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Research Article

Solvability of *N*th Order Linear Boundary Value Problems

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This paper presents a method that provides necessary and sufficient conditions for the existence of solutions of nth order linear boundary value problems. The method is based on the recursive application of a linear integral operator to some functions and the comparison of the result with these same functions. The recursive comparison yields sequences of bounds of extremes that converge to the exact values of the extremes of the BVP for which a solution exists.

1. Introduction

Let I be a compact interval in $\mathbb{R}$ and let us consider the differential operator $L : C^n(I) \to C(I)$ defined by

$$Ly = a_n(x)\, y^{(n)}(x) + a_{n-1}(x)\, y^{(n-1)}(x) + \cdots + a_0(x)\, y(x), \quad x \in I, \tag{1}$$

where $a_i(x) \in C(I)$, $0 \le i \le n$. Let $k \in \mathbb{N}$ be such that $1 \le k \le n$ and let us define the sets $\Omega^k \in \mathbb{N}^k$ and $\Omega^{n-k} \in \mathbb{N}^{n-k}$ as

$$\begin{aligned} \Omega^k &= \left\{(\alpha_1, \alpha_2, \dots, \alpha_k),\ 0 \le \alpha_i \le n-1\right\}, \\ \Omega^{n-k} &= \left\{(\alpha_1, \alpha_2, \dots, \alpha_{n-k}),\ 0 \le \alpha_i \le n-1\right\}. \end{aligned} \tag{2}$$

The purpose of this paper is to investigate the existence of solutions of the nth order boundary value problem

$$Ly = \sum_{i=0}^{\mu} c_i(x)\, y^{(i)}(x), \quad x \in\,]a, b[\,, \tag{3}$$

$$T(\alpha, \beta, a, b)\, y = 0, \tag{4}$$

where $[a, b] \subset I$, $\mu < n$, and $c_i(x) \in C[a, b]$ for $0 \le i \le \mu$ are functions with properties to be determined and $\alpha \in \Omega^k$, $\beta \in \Omega^{n-k}$, and $T(\alpha, \beta, a, b)y = 0$ are boundary conditions defined by

$$\begin{aligned} y^{(\alpha_i)}(a) &= 0, \quad (\alpha_1, \alpha_2, \dots, \alpha_k) \in \Omega^k, \\ y^{(\beta_i)}(b) &= 0, \quad (\beta_1, \beta_2, \dots, \beta_{n-k}) \in \Omega^{n-k}. \end{aligned} \tag{5}$$

In particular, we will provide necessary and sufficient conditions for the existence of solutions of (3) and (4), conditions that will be expressed as integral inequalities depending on the extremes a and b (i.e., Lyapunov-like inequalities, for the case of necessary conditions). Note that (4) covers conjugate boundary conditions (taking $\alpha = (0, 1, \dots, k)$ and $\beta = (0, 1, \dots, n-k)$), focal boundary conditions (taking $\alpha = (0, 1, \dots, k-1)$ and $\beta = (k, k+1, \dots, n-1)$ for right focality and $\alpha = (n-k, n-k+1, \dots, n-1)$ and $\beta = (0, 1, \dots, n-k-1)$ for left focality, according to Muldowney's definition), and a range of many other boundary conditions in between.

The procedure used for that will be an extension of the approach presented in [1] for the second-order linear differential equation. To this end, we will apply recursively the operator $M : C^{\mu}[a, b] \to C^n[a, b]$ defined by

$$Mf = \int_a^b G(x, t) \sum_{i=0}^{\mu} c_i(t)\, f^{(i)}(t)\, dt, \tag{6}$$

where $G(x,t)$ is the Green function of the problem

$$\begin{aligned} Ly &= g(x), \\ T(\alpha,\beta,a,b)\,y &= 0, \end{aligned} \tag{7}$$

provided that the corresponding homogeneous equation

$$\begin{aligned} Ly &= 0, \\ T(\alpha,\beta,a,b)\,y &= 0 \end{aligned} \tag{8}$$

has only the trivial solution, assumption that we will keep in the rest of the paper.

We will show that the comparison of this operator with the functions to which it is applied provides lower and upper bounds for the extremes a and b which allow (3) and (4) to have a nontrivial solution and which converge to the values of these extremes, a and b, as the recursivity index grows. It is important to remark that the method is not restricted to self-adjoint problems but it can be applied to any kind of problem of the form (3) and (4), as long as the boundary conditions $T(\alpha,\beta,a,b)y$ and the functions $c_i(x)$ satisfy certain general hypotheses. Examples of this kind of problems appear in the study of the deflections of beams, both straight ones with nonhomogeneous cross-sections in free vibration, for which the fourth-order linear Euler-Bernoulli equation is applicable, and curved ones with different shapes. Some of these and other applications can be found in [2, Chapter IV].

The study of the existence of solutions of (3) and (4) has been historically linked to the analysis of the distribution of zeroes of equations of nth order, which dates from the work of Frobenius on Wronskians of systems of solutions in the 19th century [3]. In 1922, Pólya [4] introduced the concept of disconjugacy in an interval, defined as the absence of a solution of an nth order differential equation with at least n zeroes, counting multiplicities, in that interval. He also obtained alternative representations of nth order disconjugate equations. Later, Sherman [5] extended the concept of disfocality to the nth order equation, showing the relationship between focal and conjugate points, and Muldowney [6] characterized disfocal equations in a similar manner as Pólya had done for disconjugate equations. Levin [7, 8] and Nehari [9] provided the first Sturmian comparison and separation theorems for this type of equations and presented the main properties of the conjugate and focal points. Further research on this area was continued by Ahmad and Lazer [10, 11], Elias [12–14], and other authors. Some of these results can be found in the excellent monography of Coppel [15] on the topic.

One of the most successful techniques introduced in this analysis has been the combination of the modern cone theory with the positivity and monotonicity characteristics displayed by many of the Green functions resulting from (7). In the books of Krasnosel'skii [16] and Deimling [17], one can find the grounds for this approach, which originated in Krasnosel'skii and Krein and Rutman's works (see [16, 18]) and was later pursued by Gentry and Travis [19], Schmitt and Smith [20], Keener and Travis [21], Tomastik [22, 23], Kreith [24], Hankerson and Peterson [25], Hankerson and Henderson [26], Eloe [27–32], and Diaz [33] and more recently by Graef [34, 35], Zhang et al. [36], Zhang [37], Sun et al. [38], or Hao et al. [39], among many others. Most of these papers are focused on obtaining comparison theorems for different problems and different boundary conditions, whereas some of them deal with the existence of solutions for specific boundary value problems. This is the case, for instance, of [21, 31, 35, 37, 38] or [39]. This paper shares that objective combining some ideas of Keener and Travis with the approach of [1] in order to yield an iterative sequence of necessary and sufficient conditions in integral form, each one more precise than the previous one, and setting a framework that allows covering a range of problems much wider than the ones originally assessed in [1, 21]. Let us also remark that although the iteration of operators has been used in several papers to establish the existence of solutions, it does not seem to have been applied as in this paper to determine which extremes a and b of a boundary value problem allow exactly the existence of solutions in the corresponding interval.

In terms of nomenclature, we will use the notation M to name the operator defined in (6), Mf or $M\{f\}$ to name the function with domain $[a,b]$ resulting from the application of M to $f(x) \in C^{\mu}[a,b]$, $Mf(x)$ or $M\{f\}(x)$ to name the value of the function Mf at the point x, and M_{ab} to name the operator M considered as a function of the extremes a, b.

The organization of the paper is as follows. Section 2 will use the cone theory to assess the main properties of the operator M and develop a generic method to obtain necessary and sufficient conditions for problem (3) and (4) to have a nontrivial solution. Section 3 will apply the method presented in Section 2 to a broad problem of the type (3) and (4) studied by Eloe and Henderson in [30], yielding concrete necessary and sufficient conditions in integral forms. Section 4 will apply the results of Section 3 to several examples. Finally Section 5 will discuss the significance of the results of previous sections as well as their limitations.

2. The Application of the Cone Theory to the Operator M

The purpose of this section will be to set up the theoretical framework for the determination of the existence of solutions of general boundary value problem (3) and (4) based on the application of the cone theory to the operator M defined in (6). As a first step, we will make the obvious observation that (3) and (4) can have a solution if and only if there is $y(x) \in C^n[a,b]$ such that

$$My(x) = y(x), \quad x \in [a,b], \tag{9}$$

which is equivalent to the problem of finding an eigenfunction of the problem

$$Mu(x) = \lambda u(x), \quad x \in [a,b], \tag{10}$$

such that $\lambda = 1$. We will denote the spectral radius of M by $r(M)$ (in other words, $r(M)$ is the supremum of the spectrum of M). As a lot of literature shows, eigenvalue problem (10) admits an attack based on Krasnosel'skii cone theory if we manage to find a Banach space of functions defined on $[a,b]$ and a cone inside for which the operator M satisfies certain positivity properties. In order to make this paper self-contained, let us recall that, given a Banach space B, a cone $P \subset B$ is a nonempty closed set defined by the following conditions:

(1) If $u, v \in P$, then $cu + dv \in P$ for any real numbers $c, d \geq 0$.

(2) If $u \in P$ and $-u \in P$, then $u = 0$.

We will denote the interior of the cone P by P^0, and we will say that the cone P is reproducing if any $y \in B$ can be expressed as $y = u - v$ with $u, v \in P$. Note that the function $f(x) \equiv 0$, where $x \in [a,b]$ belongs to both the Banach space B (whatever this is) and the cone P.

One of the advantages of the existence of a cone is that it allows defining a partial ordering relationship in the Banach space B by setting $u \leq v$ if and only if $v - u \in P$ (it is straightforward to prove that this relationship satisfies the criteria for a partial ordering). Accordingly, we will say that the operator M is u_0-positive if there exists $u_0 \in P$ such that for any $v \in P \setminus \{0\}$ one can find constants δ_1 and δ_2 such that $\delta_1 u_0 \leq v \leq \delta_2 u_0$ (note that the constants δ_1 and δ_2 do not need to be the same for all v). We will also write $u > v$ when $u - v \in P^0$. With this in mind, we can start presenting the main results of this section.

Lemma 1. *$r(M_{ab})$ is continuous with a and b.*

Proof. Let us consider a fixed interval $[A,B] \subset I$ such that $[a,b] \subset [A,B]$ and the associated Banach space $C^{\mu}[A,B]$, together with the norm

$$\|f\| = \max\left\{\sup\left\{\left|\frac{\partial^l f(x)}{\partial x^l}\right|,\ x \in [A,B]\right\},\ 0 \leq l \leq \mu\right\}. \tag{11}$$

We will extend the definition of M_{ab} given in (6) onto the full interval $[A,B]$ as follows:

$$M^*_{ab} f = \begin{cases} \displaystyle\sum_{i=0}^{\mu} \frac{\partial^i M_{ab} f(a)}{\partial x^i} \frac{(x-a)^i}{i!}, & x \in [A,a]; \\ M_{ab} f(x), & x \in [a,b]; \\ \displaystyle\sum_{i=0}^{\mu} \frac{\partial^i M_{ab} f(b)}{\partial x^i} \frac{(x-b)^i}{i!}, & x \in [b,B]. \end{cases} \tag{12}$$

From (12) it is immediate to show that $M^*_{ab} f \in C^{\mu}[A,B]$.

Next, following [40, Theorem 2.5], the mapping $(a,b) \to r(M^*_{ab})$ will turn out to be continuous if we manage to show that M^*_{ab} is compact, bounded, and linear and the mappings $a \to M^*_{ab}$ and $b \to M^*_{ab}$ are continuous on the uniform operator topology.

The compactness of M^*_{ab} can be proven using a typical Arzela-Ascoli argument and the linearity is evident by simple inspection of (6) and (12). The boundedness can also be easily proven from (11) and (12), taking in consideration that

$$\begin{aligned} \left|\frac{\partial^l M^*_{ab} f(x)}{\partial x^l}\right| &\leq \|f\| \\ &\cdot \max\left\{\sup\left\{\int_a^b \left|\frac{\partial^j G_{a,b}(x,t)}{\partial x^j}\right| \sum_{i=0}^{\mu} |c_i(t)|\, dt,\ x \in [a,b]\right\},\right. \\ &\left. 0 \leq j \leq \mu\right\} \max\left\{1,\ \sum_{i=l}^{\mu} \frac{(a-A)^{i-l}}{(i-l)!},\ \sum_{i=l}^{\mu} \frac{(B-b)^{i-l}}{(i-l)!}\right\}, \end{aligned} \tag{13}$$

for $x \in [A,B]$ and $0 \leq l \leq \mu < n$.

