1,2,j} \left|a_{jk}\right| \\ &\quad \leq \left|a_{j2}\right| + \left|a_{j1}\right| + \sum_{k\neq 1,2,j} \left|a_{jk}\right| \leq a_{jj}.\end{aligned} \tag{33}$$

Since B is DD and, besides, all its diagonal entries are nonnegative, we can apply Lemma 13 and conclude that $\det B \geq 0$. Consequently $A_{11} \pm A_{12} \geq 0$; then $A_{11} = |A_{11}| \geq |A_{12}|$.

Finally, in the general case, when there are different signs in the diagonal entries, we build the sign matrix $\overline{D} = \operatorname{sgn}(\operatorname{diag}(D))$ such that the matrix $F = A\overline{D}$ is DD and its diagonal entries are all positive. Applying the result of the first part of the proof to the matrix F we conclude that $|F_{ii}| \geq |F_{ij}|$ for all $j \neq i$. Finally, since

$$F_{ij} = A_{ij} \prod_{j\neq i} \operatorname{sgn}\left(a_{jj}\right), \tag{34}$$

we conclude that $|A_{ii}| = |F_{ii}| \geq |F_{ij}| = |A_{ij}|$, $\forall i, j \in N$, $j \neq i$. □

An immediate consequence of this result is the extension of Theorem 2.5.12 of [11] to DD matrices.

Theorem 16. *Let A be a nonsingular and diagonally dominant matrix. Then $A^{-1} = [\alpha_{ij}]$ is diagonally dominant of its column entries. That is,*

$$\left|\alpha_{ii}\right| \geq \left|\alpha_{ji}\right|, \quad \forall i, j \in N. \tag{35}$$

Proof. We know that a nonsingular and DD matrix does not have null diagonal entries (see [17]). Then, we can apply Theorem 15 to matrix A and then inequalities (29) become inequalities (35). □

Theorem 15 cannot be extended to GDD and to GSDD matrices because $(AD)_{ii} = A_{ii}D_{ii}$ but $(AD)_{ij} = A_{ij}D_{jj}$. The following example illustrates this fact.

Example 17. The matrix

$$A = \begin{bmatrix} 3 & 4 & -3 \\ 4 & 10 & -3 \\ 1 & 4 & -24 \end{bmatrix} \tag{36}$$

is GSDD, because when it is multiplied by the diagonal matrix $D = \operatorname{diag}(1, 1/2, 1/4)$ we have the following SDD matrix:

$$AD = \begin{bmatrix} 3 & 2 & -\dfrac{3}{4} \\ 4 & 5 & -\dfrac{3}{4} \\ 1 & 2 & -6 \end{bmatrix}. \tag{37}$$

Though the inverse matrix of AD satisfies inequality (35), the matrix A^{-1} does not satisfy it since $\alpha_{12} = -84/\det A$ and $\alpha_{22} = -69/\det A$; $\alpha_{13} = 18/\det A$ and $\alpha_{33} = 14/\det A$. That is, $|\alpha_{22}| < |\alpha_{12}|$ and $|\alpha_{33}| < |\alpha_{13}|$.

We are almost in position to extend the result on the combined matrix to nonsingular H-matrices of the mixed class.

Theorem 18. *Let $A \in \mathscr{H}_M$ be a nonsingular and irreducible matrix. Then $C(A)$ is diagonally dominant.*

Proof. Since $\det A \neq 0$, we have to prove

$$\left|a_{ii}A_{ii}\right| \geq \sum_{j\neq i} \left|a_{ij}\right| \left|A_{ij}\right|, \quad \forall i \in N. \tag{38}$$

Since A is an irreducible H-matrix of the mixed class, we know that A is GDD by Theorem 7. As $C(AD) = C(A)$, we can suppose that A is DD; that is,

$$\left|a_{ii}\right| \geq \sum_{j\neq i} \left|a_{ij}\right|, \quad \forall i \in N. \tag{39}$$

According to Theorem 15,

$$\left|A_{ii}\right| \geq \left|A_{ij}\right|, \quad \forall i, j \in N. \tag{40}$$

Taking into account this last inequality in (39) we obtain (38) and consequently the combined matrix is DD. □

Corollary 19. *If A is a nonsingular H-matrix, then its combined matrix $C(A)$ is an H-matrix.*

Proof. Suppose that A is written in Frobenius normal form as it is shown in (5). Let us work with its diagonal blocks. If a diagonal block of A belongs to $\mathscr{H}_I$, then its corresponding block in $C(A)$ (see expression (9)) is SDD by Theorem 9 and belongs to $\mathscr{H}_I$ by Corollary 10. Further, if an irreducible diagonal block of A is in $\mathscr{H}_M$, then its corresponding block in $C(A)$ is DD by Theorem 18. Then $C(A)$ is an H-matrix. □

Example 20. Let us consider the reducible and nonsingular H-matrix A of the mixed class

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 7 & 3 & 3 & 1 & 2 \\ 0 & 2 & 1 & 5 & 4 \\ 6 & 0 & 1 & 1 & -2 \end{bmatrix} \tag{41}$$

written in normal form, where the first 2×2 diagonal block belongs to $\mathscr{H}_I$ and the second 3×3 diagonal block belongs to $\mathscr{H}_M$.

The combined matrix is

$$C(A) = \frac{1}{132} \begin{bmatrix} 88 & 44 & 0 & 0 & 0 \\ 44 & 88 & 0 & 0 & 0 \\ 0 & 0 & 126 & -18 & 24 \\ 0 & 0 & -12 & 120 & 24 \\ 0 & 0 & 18 & 30 & 84 \end{bmatrix}. \tag{42}$$

Notice that $C(A)$ is block diagonal. Each diagonal block is the combined matrix of the corresponding diagonal block of A. Specifically, both blocks are SDD and then $C(A) \in \mathscr{H}_I$.

In conclusion, in this work, we have proved that the combined matrix of a nonsingular H-matrix of the class either $\mathscr{H}_I$ or $\mathscr{H}_M$ is an H-matrix. Actually, this is an extension of a well known result for nonsingular M-matrices.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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