As for the continuity of M^*_{ab} with the extremes, let us focus first on investigating the continuity of $G_{ab}(x,t)$ with a. Accordingly, let us recall that, following [15, Chapter 3], if we denote by $y_j(x)$ the solution of $Ly = 0$ in $[A,B]$ which satisfies the initial conditions

$$y_j^{(i-1)}(a) = \delta_{ij} \tag{14}$$

for $1 \leq j \leq n$, then the Green function of problem (7) is defined by

$$G(x,t) = \begin{cases} \gamma_1(t) y_1(x) + \gamma_2(t) y_2(x) + \cdots + \gamma_n(t) y_n(x), & a \leq x \leq t \leq b; \\ \eta_1(t) y_1(x) + \eta_2(t) y_2(x) + \cdots + \eta_n(t) y_n(x), & a \leq t < x \leq b, \end{cases} \tag{15}$$

with $\gamma_j(t)$, $\eta_j(t)$ being continuous on $[a,b]$ such that $G(x,t)$ satisfies (on x) the boundary conditions $T(\alpha,\beta,a,b)G = 0$ and the continuity conditions (on t)

$$\sum_{j=1}^{n} \gamma_j(t) y_j^{(l)}(t) = \sum_{j=1}^{n} \eta_j(t) y_j^{(l)}(t), \quad 0 \leq l \leq n-2,\ t \in [a,b],$$

$$\sum_{j=1}^{n} \left(\eta_j(t) y_j^{(n-1)}(t) - \gamma_j(t) y_j^{(n-1)}(t)\right) = 1, \quad t \in [a,b]. \tag{16}$$

The fact that $y_j^{(i-1)}(x)$ is continuous with a for $1 \leq i, j \leq n$ is a consequence of [41, Theorem V.2.1]. Accordingly, setting

$$\chi_j(t) = \eta_j(t) - \gamma_j(t), \quad 1 \leq j \leq n, \tag{17}$$

from (16) one has that $\chi(t) = \{\chi_j(t), 1 \le j \le n\} \in \mathbb{R}^{n\times 1}$ is the solution of a system of equations of the form $Y(t)\chi(t) = B$, where

$$B = (0, 0, \ldots, 0, 1)^t \in \mathbb{R}^{n\times 1} \tag{18}$$

and $Y(t) \in \mathbb{R}^{n\times n}$ is defined by

$$(Y(t))_{ij} = \left(y_j^{(i-1)}(t)\right). \tag{19}$$

The solution of such a system is $\chi(t) = Y^{-1}(t)B$ which, given that the components of $\mathrm{Y}(t)$ are continuous with a and $\det Y(t) = 1$ (it is the Wronskian of the solutions of $Ly = 0$ satisfying (14)), by Cramer's rule implies the continuity of all $\chi_j(t)$ with a. Replacing the obtained $\chi_j(t)$ in (17) and applying the resulting identities to (15), together with the boundary conditions $T(\alpha, \beta, a, b)G = 0$, one gets a system of equations for the unknown $\gamma_j(t)$ which has all known terms continuous on a and whose associated matrix has a nonzero determinant (if the determinant was zero, setting $\eta_j(t) = \gamma_j(t)$ for $1 \le j \le n$ in (15) and applying the boundary conditions (4), one would obtain a nontrivial $G(x,t) \in C^n[a,b]$ which would be a solution of (8) on the variable x, contradicting the hypothesis). As a result, applying again Cramer's rule, one has that $\gamma_j(t)$ and $\eta_j(t)$ are also continuous on a and therefore, from (15), $G(x,t)$ must be also continuous with a. This assertion is also valid for the partial derivatives $\partial G^l_{ab}(x,t)/\partial x^l$ up to $l = n-1$, since the functions $y_j(t)$ and their derivatives up to $(n-1)$th order were shown to be continuous on a before. Repeating the argument with b, one gets a similar result.

Once continuity of $G_{ab}(x,t)$ with a and b has been confirmed, we will study the continuity of M^* with a. To this end, let us pick $\epsilon > 0$ and let us divide the interval $[A, B]$ in the subintervals $[A, a]$, $[a, a+\delta_1]$, $[a+\delta_1, b]$, and $[b, B]$, where $\delta_1 > 0$ is a value to be determined later. Let us focus first on the interval $[a+\delta_1, b]$. From (6), (11), and (12) we have

$$\begin{aligned} \left| \frac{\partial^l M^*_{a,b} f(x)}{\partial x^l} - \frac{\partial^l M^*_{a+\delta_1,b} f(x)}{\partial x^l} \right| &\le \|f\| \\ &\cdot \left\{ \int_a^{a+\delta_1} \left| \frac{\partial^l G_{a,b}(x,t)}{\partial x^l} \right| \sum_{i=0}^{\mu} |c_i(t)|\, dt \right. \\ &+ \int_{a+\delta_1}^{x} \left| \frac{\partial^l G_{a,b}(x,t)}{\partial x^l} - \frac{\partial^l G_{a+\delta_1,b}(x,t)}{\partial x^l} \right| \sum_{i=0}^{\mu} |c_i(t)|\, dt \\ &+ \left. \int_x^b \left| \frac{\partial^l G_{a,b}(x,t)}{\partial x^l} - \frac{\partial^l G_{a+\delta_1,b}(x,t)}{\partial x^l} \right| \sum_{i=0}^{\mu} |c_i(t)|\, dt \right\}, \\ & x \in [a+\delta_1, b], \end{aligned} \tag{20}$$

for $0 \le l \le \mu$. As a result, by Heine's theorem, we can find a $\delta_2 > 0$ such that if $\delta_1 < \delta_2$ then

$$\begin{aligned} \left| \frac{\partial^l M^*_{a,b} f(x)}{\partial x^l} - \frac{\partial^l M^*_{a+\delta_1,b} f(x)}{\partial x^l} \right| &< \epsilon \|f\|, \\ x \in [a+\delta_1, b], \quad 0 \le l \le \mu. \end{aligned} \tag{21}$$

Next, applying (12) to the first subinterval, it is clear that

$$\begin{aligned} &\left| \frac{\partial^l M^*_{a,b} f(x)}{\partial x^l} - \frac{\partial^l M^*_{a+\delta_1,b} f(x)}{\partial x^l} \right| \\ &\le \sum_{i=l}^{\mu} \left| \frac{\partial^i M_{a,b} f(a)}{\partial x^i} \frac{(x-a)^{i-l}}{(i-l)!} \right. \\ &\quad \left. - \frac{\partial^i M_{a+\delta_1,b} f(a+\delta_1)}{\partial x^i} \frac{(x-a-\delta_1)^{i-l}}{(i-l)!} \right| \\ &\le \sum_{i=l}^{\mu} \left| \frac{\partial^i M_{a,b} f(a)}{\partial x^i} - \frac{\partial^i M_{a,b} f(a+\delta_1)}{\partial x^i} \right| \frac{(a-A)^{i-l}}{(i-l)!} \\ &\quad + \sum_{i=l}^{\mu} \left| \frac{\partial^i M_{a,b} f(a+\delta_1)}{\partial x^i} - \frac{\partial^i M_{a+\delta_1,b} f(a+\delta_1)}{\partial x^i} \right| \\ &\quad \cdot \frac{(a-A)^{i-l}}{(i-l)!} + \sum_{i=l}^{\mu} \left| \frac{\partial^i M_{a+\delta_1,b} f(a+\delta_1)}{\partial x^i} \right| \left| \frac{(x-a)^{i-l}}{(i-l)!} \right. \\ &\quad \left. - \frac{(x-a-\delta_1)^{i-l}}{(i-l)!} \right|, \end{aligned} \tag{22}$$

for $x \in [A, a]$, $0 \le l \le \mu$.

From the continuity of $\partial^l M_{a,b} f(x)/\partial x^l$ on a, (13), (21), and (22), there must exist $\delta_3 > 0$ such that if $\delta_1 < \delta_3$ then

$$\begin{aligned} \left| \frac{\partial^l M^*_{a,b} f(x)}{\partial x^l} - \frac{\partial^l M^*_{a+\delta_1,b} f(x)}{\partial x^l} \right| &< \epsilon \|f\|, \\ x \in [A, a], \quad 0 \le l \le \mu. \end{aligned} \tag{23}$$

A similar (and simpler) analysis in the fourth interval shows us that there must exist $\delta_4 > 0$ such that if $\delta_1 < \delta_4$ then

$$\begin{aligned} \left| \frac{\partial^l M^*_{a,b} f(x)}{\partial x^l} - \frac{\partial^l M^*_{a+\delta_1,b} f(x)}{\partial x^l} \right| &< \epsilon \|f\|, \\ x \in [b, B], \quad 0 \le l \le \mu. \end{aligned} \tag{24}$$

Last but not least, for the second subinterval $[a, a+\delta_1]$, we have

$$\begin{aligned}\left|\frac{\partial^l M^*_{a,b} f(x)}{\partial x^l} - \frac{\partial^l M^*_{a+\delta_1,b} f(x)}{\partial x^l}\right| &= \left|\frac{\partial^l M^*_{a,b} f(x)}{\partial x^l}\right. \\ &\quad \left. - \sum_{i=l}^{\mu} \frac{\partial^i M_{a+\delta_1,b} f(a+\delta_1)}{\partial x^i} \frac{(x-a-\delta_1)^{i-l}}{(i-l)!}\right| \\ &\le \left|\frac{\partial^l M^*_{a,b} f(x)}{\partial x^l} - \sum_{i=l}^{\mu} \frac{\partial^i M_{a,b} f(a)}{\partial x^i} \frac{(x-a)^{i-l}}{(i-l)!}\right| \\ &\quad + \left|\sum_{i=l}^{\mu} \frac{\partial^i M_{a,b} f(a)}{\partial x^i} \frac{(x-a)^{i-l}}{(i-l)!}\right. \\ &\quad \left. - \sum_{i=l}^{\mu} \frac{\partial^i M_{a+\delta_1,b} f(a+\delta_1)}{\partial x^i} \frac{(x-a-\delta_1)^{i-l}}{(i-l)!}\right|,\end{aligned} \tag{25}$$

for $x \in [a, a+\delta_1]$ and $0 \le l \le \mu$. Taylor's theorem allows us to bound the first term of the right hand side of (25) as

$$\begin{aligned}\left|\frac{\partial^l M^*_{a,b} f(x)}{\partial x^l} - \sum_{i=l}^{\mu} \frac{\partial^i M_{a,b} f(a)}{\partial x^i} \frac{(x-a)^{i-l}}{(i-l)!}\right| &\le \left|\frac{\partial^{\mu+1} M_{a,b} f(d(x))}{\partial x^{\mu+1}}\right| \frac{\delta_1^{\mu-l+1}}{(\mu-l+1)!} \\ &\le \begin{cases} \|f\| \dfrac{\delta_1^{\mu-l+1}}{(\mu-l+1)!} \displaystyle\int_a^b \left|\frac{\partial^{\mu+1} G_{ab}(d(x),t)}{\partial x^{\mu+1}}\right| \sum_{i=0}^{\mu} \left|c_i(t)\right| dt, & \mu+1 < n, \\ \|f\| \dfrac{\delta_1^{\mu-l+1}}{(\mu-l+1)!} \left[\displaystyle\sum_{i=0}^{\mu} \left|c_i(d(x))\right| + \int_a^b \left|\frac{\partial^{\mu+1} G_{ab}(d(x),t)}{\partial x^{\mu+1}}\right| \sum_{i=0}^{\mu} \left|c_i(t)\right| dt\right], & \mu+1 = n, \end{cases}\end{aligned} \tag{26}$$

with $d(x) \in]a, a+\delta_1[$. From (25) and (26) and following the same reasoning used in (22) and (23), it is possible to show that there exists $\delta_5 > 0$ such that, for $\delta_1 < \delta_5$, one has

$$\left|\frac{\partial^l M^*_{a,b} f(x)}{\partial x^l} - \frac{\partial^l M^*_{a+\delta_1,b} f(x)}{\partial x^l}\right| < \epsilon \|f\| \qquad x \in [a, a+\delta_1], \ 0 \le l \le \mu. \tag{27}$$

After covering all subinterval cases, if we set $\delta_1 < \min\{\delta_2, \delta_3, \delta_4, \delta_5\}$, from (21), (23), (24), and (27), we get to

$$\left\|M^*_{a,b} f - M^*_{a+\delta_1,b} f\right\| < \epsilon \|f\|, \tag{28}$$

which implies that M^*_{ab} is continuous with a on the uniform operator topology. This completes the proof that $r(M^*_{ab})$ is continuous with a. The proof for continuity with b follows exactly the same steps and will not be repeated.

Finally, the eigenfunction u of M_{ab} associated with the eigenvalue $r(M_{ab})$ and defined on $[a,b]$ can be extended onto the interval $[A,B]$ as we did with M^*; that is,

$$u^* = \begin{cases} \displaystyle\sum_{i=0}^{\mu} u^{(i)}(a) \frac{(x-a)^i}{i!}, & x \in [A,a], \\ u(x), & x \in [a,b], \\ \displaystyle\sum_{i=0}^{\mu} u^{(i)}(b) \frac{(x-b)^i}{i!}, & x \in [b,B]. \end{cases} \tag{29}$$

As a result, one has $M^*_{ab} u^* = r(M_{ab}) u^*$, which implies $r(M_{ab}) \le r(M^*_{ab})$. Given that $r(M^*_{ab}) \le r(M_{ab})$ as both operators coincide in the interval $[a,b]$, one finally gets to $r(M_{ab}) = r(M^*_{ab})$, which guarantees the continuity of $r(M_{ab})$ with a and b. □

Theorem 2. *Let us suppose that there is a Banach space B and a reproducing cone P therein for which $M(P) \subset P$ and M is u_0-positive. Then eigenvalue problem (10) has a solution $u \in P$ and its associated eigenvalue λ is positive, simple, and bigger in absolute value than any other eigenvalue of (10).*

In addition, if $r(M)$ is strictly increasing with the length of the interval $[a,b]$ (i.e., if fixed a, $r(M_{ab})$ is increasing with b and fixed b, $r(M_{ab})$ is decreasing with a) and $\lim_{b \to a^+} r(M_{ab}) = \lim_{a \to b^-} r(M_{ab}) = 0$, one has the following:

(i) *If there is no nontrivial solution of (3) that satisfies (4) either at a, b or at any a', b' interior to $[a,b]$, then*

$$\lim_{k \to \infty} M^k v = 0; \tag{30}$$

for any $v \in P$ and for any $v, w \in P \setminus \{0\}$, there exists $k_0 \ge 1$ such that

$$M^k v \le M w, \quad k \ge k_0, \tag{31}$$

and there cannot be any $v \in P \setminus \{0\}$ and any $k_1 \ge 1$ such that

$$M^{k_1} v \ge v. \tag{32}$$

(ii) *If there is a nontrivial solution of (3) that satisfies (4) at some a', b' interior to $[a,b]$, then, for any $v, w \in P \setminus \{0\}$, there exists $k_2 \ge 1$ such that*

$$M^k v \ge M w, \quad k \ge k_2, \tag{33}$$

and there cannot be any $v \in P \setminus \{0\}$ and any $k_3 \ge 1$ such that

$$M^{k_3} v \le v. \tag{34}$$

Proof. Since P is a reproducing cone and M is linear, compact, and u_0-positive, the existence of a solution $u \in P$ of eigenvalue problem (10) whose eigenvalue λ is positive, simple, and bigger in absolute value than any other eigenvalue of (10) is a consequence of Theorems 2.5, 2.10, 2.11, and 2.13 of Krasnosel'skii [16], summarized in [25, Theorem 2].

Moreover, if M is u_0-positive, it will also be u-positive. To see this, let us recall that there must be $\delta_3, \delta_4 > 0$ such that $\delta_3 u_0 \leq Mu \leq \delta_4 u_0$. Since $Mu = \lambda u$, once has $(\lambda/\delta_4)u \leq u_0 \leq (\lambda/\delta_3)u$ and therefore $(\delta_1\lambda/\delta_4)u \leq Mv \leq (\delta_2\lambda/\delta_3)u$ for any other nonzero $v \in P$. As a result, for any nonzero $v \in P$, one has

$$0 \leq M^k v = M^{k-1}Mv \leq M^{k-1}\left\{\frac{\delta_2\lambda}{\delta_3}u\right\} = \frac{\delta_2\lambda^k}{\delta_3}u, \quad (35)$$

$$M^k v = M^{k-1}Mv \geq M^{k-1}\left\{\frac{\delta_1\lambda}{\delta_4}u\right\} = \frac{\delta_1\lambda^k}{\delta_4}u \geq 0. \quad (36)$$

Now, let us suppose that there is no nontrivial solution of (3) that satisfies (4) either at a, b or at any a', b' interior to $[a, b]$. Then the eigenvalue $\lambda = r(M_{ab}) < 1$. To see this, let us first fix the lower extreme a and make b variable, and let us recall that, from Lemma 1, $r(M_{ab})$ is continuous with b. Given that $r(M_{ab})$ is increasing with b and $\lim_{b\to a^+} r(M_{ab}) = 0$, if $\lambda \geq 1$ it would be possible to find a b' with $a < b' \leq b$ such that $\lambda_{ab'} = r(M_{ab'}) = 1$ and therefore problem (9) (ergo (3) and (4)) would have a solution with the extremes a and b', contrary to the assumption. A similar result can be obtained fixing the upper extreme b and making a variable. As M is compact, P is closed, and $\lambda < 1$, then

$$\lim_{k\to\infty} M^k v = \lim_{k\to\infty} \frac{\delta_2\lambda^k}{\delta_3}u = 0, \quad (37)$$

which is (30). Given that M is u-positive, for any $w \in P \setminus \{0\}$, we have

$$Mw \geq \delta_5 u. \quad (38)$$

Combining (35) and (38), one has

$$M^k v \leq \frac{\delta_2\lambda^k}{\delta_3\delta_5}Mw, \quad (39)$$

which gives (31). On the other hand, let us suppose that there exists $w \in P \setminus \{0\}$ such that

$$M^{k_1}w \geq w. \quad (40)$$

From [21, Theorem 2.2], the fact that $M^{k_1}u = \lambda^{k_1}u$, and (40), one has that $\lambda \geq 1$, which contradicts the hypothesis. This proves (32).

Let us now suppose that there is a nontrivial solution of (3) that satisfies (4) at some a', b' interior to $[a, b]$. Then the eigenvalue $\lambda = r(M_{ab}) > 1$. Otherwise, fixing a and making b variable, since $r(M_{ab})$ is increasing with b, there could not be a solution of (10) with $\lambda_{ab'} = r(M_{ab'}) = 1$ and $a < b' < b$ (a similar argument can be used fixing b and making a variable). Given that M is u-positive, for any $w \in P$, one has

$$Mw \leq \delta_6 u. \quad (41)$$

Combining (36) and (41), one has

$$M^k v \geq \frac{\delta_1\lambda^k}{\delta_4\delta_6}Mw, \quad (42)$$

from which (33) is immediate. To complete the proof, let us suppose that there exists $w \in P$ such that

$$M^{k_3}w \leq w. \quad (43)$$

From [21, Theorem 2.2], the fact that $M^{k_3}u = \lambda^{k_3}u$ and (43), one has that $\lambda \leq 1$, which contradicts the hypothesis. This proves (34). □

Corollary 3. *The conclusions of Theorem 2 are also valid if M maps P into P^0 and $r(M)$ is strictly increasing with the length of the interval $[a, b]$.*

Proof. It is a straightforward consequence of the facts that if P^0 exists, then P is reproducing cone ([17, Proposition 19.1.a]), and that if $M(P) \in P^0$, then M is u_0-positive ([33, Lemma 1.5]). □

Corollary 4. *Under the hypotheses of Theorem 2, a sufficient condition for problem (3) not to have a nontrivial solution that satisfies (4) either at a, b or at any a', b' interior to $[a, b]$ is the existence of $v \in P$ and $k, j \geq 0$ with $k > j$ such that*

$$M^k v \leq M^j v. \quad (44)$$

Likewise, a sufficient condition for the problem (3) and (4) to have a solution in the extremes a, b or extremes inner to a and b is the existence of $v \in P$ and $k, j \geq 0$ with $k > j$ such that

$$M^k v \geq M^j v. \quad (45)$$

Proof. Equations (44) and (45) follow from the application of (34) and (32), respectively, to $M^j v$. □

Remark 5. Theorem 2 and Corollaries 3 and 4 provide methods to determine if there exists a solution of (3) and (4) in extremes a', b' interior or exterior to given extremes a, b. In particular,

(i) if there is no nontrivial solution of (3) that satisfies (4) either at a, b or at any a', b' interior to $[a, b]$, then from (30) there may exist $k_0 \geq 1$ and $v \in P$ such that $M^k v \leq v$ for $k \geq k_0$, violating (34), and from (31) for any $z \in P \setminus \{0\}$ there will exist k_1, k_2, and j such that $M^k z \leq Mz$, $k \geq k_1$, and $M^k z \leq M^j z$, $k \geq k_2$, violating also (34) and verifying sufficient condition (44), respectively;

(ii) if there is a nontrivial solution of (3) that satisfies (4) at some a', b' interior to $[a, b]$, then from (33) there may exist $k_0 \geq 1$ and $v \in P$ such that $M^k v \geq v$, $k \geq k_0$, violating (32), and for any $z \in P \setminus \{0\}$ there will exist k_1, k_2, and j such that $M^k z \geq Mz$, $k \geq k_1$, and $M^k z \geq M^j z$, $k \geq k_2$, violating also (32) and verifying sufficient condition (45), respectively.

The key of the method will be to find out functions $v, z \in P$ for which the aforementioned inequalities appear with low indexes k, j.

Remark 6. In the case where it is not possible to prove the two monotonicity conditions of $r(M_{ab})$ with a and b but only one, it is straightforward to establish variants of Theorem 2 and Corollaries 3 and 4 which consider the extreme for which the monotonicity of $r(M_{ab})$ cannot be proven as fixed in terms of application of the corresponding boundary condition, making the existence or nonexistence of the solutions of (3) and (4) occurs in the interval between that fixed extreme and other points lower than, equal to, or higher than the other extreme.

Remark 7. The arguments of Theorem 2 and Corollary 4 can also be used to provide upper and lower bounds for the largest eigenvalue of problem (10). In particular, any $\mu > 0$ for which we can find $v \in P$ and $k > j \geq 0$ such that $M^k v \leq \mu^{k-j} M^j v$ will be an upper bound for the eigenvalue λ. Likewise, any $\mu > 0$ for which we can find $v \in P$ and $k > j \geq 0$ such that $M^k v \geq \mu^{k-j} M^j v$ will be a lower bound for the eigenvalue λ.

3. Application to a Broad Boundary Value Problem

This section will be devoted to applying the results of Section 2 to a problem of the type (3) and (4) characterized by

$$\begin{aligned} Ly &= \sum_{i=0}^{\mu} c_i(x)\, y^{(i)}(x), \quad x \in\,]a,b[; \\ T(\alpha,\beta,a,b)\, y &= 0, \end{aligned} \tag{46}$$

with L being defined as in (1) with

$$a_n(x) = 1, \quad x \in [a,b], \tag{47}$$

$$L \text{ being right-disfocal on } [a,b], \tag{48}$$

$$\alpha = (0,1,2,\ldots,k-1), \tag{49}$$

$$\beta = (\beta_1,\beta_2,\ldots,\beta_{n-k}), \quad 0 \leq \beta_1 < \cdots < \beta_{n-k} < n-1, \tag{50}$$

$$\mu \leq \beta_1, \tag{51}$$

$$(-1)^{n-k} c_i(x) \geq 0, \quad x \in [a,b],\ 0 \leq i \leq \mu, \tag{52}$$

$$(-1)^{n-k} c_j(x) > 0 \quad \text{a.e. on } [a,b] \tag{53}$$

for at least one j such that $0 \leq j \leq \mu$.

Problem (46)–(53) was studied by Eloe et al. in two papers [30, 42]. In particular, in [42], Eloe and Ridenhour analysed the positivity and monotonicity properties of the Green function of problem (7), subject to the constraints (47)–(50), whereas, in [30], Eloe and Henderson applied these properties to address the eigenvalue problem $Mu = \lambda u$ with M being defined as in (6) and $G(x,t)$ being the aforementioned Green function. In order to analyse problem (46)–(53), we will use a Banach space B and a cone P slightly different from those used by Eloe and Henderson, concretely:

$$B = \left\{ y \in C^{\beta_1 - 1}[a,b],\ y^{(\beta_1)}(x) \text{ piecewise continuous on } [a,b],\ y^{(i)}(a) = 0,\ 0 \leq i \leq \beta_1 - 1 \right\} \tag{54}$$

(the Banach space considered by Eloe and Henderson satisfied also $y \in C^{\beta_1}[a,b]$ and $y^{(\beta_1)}(a) = 0$) and

$$P = \left\{ y \in B : (-1)^{n-k} y^{(\beta_1)}(x) \geq 0;\ x \in [a,b] \right\}. \tag{55}$$

From the definition of P, it is clear that $(-1)^{n-k} y^{(i)}(x) \geq 0$, $x \in [a,b]$, $0 \leq i \leq \beta_1$. With the help of the cone P, it is possible to prove the following theorem.

Theorem 8. *The conclusions of Theorem 2 and Corollary 4 are applicable for problem (46)–(53) and the cone P defined in (55).*

Proof. We just need to prove that the conditions of Theorem 2 are applicable to this problem and cone. Thus, if we use the notation

$$\begin{aligned} \{u(x)\}^{+} &= (-1)^{n-k} \max\left\{ (-1)^{n-k} u(x), 0 \right\}, \\ \{u(x)\}^{-} &= (-1)^{n-k} \max\left\{ -(-1)^{n-k} u(x), 0 \right\}, \end{aligned} \tag{56}$$

for $x \in [a,b]$, that P is a reproducing cone follows from

$$y^{(\beta_1)}(x) = \left\{ y^{(\beta_1)}(x) \right\}^{+} - \left\{ y^{(\beta_1)}(x) \right\}^{-}, \quad x \in [a,b], \tag{57}$$

so that

$$\begin{aligned} y(x) &= \int_a^x \frac{(x-t)^{\beta_1 - 1}}{(\beta_1 - 1)!} \left\{ y^{(\beta_1)}(t) \right\}^{+} dt \\ &\quad - \int_a^x \frac{(x-t)^{\beta_1 - 1}}{(\beta_1 - 1)!} \left\{ y^{(\beta_1)}(t) \right\}^{-} dt, \quad x \in [a,b], \end{aligned} \tag{58}$$

for any $y \in B$. Obviously the two terms of the right hand side of (58) are functions which belong to P.

To prove the u_0-positivity of M, let us consider the auxiliary Banach space $\mathscr{B}$, defined by

$$\mathscr{B} = \left\{ y \in C^{n-1}[a,b] : y^{(i)}(a) = 0,\ 0 \leq i \leq k - 1;\ y^{(\beta_i)}(b) = 0,\ 1 \leq i \leq n-k \right\}, \tag{59}$$

and the auxiliary cone $\mathscr{P}$, defined by

$$\mathscr{P} = \left\{ y \in \mathscr{B} : (-1)^{n-k} y^{(\beta_1)}(x) \geq 0,\ x \in [a,b] \right\}. \tag{60}$$

If we denote by m the lowest integer that satisfies $m > \beta_1$ and $m \neq \beta_2, \ldots, \beta_{n-k}$, then, following an argument of Diaz (see [33, Lemma 2.8]), it is possible to prove that the interior of $\mathscr{P}$ is

$$\mathscr{P}^0 = \left\{ y \in \mathscr{B} : (-1)^{n-k} y^{(\beta_1)}(x) > 0,\ x \in\,]a,b[;\ (-1)^{n-k} y^{(k)}(a) > 0;\ (-1)^{n-k-m+\beta_1} y^{(m)}(b) > 0 \right\}. \tag{61}$$

M maps $P\setminus\{0\}$ into $\mathscr{P}^0$. To see this, let us take $v\in P\setminus\{0\}$. Obviously $Mv\in\mathscr{B}$ by the construction of M. Since $v^{(j)}(a)=0$ for $0\le j\le\beta_1-1$, and $(-1)^{n-k}v^{(\beta_1)}(x)$ is piecewise continuous, nonnegative, and nonidentically zero on $[a,b]$ (otherwise $v\equiv 0$), $(-1)^{n-k}v^{(j)}(x)$ must also be piecewise continuous, nonnegative, and nonidentically zero on $[a,b]$ for $0\le j\le\mu$. From here, the hypotheses (52) and (53), and the fact that

$$(-1)^{n-k}\frac{\partial^{\beta_1}G(x,t)}{\partial x^{\beta_1}}>0 \tag{62}$$

on $(x,t)\in\,]a,b[\,\times\,]a,b[$ from [42, Theorem 2.7], one has

$$\begin{aligned}(-1)^{n-k}&(Mv)^{(\beta_1)}(x)\\ &=(-1)^{n-k}\int_a^b\frac{\partial^{\beta_1}G(x,t)}{\partial x^{\beta_1}}\sum_{i=0}^{\mu}c_i(t)\,v^{(i)}(t)\,dt>0,\\ &\qquad x\in\,]a,b[\,.\end{aligned} \tag{63}$$

In the same manner, also from [42, Theorem 2.7], one has

$$(-1)^{n-k}\frac{\partial^{k}G(a,t)}{\partial x^{k}}>0,\quad t\in\,]a,b[\,, \tag{64}$$

which, together with hypotheses (52) and (53), yield

$$\begin{aligned}(-1)^{n-k}&(Mv)^{(k)}(a)\\ &=(-1)^{n-k}\int_a^b\frac{\partial^{k}G(a,t)}{\partial x^{k}}\sum_{i=0}^{\mu}c_i(t)\,v^{(i)}(t)\,dt>0.\end{aligned} \tag{65}$$

Finally from [42, Lemma 2.4] one has

$$(-1)^{n-k-m+\beta_1}\frac{\partial^{m}G(b,t)}{\partial x^{m}}>0,\quad t\in\,]a,b[\,, \tag{66}$$

which combined with (52) and (53) give

$$\begin{aligned}(-1)^{n-k-m+\beta_1}&(Mv)^{(m)}(b)\\ &=(-1)^{n-k-m+\beta_1}\int_a^b\frac{\partial^{m}G(b,t)}{\partial x^{m}}\sum_{i=0}^{\mu}c_i(t)\,v^{(i)}(t)\,dt>0.\end{aligned} \tag{67}$$

Therefore, $Mv\in\mathscr{P}^0$ for any $v\in P\setminus\{0\}$, and if we pick any $u_0\in\mathscr{P}^0$, there must be $\epsilon_1>0$ such that

$$Mv-\epsilon_1u_0\in\mathscr{P}, \tag{68}$$

which implies

$$Mv\ge\epsilon_1u_0\quad(\text{w.r.t. }\mathscr{P}). \tag{69}$$

Likewise, there must be $\epsilon_2>0$ such that

$$u_0-\epsilon_2Mv\in\mathscr{P}, \tag{70}$$

which implies

$$u_0\ge\epsilon_2Mv\quad(\text{w.r.t. }\mathscr{P}). \tag{71}$$

Combining (69) and (71), one gets to

$$\epsilon_1u_0\le Mv\le\frac{u_0}{\epsilon_2}\quad(\text{w.r.t. }\mathscr{P}). \tag{72}$$

As $u_0\in\mathscr{P}^0$ and $\mathscr{P}\subset P$ (this follows from the fact that $\mathscr{B}\subset B$), (72) proves that M is u_0-positive in P.

Next, fixed a, the increasing monotonicity of $r(M_b)$ as a function of b, follows from [30, Theorem 8] (in the proof of such Eloe's theorem, the difference between our cone P and that used by Eloe does not have any effect). Eloe and Henderson did not get to prove an equivalent result for the monotonicity of $r(M_a)$ with a, but we can replicate the steps used for b to obtain a similar result. To do so, let us define the Banach space

$$\mathtt{B}_a=\left\{y\in C^m[a,b]: y^{(\beta_i)}(b)=0,\ 1\le i\le n-k,\ \beta_i<m\right\} \tag{73}$$

and the cone

$$\mathtt{P}_a=\left\{y\in\mathtt{B}_a:(-1)^{n-k}y^{(j)}(x)\ge 0,\ x\in[a,b],\ 0\le j\le\beta_1\right\}, \tag{74}$$

whose interior is given by

$$\begin{aligned}\mathtt{P}_a^0=\Big\{&y\in\mathtt{B}_a:(-1)^{n-k}y^{(j)}(x)>0,\ x\in[a,b],\ 0\le j\le\beta_1-1;\ (-1)^{n-k}y^{(\beta_1)}(x)>0,\ x\\ &\in[a,b[\,;\ (-1)^{n-k-m+\beta_1}y^{(m)}(b)>0\Big\}.\end{aligned} \tag{75}$$

Like we did before, one can show that the cone $\mathscr{P}_a$ of (60) satisfies $\mathscr{P}_a\subset\mathtt{P}_a$, $M(\mathtt{P}_a)=\mathscr{P}_a^0$ and that M is u_0-positive in $\mathtt{P}_a$. Now, let us take $a'<a$, and let us notice that, from [25, Theorem 2] and the previous arguments, there is $u_a\in\mathscr{P}_a^0\subset\mathtt{P}_a$ such that $M_{ab}u_a=\lambda_au_a$. We can extend the definition of M_{ab} to make it map $C^{\mu}[a,b]$ into $C^{k-1}[a',b]$ by setting

$$M_{a'b}^{*}f(x)=\begin{cases}0, & x\in\left[a',a\right],\\ M_{ab}f(x), & x\in[a,b].\end{cases} \tag{76}$$

If we extend u_a similarly to $[a',b]$ by setting

$$u_{a'}(x)=\begin{cases}0, & x\in\left[a',a\right],\\ u_a(x), & x\in[a,b],\end{cases} \tag{77}$$

then it is clear that $M_{a'b}^{*}u_{a'}=\lambda_au_{a'}$ and

$$\begin{aligned}&M_{a'b}u_{a'}(x)-M_{a'b}^{*}u_{a'}(x)\\ &=\begin{cases}\displaystyle\int_{a'}^{b}G_{a'b}(x,t)\sum_{i=0}^{\mu}c_i(t)\,u_{a'}^{(i)}(t)\,dt, & x\in\left[a',a\right],\\ \displaystyle\int_{a}^{b}\left(G_{a'b}(x,t)-G_{ab}(x,t)\right)\sum_{i=0}^{\mu}c_i(t)\,u_{a'}^{(i)}(t)\,dt, & x\in[a,b].\end{cases}\end{aligned} \tag{78}$$

We will prove now that the restriction of $M_{a'b}u_{a'} - M^*_{a'b}u_{a'}$ to $[a,b]$ belongs to P^0_a. To do so, we need to obtain first some of the properties of $\partial G_{ab}(x,t)/\partial a$. Following similar reasoning to that of [42, Lemma 2.5] for $\partial G_{ab}(x,t)/\partial b$, it is possible to prove that $\partial G_{ab}(x,t)/\partial a$ is a solution of the BVP

$$\begin{aligned} &Ly = 0; \quad x \in [a,b]; \\ &y^{(i)}(a) = 0, \quad 0 \le i \le k-2; \\ &y^{k-1}(a) = -\frac{\partial^k G_{ab}(a,t)}{\partial x^k}; \\ &y^{(\beta_i)}(b) = 0, \quad 1 \le i \le n-k. \end{aligned} \tag{79}$$

From [42, Lemma 2.3], one has that $y^{(i)}(x)$ does not change sign on $]a,b[$ for $i = 0,1,\ldots,\beta_1$ and that we can extend the sign of $y^{(i)}(x)$ to $]a,b]$ for $i = 0,1,\ldots,\beta_1 - 1$, given that $y^{(i)}(b) \neq 0$ for these indices. In addition, from [42, Theorem 2.7], we know that

$$(-1)^{n-k}\frac{\partial^k G_{ab}(a,t)}{\partial x^k} > 0, \quad t \in]a,b[. \tag{80}$$

Accordingly, applying Taylor's theorem and the boundary conditions on a of (79), one has that $(-1)^{n-k}y^{(i)}(x) < 0$, for $x \in]a,b]$ and $i = 0,1,\ldots,\beta_1 - 1$, and that $(-1)^{n-k}y^{(\beta_1)}(x) < 0$, for $x \in]a,b[$, which implies

$$(-1)^{n-k}\frac{\partial^i G_{a'b}(x,t)}{\partial x^i} > (-1)^{n-k}\frac{\partial^i G_{ab}(x,t)}{\partial x^i}, \tag{81}$$

for $(x,t) \in]a,b] \times]a,b[$ and $i = 0,1,\ldots,\beta_1 - 1$, and

$$(-1)^{n-k}\frac{\partial^{\beta_1} G_{a'b}(x,t)}{\partial x^{\beta_1}} > (-1)^{n-k}\frac{\partial^{\beta_1} G_{ab}(x,t)}{\partial x^{\beta_1}}, \tag{82}$$

for $(x,t) \in]a,b[\times]a,b[$. Likewise, from property (48) and (79) it cannot happen that $y^{(m)}(b) = 0$. If $(-1)^{n-k-m+\beta_1}y^{(m)}(b) > 0$, from the boundary conditions on b of (79) and Taylor's theorem, one would have $(-1)^{n-k}y^{(\beta_1)}(x) > 0$ for $x \in]a,b[$, contradicting (82). Therefore, $(-1)^{n-k-m+\beta_1}y^{(m)}(b) < 0$, which in turn implies

$$\begin{aligned} &(-1)^{n-k-m+\beta_1}\frac{\partial^m G_{a'b}(b,t)}{\partial x^m} \\ &\quad > (-1)^{n-k-m+\beta_1}\frac{\partial^m G_{ab}(b,t)}{\partial x^m}, \end{aligned} \tag{83}$$

for $t \in]a,b[$. It remains to prove what happens at $x = a$. Applying [41, Theorem V.3.1] recursively to BVP (79), one has that $\partial^l G_{ab}(x,t)/\partial a^l$ is a solution of the initial value problem:

$$\begin{aligned} &Ly = 0; \quad x \in [a,b]; \\ &y^{(i)}(a) = 0, \quad 0 \le i \le k-l-1; \\ &y^{k-l}(a) = (-1)^l\frac{\partial^k G_{ab}(a,t)}{\partial x^k}. \end{aligned} \tag{84}$$

Therefore, we can fix t, b, and $x = a$ and apply Taylor's theorem with regard to the variable a to obtain

$$\begin{aligned} &\frac{\partial^i G_{a'b}(a,t)}{\partial x^i} - \frac{\partial^i G_{ab}(a,t)}{\partial x^i} \\ &\quad = \sum_{j=1}^{k-i-1}(-1)^j\frac{\left(a-a'\right)^j}{j!}\frac{\partial^i}{\partial x^i}\frac{\partial^j G_{ab}(a,t)}{\partial a^j} \\ &\quad\quad + (-1)^{k-i}\frac{\left(a-a'\right)^{k-i}}{(k-i)!}\frac{\partial^i}{\partial x^i}\frac{\partial^{k-i} G_{cb}(a,t)}{\partial a^{k-i}} \\ &\quad = (-1)^{k-i}\frac{\left(a-a'\right)^{k-i}}{(k-i)!}\frac{\partial^i}{\partial x^i}\frac{\partial^{k-i} G_{cb}(a,t)}{\partial a^{k-i}}, \end{aligned} \tag{85}$$

for $t \in]a,b[$, $0 \le i \le k-1$, with $c \in]a',a[$. By continuity, we can find $\epsilon > 0$ such that, for $a' > a - \epsilon$, the sign of $(\partial^i/\partial x^i)(\partial^{k-i}G_{cb}(a,t)/\partial a^{k-i})$ and $(\partial^i/\partial x^i)(\partial^{k-i}G_{ab}(a,t)/\partial a^{k-i})$ coincide, sign which is the same as that of $(-1)^{k-i}(\partial^k G_{ab}(a,t)/\partial x^k)$ according to (84). From here, the fact that $\beta_1 \le k-1$, (80), and (85), we have that

$$\begin{aligned} &(-1)^{n-k}\left(\frac{\partial^i G_{a'b}(a,t)}{\partial x^i} - \frac{\partial^i G_{ab}(a,t)}{\partial x^i}\right) > 0, \\ &\qquad\qquad t \in]a,b[, \ 0 \le i \le \beta_1. \end{aligned} \tag{86}$$

From (52), (53), (75), (78), (81), (82), (83), and (86), it follows that the restriction of $M_{a'b}u_{a'} - M^*_{a'b}u_{a'}$ to $[a,b]$ belongs to P^0_a, as indicated before. Accordingly there exists $\delta > 0$ such that

$$M_{a'b}u_{a'} - M^*_{a'b}u_{a'} \ge \delta u_{a'} \quad (\text{w.r.t. } \mathrm{P}_a); \tag{87}$$

that is

$$M_{a'b}u_{a'} \ge \left(\lambda_a + \delta\right)u_{a'} \quad (\text{w.r.t. } \mathrm{P}_a). \tag{88}$$

From [42, Theorem 2.7] we know that

$$(-1)^{n-k}\frac{\partial^i G_{a'b}(x,t)}{\partial x^i} > 0, \quad 0 \le i \le \beta_1, \tag{89}$$

for $(x,t) \in]a',b[\times]a',b[$, which together with (52), (53), and (78) gives

$$\begin{aligned} &(-1)^{n-k}\left(\frac{\partial^i M_{a'b}u_{a'}(x)}{\partial x^i} - \frac{\partial^i M^*_{a'b}u_{a'}(x)}{\partial x^i}\right) \ge 0, \\ &\qquad\qquad x \in \left[a',a\right], \ 0 \le i \le \beta_1. \end{aligned} \tag{90}$$

The combination of (77), (88), and (90) yields

$$M_{a'b}u_{a'} \ge \left(\lambda_a + \delta\right)u_{a'} \quad (\text{w.r.t. } \mathrm{P}_{a'}). \tag{91}$$

From here, the u_0-positivity of M in $\mathrm{P}_{a'}$, and [21, Theorem 2.2], we can conclude that $\lambda_{a'} = r(M_{a'b}) > \lambda_a$ and therefore that $r(M_{ab})$ is decreasing as a function of a.

Finally, if we take the norm $\|f\| = \max\{\sup\{|f^{(j)}(x)|,\ x \in [a,b]\};\ 0 \le j \le \mu\}$, then from (6) one has

$$\begin{aligned} &\left|(M_{ab}f)^{(j)}(x)\right| \\ &\quad \le \int_a^b \left|\frac{\partial^j G_{ab}(x,t)}{\partial x^j}\right| \sum_{i=0}^{\mu} |c_i(t)| \left|f^{(i)}(t)\right| dt \\ &\quad \le \|f\| \int_a^b \left|\frac{\partial^j G_{ab}(x,t)}{\partial x^j}\right| \sum_{i=0}^{\mu} |c_i(t)|\, dt, \quad 0 \le j \le \mu. \end{aligned} \tag{92}$$

Therefore

$$\begin{aligned} &\|M_{ab}f\| \le \|f\| \\ &\quad \cdot \max\left\{\sup\left\{\int_a^b \left|\frac{\partial^j G_{ab}(x,t)}{\partial x^j}\right| \sum_{i=0}^{\mu} |c_i(t)|\, dt,\ x \in [a,b]\right\},\right. \\ &\quad \left. 0 \le j \le \mu\right\}. \end{aligned} \tag{93}$$

Taking $[A,B] \subset I$ such that $G_{AB}(x,t)$ exists and $[a,b] \subseteq [A,B]$, from the previous results on the monotonicity of $G_{ab}(x,t)$ with a, b, one has

$$\begin{aligned} &\|M_{ab}\| \\ &\quad \le \max\left\{\sup\left\{\int_a^b \left|\frac{\partial^j G_{AB}(x,t)}{\partial x^j}\right| \sum_{i=0}^{\mu} |c_i(t)|\, dt,\ x \in [a,b]\right\},\right. \\ &\quad \left. 0 \le j \le \mu\right\}. \end{aligned} \tag{94}$$

Now, from [43, Theorems 3.6.1 and 3.7.8], one has $r(M_{ab}) \le \|M_{ab}\|$. From here, (94), and the continuity of $c_i(x)$ on $[a,b]$, one gets

$$\lim_{b \to a^+} r(M_{ab}) = \lim_{a \to b^-} r(M_{ab}) = 0. \tag{95}$$

This completes the proof. □

In order to apply Theorem 2, we can consider, for instance, the function

$$v(x) = (-1)^{n-k} \frac{(x-a)^{\beta_1}}{\beta_1!}, \quad x \in [a,b], \tag{96}$$

which satisfies

$$\begin{aligned} v^{(\beta_1)}(x) &= (-1)^{n-k}, \quad x \in [a,b]; \\ v(a) &= v'(a) = \cdots = v^{(\beta_1-1)}(a) = 0. \end{aligned} \tag{97}$$

From the definition of $v(x)$, Theorem 2, and Corollary 4, we can obtain the following theorem.

Theorem 9. *A sufficient condition for problem (46)–(53) not to have a nontrivial solution in extremes interior to a, b is the existence of $k, j \ge 0$ with $k > j$ such that*

$$\begin{aligned} &\int_a^b \frac{\partial^{\beta_1} G(x,t)}{\partial x^{\beta_1}} \sum_{i=0}^{\mu} c_i(t) \left(M^k\left\{\frac{(x-a)^{\beta_1}}{\beta_1!}(t)\right\}\right)^{(i)} dt \\ &\quad \le \int_a^b \frac{\partial^{\beta_1} G(x,t)}{\partial x^{\beta_1}} \sum_{i=0}^{\mu} c_i(t) \left(M^j\left\{\frac{(x-a)^{\beta_1}}{\beta_1!}(t)\right\}\right)^{(i)} dt, \end{aligned} \tag{98}$$

for all $x \in [a,b]$, or, in particular, the existence of $k \ge 0$ such that

$$\begin{aligned} &\int_a^b \frac{\partial^{\beta_1} G(x,t)}{\partial x^{\beta_1}} \sum_{i=0}^{\mu} c_i(t) \left(M^k\left\{\frac{(x-a)^{\beta_1}}{\beta_1!}\right\}(t)\right)^{(i)} dt \\ &\quad \le 1, \end{aligned} \tag{99}$$

for all $x \in [a,b]$.

A sufficient condition for problem (46)–(53) to have a nontrivial solution in the extremes a, b or extremes inner to a and b is the existence of $k, j \ge 0$ with $k > j$ such that

$$\begin{aligned} &\int_a^b \frac{\partial^{\beta_1} G(x,t)}{\partial x^{\beta_1}} \sum_{i=0}^{\mu} c_i(t) \left(M^k\left\{\frac{(x-a)^{\beta_1}}{\beta_1!}(t)\right\}\right)^{(i)} dt \\ &\quad \ge \int_a^b \frac{\partial^{\beta_1} G(x,t)}{\partial x^{\beta_1}} \sum_{i=0}^{\mu} c_i(t) \\ &\quad \cdot \left(M^j\left\{\frac{(x-a)^{\beta_1}}{\beta_1!}(t)\right\}\right)^{(i)} dt, \end{aligned} \tag{100}$$

for all $x \in [a,b]$.

The advantage of Theorem 9 is that it provides a simpler sufficient condition (99) for the nonexistence of a solution in extremes inner to a, b. If we extend a technique devised by Nehari and described by Keener and Travis in [21], we can also obtain a simple sufficient condition for the existence of a solution in the extremes a, b or extremes inner to a and b. To this end, let us pick ξ_1, ξ_2 such that $a < \xi_1 < \xi_2 < b$, and let us consider the function

$$h(x) = \begin{cases} 0, & a \le x \le \xi_1, \\ \dfrac{(-1)^{n-k}}{(\beta_1-1)!} \displaystyle\int_{\xi_1}^{x} (x-t)^{\beta_1-1}\, dt, & \xi_1 \le x \le \xi_2, \\ \dfrac{(-1)^{n-k}}{(\beta_1-1)!} \displaystyle\int_{\xi_1}^{\xi_2} (x-t)^{\beta_1-1}\, dt, & \xi_2 \le x \le b, \end{cases} \tag{101}$$

which satisfies $h^{(i)}(a) = 0$ for $0 \le i \le \beta_1 - 1$ and

$$h^{(\beta_1)}(x) = \begin{cases} 0, & a \le x < \xi_1, \\ (-1)^{n-k}, & \xi_1 < x < \xi_2, \\ 0, & \xi_2 < x \le b. \end{cases} \tag{102}$$

From (102), Theorem 2, and Corollary 4, one gets the following theorem.

Theorem 10. *A sufficient condition for problem (46)–(53) not to have a nontrivial solution in extremes interior to a, b is the existence of k, $j \geq 0$ with $k > j$ such that*

$$\begin{aligned} &(-1)^{n-k} \int_a^b \frac{\partial^{\beta_1} G(x,t)}{\partial x^{\beta_1}} \sum_{i=0}^{\mu} c_i(t) \left(M^k \{h\}(t)\right)^{(i)} dt \\ &\quad \leq (-1)^{n-k} \int_a^b \frac{\partial^{\beta_1} G(x,t)}{\partial x^{\beta_1}} \sum_{i=0}^{\mu} c_i(t) \left(M^j \{h\}(t)\right)^{(i)} dt, \end{aligned} \tag{103}$$

for all $x \in [a,b]$.

A sufficient condition for problem (46)–(53) to have a nontrivial solution in the extremes a, b or extremes inner to a and b is the existence of k, $j \geq 0$ with $k > j$ such that

$$\begin{aligned} &(-1)^{n-k} \int_a^b \frac{\partial^{\beta_1} G(x,t)}{\partial x^{\beta_1}} \sum_{i=0}^{\mu} c_i(t) \left(M^k \{h\}(t)\right)^{(i)} dt \\ &\quad \geq (-1)^{n-k} \int_a^b \frac{\partial^{\beta_1} G(x,t)}{\partial x^{\beta_1}} \sum_{i=0}^{\mu} c_i(t) \left(M^j \{h\}(t)\right)^{(i)} dt, \end{aligned} \tag{104}$$

for all $x \in [a,b]$, or, in particular, the existence of a $k \geq 0$ such that

$$(-1)^{n-k} \int_a^b \frac{\partial^{\beta_1} G(x,t)}{\partial x^{\beta_1}} \sum_{i=0}^{\mu} c_i(t) \left(M^k \{h\}(t)\right)^{(i)} dt \geq 1, \tag{105}$$

for all $x \in [\xi_1, \xi_2]$.

In [42], Eloe and Ridenhour used arguments of symmetry with regard to problem (46)–(53) to also provide positivity and monotonicity properties for the Green function of the problem

$$Ly = 0, \quad x \in [a,b], \tag{106}$$

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k), \quad 0 \leq \alpha_1 < \cdots < \alpha_k < n-1, \tag{107}$$

$$\beta = (0, 1, 2, \ldots, n-k-1), \tag{108}$$

with L defined as in (1), left-disfocal on $[a,b]$, and having $a_n(x) = 1$. This allows us to extend the results of Theorem 2 to the problem

$$\begin{gathered} Ly = \sum_{i=0}^{\mu} c_i(x) y^{(i)}(x), \quad x \in]a,b[; \\ T(\alpha, \beta, a, b) y = 0, \end{gathered} \tag{109}$$

$$a_n(x) = 1, \quad x \in [a,b], \tag{110}$$

$$L \text{ being left-disfocal on } [a,b], \tag{111}$$

$$\mu \leq \alpha_1, \tag{112}$$

$$(-1)^{n-k-i} c_i(x) \geq 0, \quad x \in [a,b], \ 0 \leq i \leq \mu, \tag{113}$$

$$(-1)^{n-k-i} c_j(x) > 0 \quad \text{a.e. on } [a,b] \tag{114}$$

for at least one j such that $0 \leq j \leq \mu$, using the Banach space

$$\begin{aligned} B = \{ &y \in C^{\alpha_1 - 1}[a,b], \ y^{(\alpha_1)}(x) \\ &\text{piecewise continuous on } [a,b], \ y^{(i)}(b) = 0, \ 0 \\ &\leq i \leq \alpha_1 - 1 \} \end{aligned} \tag{115}$$

and the cone

$$P = \left\{ y \in B : (-1)^{n-k-\alpha_1} y^{(\alpha_1)}(x) \geq 0; \ x \in [a,b] \right\}. \tag{116}$$

From the definition of P, it is clear that $(-1)^{n-k-i} y^{(i)}(x) \geq 0$, $x \in [a,b]$, $0 \leq i \leq \alpha_1$. Note that Eloe and Henderson did not get to apply these properties of $G(x,t)$ to problem (109)–(114) in any paper, as far as the authors are aware.

Theorem 11. *The conclusions of Theorem 2 and Corollary 4 are applicable for problem (109)–(114) and the cone P defined in (116).*

Proof. The proof is very similar to that of Theorem 8, just using an argument similar to (58) to show that P is a reproducing cone and [42, Theorem 3.1] and the equivalent result of [42, Lemma 2.4] to prove that the operator M associated with (109)–(114) is u_0-positive and applying [42, Theorem 3.2] to the arguments of [30, Theorem 8] to determine the decreasing monotonicity of $r(M_a)$ as a function of a and [42, Theorem 3.4] and the arguments of Theorem 8 to show the increasing monotonicity of $r(M_b)$ as a function of b. □

A consequence of Theorem 11 is that the sufficient conditions of Theorems 9 and 10 are also applicable to problem (109)–(114), changing the function $v(x)$ defined in (96) by

$$v(x) = (-1)^{n-k} \frac{(b-x)^{\alpha_1}}{\alpha_1!}, \quad x \in [a,b] \tag{117}$$

and the function $h(x)$ defined in (101) by

$$h(x) = \begin{cases} \dfrac{(-1)^{n-k}}{(\alpha_1 - 1)!} \displaystyle\int_{\xi_1}^{\xi_2} (t-x)^{\alpha_1 - 1} dt, & a \leq x \leq \xi_1, \\ \dfrac{(-1)^{n-k}}{(\alpha_1 - 1)!} \displaystyle\int_{x}^{\xi_2} (t-x)^{\alpha_1 - 1} dt, & \xi_1 \leq x \leq \xi_2, \\ 0, & \xi_2 \leq x \leq b. \end{cases} \tag{118}$$

At this point the reader will have noticed that (46) and (47) are just a way of representing a subrange of problems of the type

$$\begin{gathered} y^{(n)}(x) + p_{n-1}(x) y^{(n-1)}(x) + \cdots + p_0(x) y(x) = 0, \\ x \in [a,b]; \end{gathered} \tag{119}$$

$$T(\alpha, \beta, a, b) y = 0, \tag{120}$$

in a way that allows the existence and calculation of the Green function of the problem $Ly = 0$, $T(\alpha, \beta, a, b) y = 0$, while guaranteeing the right disfocality of $Ly = 0$ on $[a,b]$ and at the same time yielding functions $c_i(x)$, $0 \leq i \leq \mu$, which satisfy the conditions for the application of the method. However, there still exists some freedom in the choice of $a_i(x)$ and $c_i(x)$ as long as $p_i(x) = a_i(x) - c_i(x)$, $0 \le i \le \mu$, and one can wonder if there are choices of such functions that give better results than others. The next theorem aims to solve that question.

Theorem 12. *Let one consider the differential operators*

$$\begin{aligned} Ly &\equiv y^{(n)}(x) + a_{n-1}(x)\, y^{(n-1)}(x) + \cdots \\ &\quad + a_0(x)\, y(x), \quad x \in [a,b], \\ L' y &\equiv y^{(n)}(x) + d_{n-1}(x)\, y^{(n-1)}(x) + \cdots \\ &\quad + d_0(x)\, y(x), \quad x \in [a,b], \end{aligned} \tag{121}$$

with L and L' being right-disfocal on $[a,b]$, and the problems

$$\begin{aligned} Ly &= \sum_{i=0}^{\mu} c_i(x)\, y^{(i)}(x), \quad x \in [a,b], \\ T(\alpha,\beta,a,b)\, y &= 0, \end{aligned} \tag{122}$$

$$\begin{aligned} L' y &= \sum_{i=0}^{\mu} e_i(x)\, y^{(i)}(x), \quad x \in [a,b], \\ T(\alpha,\beta,a,b)\, y &= 0, \end{aligned} \tag{123}$$

with (α,β) defined as in (49) and (50) and $a_i(x), c_i(x), d_i(x), e_i(x)$ being continuous functions on $[a,b]$ for $0 \le i \le n-1$ such that

$$\begin{aligned} a_i(x) - c_i(x) &= d_i(x) - e_i(x), \\ &\qquad x \in [a,b],\ 0 \le i \le \mu, \end{aligned} \tag{124}$$

$$a_i(x) = d_i(x), \quad x \in [a,b],\ \mu < i \le n-1, \tag{125}$$

$$\begin{aligned} (-1)^{n-k} c_i(x) &\ge (-1)^{n-k} e_i(x) \ge 0, \\ &\qquad x \in [a,b],\ 0 \le i \le \mu, \end{aligned} \tag{126}$$

$$(-1)^{n-k} c_j(x) > (-1)^{n-k} e_j(x) > 0, \quad \text{a.e. on } [a,b], \tag{127}$$

for at least one j such that $0 \le j \le \mu$. Let $q_i(x) \in C[a,b]$ be functions that satisfy

$$\begin{aligned} &(-1)^{n-k} q_i(x) \\ &\quad \le \min\left\{(-1)^{n-k}\left(e_i(x) - d_i(x)\right),\ x \in [a,b]\right\}, \\ &\qquad 0 \le i \le \mu, \end{aligned} \tag{128}$$

for $x \in [a,b]$, if there is no solution of (119) satisfying (120) at a, b or extremes a', b' interior to $[a,b]$, and

$$\begin{aligned} &(-1)^{n-k} q_i(x) \\ &\quad \le \min\left\{(-1)^{n-k}\left(\frac{c_i(x)}{\lambda} - a_i(x)\right),\ x \in [a,b]\right\}, \\ &\qquad 0 \le i \le \mu, \end{aligned} \tag{129}$$

for $x \in [a,b]$, if there is a solution of (119) satisfying (120) at extremes a', b' interior to $[a,b]$, λ being the eigenvalue of the problem

$$\begin{aligned} Ly &= \frac{1}{\lambda}\sum_{i=0}^{\mu} c_i(x)\, y^{(i)}(x), \quad x \in [a,b], \\ T(\alpha,\beta,a,b)\, y &= 0. \end{aligned} \tag{130}$$

Assume also that the equation

$$\begin{aligned} y^{(n)}(x) + \sum_{i=\mu+1}^{n-1} a_i(x)\, y^{(i)}(x) - \sum_{i=0}^{\mu} q_i(x)\, y^{(i)}(x) = 0, \\ x \in [a,b] \end{aligned} \tag{131}$$

is right-disfocal on $[a,b]$, and that, combined with the boundary conditions

$$T(\alpha,\beta,a,b)\, y = 0, \tag{132}$$

it gives a problem whose Green function exists. Let us denote by M and N the operators of the type (6) associated with (122) and (123), respectively, and let P be the cone defined in (55). One has the following:

(i) *If there is no solution of (119) satisfying (120) at a, b or extremes a', b' interior to $[a,b]$, then for any $v \in P \setminus \{0\}$ and $\epsilon > 0$ there exists $k_0 \ge 1$ such that*

$$N^k v \le \epsilon M^k v, \quad k \ge k_0. \tag{133}$$

(ii) *If there is a solution of (119) satisfying (120) at extremes a', b' interior to $[a,b]$, then for any $v \in P \setminus \{0\}$ and $\epsilon > 0$ there exists $k_1 \ge 1$ such that*

$$M^k v \le \epsilon N^k v, \quad k \ge k_1. \tag{134}$$

Proof. As proven in Theorem 8, both M and N map the cone P defined in (55) into the interior of the cone $\mathscr{P}$ of (60) and (61), and the respective eigenvalue problems $Mu = \lambda u$ and $Nz = \nu z$ have eigenfunctions $u, z \in \mathscr{P}^0 \subset P$. From [33, Lemma 1.5], it follows that both operators are u-positive and z-positive in P; that is, for any $v \in P \setminus \{0\}$, one has

$$\begin{aligned} Mv &\ge \delta_1 u, \\ M^k v &\ge \delta_1 M^{k-1} u = \delta_1 \lambda^{k-2} M u \ge \delta_1 \delta_2 \lambda^{k-2} z, \end{aligned} \tag{135}$$

$$\begin{aligned} Nv &\le \delta_3 z, \\ N^k v &\le \delta_3 N^{k-1} z = \delta_3 \nu^{k-1} z. \end{aligned} \tag{136}$$

Combining (135) and (136), one gets

$$N^k v \le \frac{\delta_3 \lambda}{\delta_1 \delta_2}\left(\frac{\nu}{\lambda}\right)^{k-1} M^k v = K_1 \left(\frac{\nu}{\lambda}\right)^{k-1} M^k v, \tag{137}$$

with $K_1 > 0$. In a similar manner it is possible to obtain

$$M^k v \le K_2 \left(\frac{\lambda}{\nu}\right)^{k-1} N^k v, \tag{138}$$

with $K_2 > 0$.

Next, if there is no solution of (119) satisfying (120) at a, b or extremes a', b' interior to $[a,b]$, then one must have $\nu < \lambda < 1$. To see this, given that we already know from Theorem 2 that $\nu, \lambda < 1$, let us suppose that $1 > \nu \geq \lambda$. From (124), it follows that

$$\begin{aligned}\frac{c_i(x)}{\lambda} - a_i(x) &= \frac{a_i(x) - d_i(x) + e_i(x)}{\lambda} - a_i(x) \\ &= \frac{a_i(x) - d_i(x) + e_i(x)}{\lambda} - a_i(x) \\ &\quad + d_i(x) - d_i(x) \\ &= \frac{e_i(x)}{\lambda} + \left(\frac{1}{\lambda} - 1\right)(a_i(x) - d_i(x)) \\ &\quad - d_i(x),\end{aligned} \tag{139}$$

which, together with (124), (126), (127), and our assumption, $1 > \nu \geq \lambda$ yield

$$\begin{aligned}&(-1)^{n-k}\left(\frac{c_i(x)}{\lambda} - a_i(x)\right) \\ &\quad \geq (-1)^{n-k}\left(\frac{e_i(x)}{\nu} - d_i(x)\right),\end{aligned} \tag{140}$$

for $0 \leq i \leq \mu$, with the inequality being strict for at least one i, and $a_i(x) \equiv d_i(x)$ for $\mu < i \leq n-1$. Now, let $q_i(x)$ be the functions defined in (128), let $H(x,t)$ be the Green function of problem (131) and (132), and let us define the operators

$$\begin{aligned}&N_1 f \\ &= \int_a^b H(x,t) \sum_{i=0}^{\mu} \left(\frac{c_i(t)}{\lambda} - a_i(t) - q_i(t)\right) y^{(i)}(t)\,dt, \\ &\qquad x \in [a,b], \\ &N_2 f \\ &= \int_a^b H(x,t) \sum_{i=0}^{\mu} \left(\frac{e_i(t)}{\nu} - d_i(t) - q_i(t)\right) y^{(i)}(t)\,dt, \\ &\qquad x \in [a,b].\end{aligned} \tag{141}$$

From the definition of the eigenfunctions u and z, we have $N_1 u = u$ and $N_2 z = z$. Since, from (127), (128), (140), the fact that $\nu < 1$, and the arguments of Theorem 8, we know that N_1 is z-positive in P (N_1 is in fact u_0-positive with any $u_0 \in \mathscr{P}^0$ according to [33, Lemma 1.5]) and that $N_1 f > N_2 f$, for $f \in P \setminus \{0\}$, we can define

$$\sigma_0 = \sup\{\sigma \mid N_1(u - \sigma z) \in P\}, \tag{142}$$

with $\sigma_0 > 0$, and we have $\delta_1 z \leq N_1(u - \sigma_0 z) \leq u - \sigma_0 z$, which implies $u - (\sigma_0 + \delta_1) z \in P$, contradicting the maximal property of σ_0. Therefore, $\nu < \lambda < 1$. From here and (137), it is straightforward to obtain (133). A similar argument allows showing, in the case that there is a solution of (119) satisfying (120) at extremes a', b' interior to $[a,b]$, that $\nu > \lambda > 1$ (note in this case that the eigenvalue λ of (129) and (130) is exactly the eigenvalue associated with u such that $Mu = \lambda u$, as a direct inspection of the formulae show). This and (138) give (134). □

Remark 13. The consequence of Theorem 12 is that the use of functions $c_i(x)$ closer to zero improves the speed of growth or decrease of M^k on k so that lower values of k are required to find better bounds of the extremes for which a solution of (46)–(53) exists.

With this in mind, our recommendation will be to look for decompositions of (119) that yield a differential equation $Ly = 0$ easy to solve, like those with constant coefficients (this guarantees an easy calculation of the Green function of $Ly = 0$, $T(\alpha,\beta,a,b)y = 0$) and a set of functions $c_i(x)$, $0 \leq i \leq \mu$, as close to zero on $[a,b]$ as possible, while satisfying conditions (48), (52), and (53). For instance, a good choice starting from a problem like (119) and (120) is to pick $a_i(x) = (-1)^{n-k}\max\{(-1)^{n-k} p_i(x),\ x \in [a,b]\}$ and $c_i(x) = a_i(x) - p_i(x)$, $0 \leq i \leq \mu$, unless all $p_i(x)$ for $i = 0, \ldots, \mu$ are constant on $[a,b]$ or the resulting equations $Ly = 0$ are not right-disfocal on $[a,b]$.

Before closing this section we will provide the following Sturm comparison theorem, applicable to problem (46)–(53).

Theorem 14. *Let one consider the problems*

$$\begin{aligned}Ly &= \sum_{i=0}^{\mu} c_i(x)\, y^{(i)}(x), \quad x \in [a,b], \\ T(\alpha,\beta,a,b)\, y &= 0,\end{aligned} \tag{143}$$

$$\begin{aligned}Ly &= \sum_{i=0}^{\mu} e_i(x)\, y^{(i)}(x), \quad x \in [a,b], \\ T(\alpha,\beta,a,b)\, y &= 0,\end{aligned} \tag{144}$$

where (143) and (144) are problems of the type (46)–(53) such that

$$\begin{aligned}&(-1)^{n-k} c_i(x) \geq (-1)^{n-k} e_i(x), \\ &\qquad x \in [a,b],\ 0 \leq i \leq \mu.\end{aligned} \tag{145}$$

Then,

(i) *if there is no solution of (143) at a, b or extremes a', b' interior to $[a,b]$, there will be no solution of (144) at a, b or extremes a', b' interior to $[a,b]$;*

(ii) *if there is a solution of (144) at extremes a', b' interior to $[a,b]$, there will also be a solution of (143) at extremes a'', b'' interior to $[a,b]$.*

Proof. Let N_1, N_2 be the operators of the form (6) associated with the problems (143) and (144), respectively, and u_1, u_2 the eigenvalues in the cone P of (55) such that $N_1 u_1 = \lambda_1 u_1$ and $N_2 u_2 = \lambda_2 u_2$. From (145), one has

$$\lambda_2 u_2 = N_2 u_2 \leq N_1 u_2. \tag{146}$$

Applying [21, Theorem 2.2], one gets $\lambda_2 \leq \lambda_1$. From here and the arguments of Theorem 2 on the values of λ_1, λ_2 being lower or higher than 1 depending on the existence of solutions in extremes interior to $[a,b]$, one gets the conclusions of the theorem. □

Remark 15. The conclusions of Theorems 12 and 14 and Remark 13 are also valid for problems of the type (109)–(114) just replacing condition (48) by condition (111), conditions (126) and (145) by

$$(-1)^{n-k-i} c_i(x) \geq (-1)^{n-k-i} e_i(x) \geq 0, \quad x \in [a,b], \ 0 \leq i \leq \mu, \tag{147}$$

and condition (127) by

$$(-1)^{n-k-j} c_j(x) > (-1)^{n-k-j} e_j(x) > 0, \quad \text{a.e. on } [a,b], \tag{148}$$

for at least one j such that $0 \leq j \leq \mu$, and making $q_i(x)$ in (128) satisfy

$$(-1)^{n-k-i} q_i(x) \leq \min\left\{(-1)^{n-k-i}\left(e_i(x) - d_i(x)\right),\ x \in [a,b]\right\}, \quad 0 \leq i \leq \mu \tag{149}$$

and $q_i(x)$ in (129) satisfy

$$(-1)^{n-k-i} q_i(x) \leq \min\left\{(-1)^{n-k-i}\left(\frac{c_i(x)}{\lambda} - a_i(x)\right),\ x \in [a,b]\right\}, \quad 0 \leq i \leq \mu. \tag{150}$$

Remark 16. Although this section has focused on the problem assessed by Eloe et al., the results of Section 2 can be applied to many other problems. The choice of Eloe's problem in this paper is due to the broad range of differential equations and boundary conditions that it covers.

4. Some Examples

Throughout this section we will introduce examples where the results of Section 3 will be used to provide progressively better upper and lower bounds of the extremes that make (3) and (4) have a nontrivial solution. The examples will cover problems of the type (46)–(53) in the cases $n = 2$, 3, and 4 with different boundary conditions. The first two ones, related to $n = 2$, will show the advantages of the method of this paper versus the one of [1], similar in essence but focused only on the second-order linear BVP, namely, the ability to cover some cases not addressable in [1] as well as the possibility of selecting the functions $a_0(x)$ and $c_0(x)$ in a manner that makes the method converge faster.

In all cases, the integral calculations have been done numerically. This also includes the calculation of the derivatives $(M^k v)^{(i)}(x)$, $0 \leq i \leq \mu$, as these can be written as

$$\left(M^k v\right)^{(i)}(x) = \int_a^b \frac{\partial^i G(a,b,x,t)}{\partial x^i} \sum_{j=0}^{\mu} c_j(t) \left(M^{k-1} v(t)\right)^{(j)} dt. \tag{151}$$

The maximum number of iterations has been set to 8 in all examples and up to 3 decimal figures have been calculated for each bound.

Example 1. Let us consider the conjugate boundary value problem

$$y'' + Ae^x = 0, \quad x \geq 0, \ y(0) = 0, \ y(b) = 0, \ A > 0, \tag{152}$$

for different values of the constant A. The main interest of this example is to compare the values provided by the method of Section 3 with the values obtained for the same problem in [1], using the operator P defined by

$$Pf(x) = \int_a^b H(x,t) Ae^t f(t)\, dt, \quad x \in [a,b], \tag{153}$$

where $H(x,t)$ is the Green function of the problem

$$y'' = 0, \quad x \geq 0, \ y(0) = 0, \ y(b) = 0, \tag{154}$$

and integral inequalities somewhat different to those presented in this paper. To do the comparison, let us first rewrite (152) into

$$y'' + Ay = -A\left(e^x - 1\right) y, \quad x \geq 0, \ y(0) = 0, \ y(b) = 0, \ A > 0. \tag{155}$$

Following Coppel [15], we can obtain the following Green function for the problem

$$y'' + Ay = 0, \quad x \geq 0, \ y(0) = 0, \ y(b) = 0, \ A > 0, \tag{156}$$

namely,

$$G(x,t) = \begin{cases} \dfrac{\sin\left(\sqrt{A}x\right)\sin\left(\sqrt{A}(b-t)\right)}{\sqrt{A}\sin\left(\sqrt{A}b\right)}, & 0 \leq x < t \leq b, \\[2ex] \dfrac{\sin\left(\sqrt{A}t\right)\sin\left(\sqrt{A}(b-x)\right)}{\sqrt{A}\sin\left(\sqrt{A}b\right)}, & 0 \leq t \leq x \leq b. \end{cases} \tag{157}$$

As a remarkable exception to the conditions of Theorem 8, when dealing with Green functions of second-order linear BVPs of the form

$$y'' + q(x) y = 0, \quad x \in [a,b], \ y(a) = 0, \ y(b) = 0, \tag{158}$$

one can prove that, as long as the previous equation is disconjugate in the interval of interest, the resulting Green

Table 1: Comparison of bounds for b in Example 1.

Value of A	Used formula	Bound
$A = 3$	[1], $\int_a^b q(t)G(t,t)P^{11}\{1\}(t)dt > 1$	$b > 1.266$
	[1], $\|P^5\{v\}\|_2^2 < \|v\|_2^2$	$b < 1.287$
	Theorem 9 (98), $k = 2$, $j = 1$	$b > 1.285$
	Theorem 9 (98), $k = 5$, $j = 4$	$b > 1.286$
	Theorem 9 (99), $k = 7$	$b > 1.278$
	Theorem 9 (100), $k = 7$, $j = 0$	$b < 1.293$
	Theorem 9 (100), $k = 3$, $j = 2$	$b < 1.287$
	Theorem 10 (105), $k = 7$	$b < 1.31$
$A = 5$	[1], $\int_a^b q(t)G(t,t)P^{11}\{1\}(t)dt > 1$	$b > 1.044$
	[1], $\|P^5\{v\}\|_2^2 < \|v\|_2^2$	$b < 1.062$
	Theorem 9 (98), $k = 3$, $j = 2$	$b > 1.061$
	Theorem 9 (99), $k = 2$	$b > 1.046$
	Theorem 9 (100), $k = 3$, $j = 2$	$b < 1.062$
	Theorem 10 (105), $k = 7$	$b < 1.078$

function and its derivatives with regard to a and b satisfy the properties used in the proof of Theorem 8 to show the applicability of Theorem 2, regardless of the disfocal or nondisfocal character of the equation in such an interval. Therefore we can apply Theorems 9 and 10 (in this case using $\xi_1 = b/3$ and $\xi_2 = 2b/3$) to (155) and generate Table 1, where the values obtained in [1] are also included.

As can be shown in Table 1, Theorem 9 provides far better lower bounds than [1] with lesser iterations, which matches with the results of Theorem 12 as to the effect of the choice of $c(x)$. As for the upper bounds, in [1] there was an integral inequality (see second row of Table 1) which yields sharp upper bounds. However, as the table shows, (100) with $j = k-1$ gives similar upper bounds with a slightly lesser number of iterations than the aforementioned integral inequality of [1].

Example 2. Let us consider the following conjugate boundary value problem:

$$y'' + 3(x-1)y = 0, \quad x \geq 0, \ y(0) = 0, \ y(b) = 0. \tag{159}$$

As in the previous example, we can rewrite (159) into

$$y'' - 3y = -3xy, \quad x \geq 0, \ y(0) = 0, \ y(b) = 0. \tag{160}$$

The calculation of the Green function of

$$y'' - 3y = 0, \quad x \geq 0, \ y(0) = 0, \ y(b) = 0 \tag{161}$$

yields

$$G(x,t) = \begin{cases} \dfrac{\sinh(\sqrt{3}x)\sinh(\sqrt{3}(b-t))}{\sqrt{3}\sinh(\sqrt{3}b)}, & 0 \leq x < t \leq b, \\ \dfrac{\sinh(\sqrt{3}t)\sinh(\sqrt{3}(b-x))}{\sqrt{3}\sinh(\sqrt{3}b)}, & 0 \leq t \leq x \leq b. \end{cases} \tag{162}$$

Table 2: Comparison of bounds for b in Example 2.

Used formula	Recursivity indices	Bound
Theorem 9 (98)	$k = 7$, $j = 0$	$b > 2.626$
Theorem 9 (98)	$k = 5$, $j = 4$	$b > 2.64$
Theorem 9 (99)	$k = 7$	$b > 2.597$
Theorem 9 (100)	$k = 7$, $j = 0$	$b < 2.733$
Theorem 9 (100)	$k = 6$, $j = 5$	$b < 2.641$
Theorem 10 (103)	$k = 7$, $j = 0$	$b > 2.468$
Theorem 10 (103)	$k = 7$, $j = 6$	$b > 2.636$
Theorem 10 (104)	$k = 7$, $j = 0$	$b < 2.754$
Theorem 10 (104)	$k = 7$, $j = 6$	$b < 2.647$
Theorem 10 (105)	$k = 7$	$b < 2.913$

The application of Theorems 9 and 10 (using $\xi_1 = b/3$ and $\xi_2 = 2b/3$) gives Table 2. Note that in this case the results obtained in [1] are not applicable since $3(x-1)$ is negative for $x < 1$.

According to Table 2, in general, Theorem 9 gives better results (i.e., smaller upper bounds and bigger lower bounds) than Theorem 10 with less iterations, at least comparing the results given by (98) and (100) with those coming from (103) and (104). Another pattern worth remarking is that all inequalities, (98), (100), (103), and (104), yield sharper bounds with less iterations when taking $j = k-1$ instead of $j = 1$. Although one could think that a big difference between j and k implies a big difference in size between $(M^k f)^{(\beta_1)}(x)$ and $(M^j f)^{(\beta_1)}(x)$, what is true is that once b approaches the value for which problem (159) has exactly a solution in the extremes 0 and b, that difference may not be very big. Thus, what could be happening is that the iteration on k had the effect of making $M^k f$ converge to a function, as happened in [1], so that the differences in "shape" between $(M^k f)^{(\beta_1)}(x)$ and $(M^j f)^{(\beta_1)}(x)$ were minimal. In the case of [1], the convergence was into $\lambda^k u$, where u is the eigenfunction of $Mu = \lambda u$ and λ the corresponding eigenvalue; one can hypothesize a similar behaviour here, but it obviously requires additional proof. This seems to be reinforced by the fact that (99) and (105), which in fact are variants of (98) and (104), respectively, not applying M to the functions $v(x)$ and $h(x)$ in the right hand side of the equations, provide worse bounds, using higher j, than (98) and (104) do.

Example 3. Let us consider the following boundary value problem:

$$\begin{gathered} y''' + (4+x)y' + xy = 0, \\ x \in [0,b]; \ y(0) = y'(0) = y'(b) = 0. \end{gathered} \tag{163}$$

In order to apply Theorems 9 and 10, we need to decompose (163) in a manner that provides an equation $Ly = 0$ right-disfocal in the interval of interest. One possible option is

$$\begin{gathered} y''' + y' = (-1)^{n-k}(3+x)y' + (-1)^{n-k}xy, \\ x \in [0,b]; \ y(0) = y'(0) = y'(b) = 0, \end{gathered} \tag{164}$$

with $n = 3$, $k = 2$, $\beta_1 = 1$ and $\mu = 1$, which gives the BVP

$$y''' + y' = 0, \quad x \in [0, b]; \; y(0) = y'(0) = y'(b) = 0, \tag{165}$$

whose Green function can be calculated following Coppel [15] as

$$G(x,t) = \begin{cases} \left(\dfrac{\cos b}{\sin b} \sin t - \cos t\right)(1 - \cos x), & 0 \le x \le t \le b, \\ (1 - \cos t) + \sin t \left(\dfrac{\cos b}{\sin b}(1 - \cos x) - \sin x\right), & 0 \le t < x \le b. \end{cases} \tag{166}$$

The application of Theorems 9 and 10 (using $\xi_1 = b/3$ and $\xi_2 = 2b/3$) gives Table 3.

Table 3 shows that, as expected, the bounds get improved when the number of iterations grows. In addition, as happened in the previous example, (98) of Theorem 9 provides sharper bounds than its peer (103) in Theorem 10, for the same number of iterations. However, unlike the previous example, (104) of Theorem 10 provides sharper bounds than its peer (100) in Theorem 9. A pattern that is confirmed is that the sharpest bounds (in fact very sharp, since based on them we can state that b lies between 1.381 and 1.382) are those obtained using $j = k - 1$ in (98), (100), (103) and (104). As before, this suggests that $M^k v$ is converging to a function, potentially $\lambda^k u$ with u being the eigenfunction of $Mu = \lambda u$.

Example 4. Let us consider the following boundary value problem:

$$y^{(\mathrm{iv})} + xy'' = 0, \quad x \in [0, b]; \; y(0) = y'(0) = y''(0) = y''(b) = 0. \tag{167}$$

We can rewrite problem (167) as

$$y^{(\mathrm{iv})} = -xy'' = (-1)^{n-k} xy'', \quad x \in [0, b]; \; y(0) = y'(0) = y''(0) = y''(b) = 0, \tag{168}$$

with $n = 4$, $k = 3$, $\beta_1 = 2$, and $\mu = 2$, which, given that $y^{(\mathrm{iv})} = 0$ is always right-disfocal regardless of the interval $[a, b]$, satisfies all the conditions for the application of Theorems 9 and 10. To do that, we first need to determine the Green function of the problem

$$y^{(\mathrm{iv})} = 0, \quad x \in [0, b]; \; y(0) = y'(0) = y''(0) = y''(b) = 0. \tag{169}$$

Following Coppel [15] as we did in Example 3, the mentioned Green function can be calculated as

$$G(x,t) = \begin{cases} -\left(1 - \dfrac{t}{6}\right)\dfrac{x^3}{6}, & 0 \le x \le t \le b, \\ \left(-\dfrac{t^3}{6} + \dfrac{t^2}{2}x - \dfrac{t}{2}x^2 + \dfrac{t}{b}\dfrac{x^3}{6}\right), & 0 \le t < x \le b. \end{cases} \tag{170}$$

The application of Theorems 9 and 10 (using $\xi_1 = b/3$ and $\xi_2 = 2b/3$) gives Table 4.

In this example, almost for all recursivity indices, the equations of Theorem 10 provide sharper bounds with less iterations than their equivalent equations of Theorem 9, unlike what happened in the previous examples. As before, the sharpest bounds are obtained using $j = k - 1$ in (98), (100), (103), and (104), with the result that b must lie between 2.187 and 2.188.

5. Discussion

The method described in this paper provides a general tool to tackle the question of the existence of solutions of boundary value problems of nth order, conditioned to finding a proper reproducing cone for which the operator M can be proven to be u_0-positive (e.g., because M maps P into its interior P^0) and to the spectral radius of M satisfying certain monotonicity properties with respect to the extremes a and b where M is defined. That type of problems is quite common in the literature. In our paper, we chose to focus on two nth order linear boundary value problems studied by Eloe and Henderson in [30], namely, those of the type (3) and (4), where either (47)–(53) or (107)–(114) are satisfied. The reason for such a choice is the broad range of applications of these types of BVPs, but other cases like those analysed by Tomastik [22], Keener and Travis [21], and Diaz [33] (note that this list is not exhaustive) are also suitable candidates for the application of the method. In the case of Eloe's problems, the tool provides necessary and sufficient conditions for the existence of a solution in the form of integral inequalities. The question of whether it also yields in the process progressively better approximations of the solution of such a problem, as the extremes a and b that define M approach the extremes for which a solution exists, as the method of [1] did, is left open and could be investigated in future papers, but the fact that the sharpest bounds (in fact extremely sharp bounds) are obtained in (98), (100), (103), and (104) when using $j = k - 1$ strongly suggests that. As for what starting function works better for the method, the different examples show no preference for a concrete function, and both $v(x)$ in Theorem 9 and $h(x)$ in Theorem 10 alternate in terms of yielding the sharpest bound with the least number of iterations, depending on the concrete BVP. Obviously other function choices are possible.

Table 3: Comparison of bounds for b in Example 3.

Used formula	Recursivity indices	Bound
Theorem 9 (98)	$k = 1$, $j = 0$	$b > 1.37$
Theorem 9 (98)	$k = 2$, $j = 1$	$b > 1.38$
Theorem 9 (98)	$k = 5$, $j = 4$	$b > 1.381$
Theorem 9 (99)	$k = 0$	$b > 1.271$
Theorem 9 (99)	$k = 3$	$b > 1.35$
Theorem 9 (99)	$k = 7$	$b > 1.365$
Theorem 9 (100)	$k = 1$, $j = 0$	$b < 1.448$
Theorem 9 (100)	$k = 5$, $j = 0$	$b < 1.396$
Theorem 9 (100)	$k = 7$, $j = 0$	$b < 1.392$
Theorem 9 (100)	$k = 2$, $j = 1$	$b < 1.385$
Theorem 9 (100)	$k = 3$, $j = 2$	$b < 1.382$
Theorem 10 (103)	$k = 7$, $j = 0$	$b > 1.369$
Theorem 10 (103)	$k = 4$, $j = 3$	$b > 1.38$
Theorem 10 (103)	$k = 6$, $j = 5$	$b > 1.381$
Theorem 10 (104)	$k = 1$, $j = 0$	$b < 1.407$
Theorem 10 (104)	$k = 2$, $j = 0$	$b < 1.396$
Theorem 10 (104)	$k = 3$, $j = 0$	$b < 1.39$
Theorem 10 (104)	$k = 6$, $j = 0$	$b < 1.386$
Theorem 10 (104)	$k = 2$, $j = 1$	$b < 1.384$
Theorem 10 (104)	$k = 3$, $j = 2$	$b < 1.382$
Theorem 10 (105)	$k = 0$	$b < 1.706$
Theorem 10 (105)	$k = 2$	$b < 1.488$
Theorem 10 (105)	$k = 7$	$b < 1.421$

The advantage of this more general approach versus the one of [1], for the concrete case of the second-order linear differential equation $y'' + q(x)y = 0$, is that, by simply converting this equation into

$$y'' + q_{\min} y = -\left(q(x) - q_{\min}\right) y, \tag{171}$$

and applying the formulae of Section 3, the two drawbacks present and pointed out in [1], that is,

(i) the impossibility to cover the case where $q(x)$ is negative in a set of nonnegative measure,

(ii) the slow convergence of the method in some cases,

can be easily overcome, as Examples 1 and 2 show.

As discussed in Remark 13, this strategy can be extended to the nth order linear boundary value problem:

$$\begin{aligned} y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_0(x)\, y = 0, \\ T(\alpha, \beta, a, b)\, y = 0, \end{aligned} \tag{172}$$

by picking $a_i(x) \in C[a,b]$ such that $(-1)^{n-k} a_i(x) \geq \max\{(-1)^{n-k} p_i(x), x \in [a,b]\}$ and $Ly = 0$ is right-/left-disfocal on $[a,b]$ (depending on α, β) and setting $c_i(x) = a_i(x) - p_i(x)$, $0 \leq i \leq \mu$. Although this representation does not cover all possible problems of the type (172), it allows converting many of them to problems addressable by this method (in particular by the formulae obtained in Section 3), yields a differential equation $Ly = 0$ with constant coefficients whose Green function can be easily calculated following Coppel [15] procedure, and above all guarantees a relatively fast convergence of the integral iterations according to Theorem 12, that is, allowing determining the existence of a solution on $[a,b]$ faster. For the cases not covered (basically those where either none of the boundary conditions α, β is of the form $(0, 1, \ldots, k)$ or one boundary condition has that form while the other has the biggest component (α_k or β_{n-k}) equal to $n-1$, like the left and right focal conditions), our recommendation is to investigate the existence of M and a cone P for which the general results of Section 2 can be applied. Let us remark that the fact that $\mu \leq \beta_1$ (or $\mu \leq \alpha_1$, in the case of (107)–(114)) is not a limitation itself for the application of the method but it just complicates the explicit calculation of the Green function of (7). Nonetheless, this one can always be calculated numerically in the worst case.

Perhaps the biggest limitation, not really of this method, but of the way we have applied it to Eloe's problem, is the need for the resulting equation $Ly = 0$ to be right- or left-disfocal, depending on the boundary conditions, which reduces severely the possible choices of the functions $a_i(x)$. This constraint is a general assumption made in [42] whose removal or relaxation is not evident but it would be really welcome. Further assessment is therefore required in this area.

Table 4: Comparison of bounds for b in Example 4.

Used formula	Recursivity indices	Bound
Theorem 9 (98)	$k = 1$, $j = 0$	$b > 2.005$
Theorem 9 (98)	$k = 4$, $j = 0$	$b > 2.114$
Theorem 9 (98)	$k = 7$, $j = 0$	$b > 2.145$
Theorem 9 (98)	$k = 4$, $j = 3$	$b > 2.18$
Theorem 9 (98)	$k = 7$, $j = 6$	$b > 2.187$
Theorem 9 (99)	$k = 0$	$b > 1.79$
Theorem 9 (99)	$k = 4$	$b > 2.059$
Theorem 9 (99)	$k = 7$	$b > 2.106$
Theorem 9 (100)	$k = 1$, $j = 0$	$b < 2.972$
Theorem 9 (100)	$k = 4$, $j = 0$	$b < 2.364$
Theorem 9 (100)	$k = 7$, $j = 0$	$b < 2.288$
Theorem 9 (100)	$k = 4$, $j = 3$	$b < 2.193$
Theorem 9 (100)	$k = 7$, $j = 6$	$b < 2.189$
Theorem 10 (103)	$k = 1$, $j = 0$	$b > 2.028$
Theorem 10 (103)	$k = 4$, $j = 0$	$b > 2.139$
Theorem 10 (103)	$k = 7$, $j = 0$	$b > 2.159$
Theorem 10 (103)	$k = 4$, $j = 3$	$b > 2.185$
Theorem 10 (103)	$k = 7$, $j = 6$	$b > 2.186$
Theorem 10 (104)	$k = 1$, $j = 0$	$b < 2.41$
Theorem 10 (104)	$k = 4$, $j = 0$	$b < 2.26$
Theorem 10 (104)	$k = 7$, $j = 0$	$b < 2.229$
Theorem 10 (104)	$k = 4$, $j = 3$	$b < 2.191$
Theorem 10 (104)	$k = 7$, $j = 6$	$b < 2.188$
Theorem 10 (105)	$k = 1$	$b < 2.629$
Theorem 10 (105)	$k = 5$	$b < 2.339$
Theorem 10 (105)	$k = 7$	$b < 2.301$

A question not addressed in this paper is the existence of a kind of Courant's max-min principle that can also be used to provide sequences of integral conditions using the norm of $M^k f$, in the same manner as was done in [1] (see the integral inequality with the norm of P that was included in Table 1). The topic was partially analysed by Keener and Travis in [21, Theorem 3.1]. However the extension of their proof to the present problem is not evident, presenting gaps not easy to fill.

With all these considerations in mind, we, the authors, believe that this method can become a very powerful tool in the analysis of conjugacy and focality of equations of the type (3).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] P. Almenar and L. Jódar, "The distance between points of a solution of a second order linear differential equation satisfying general boundary conditions," *Abstract and Applied Analysis*, vol. 2014, Article ID 126713, 17 pages, 2014.

[2] M. Greguš, *Third Order Linear Differential Equations*, vol. 22, Springer, Dordrecht, The Netherlands, 1st edition, 1987.

[3] G. Frobenius, "Über adjungirte lineare Differentialausdrücke," *Journal für Mathematik*, vol. 85, pp. 185–213, 1878.

[4] G. Pólya, "On the mean-value theorem corresponding to a given linear homogeneous differential equations," *Transactions of the American Mathematical Society*, vol. 24, no. 4, pp. 312–324, 1922.

[5] T. Sherman, "Properties of solutions of nth order linear differential equations," *Pacific Journal of Mathematics*, vol. 15, no. 3, pp. 1045–1060, 1965.

[6] J. S. Muldowney, "A necessary and sufficient condition for disfocality," *Proceedings of the American Mathematical Society*, vol. 74, no. 1, pp. 49–55, 1979.

[7] A. J. Levin, "Some problems concerning the oscillation of solutions of linear differential equations," *Soviet Mathematics Doklady*, vol. 4, pp. 121–124, 1963.

[8] A. J. Levin, "Distribution of zeros of solutions of linear differential equations," *Soviet Mathematics Doklady*, vol. 5, pp. 818–822, 1964.

[9] Z. Nehari, "Disconjugate linear differential operators," *Transactions of the American Mathematical Society*, vol. 129, no. 3, pp. 500–516, 1967.

[10] S. Ahmad and A. C. Lazer, "An N-dimensional extension of the Sturm separation and comparison theory to a class of nonselfadjoint systems," *SIAM Journal on Mathematical Analysis*, vol. 9, no. 6, pp. 1137–1150, 1978.

[11] S. Ahmad and A. C. Lazer, "On nth-order Sturmian theory," *Journal of Differential Equations*, vol. 35, no. 1, pp. 87–112, 1980.

[12] U. Elias, "The extremal solutions of the equation $Ly+p(x)y=0$," *Journal of Mathematical Analysis and Applications*, vol. 50, no. 3, pp. 447–457, 1975.

[13] U. Elias, "Nonoscillation and eventual disconjugacy," *Proceedings of the American Mathematical Society*, vol. 66, no. 2, pp. 269–275, 1977.

[14] U. Elias, "Eigenvalue problems for the equation $L_y+\lambda p(x)y=0$," *Journal of Differential Equations*, vol. 29, no. 1, pp. 28–57, 1978.

[15] W. Coppel, *Disconjugacy*, vol. 220 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1st edition, 1971.

[16] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Fizmatgiz, Moscow, Russia, 1962, English Translation: P. Noordhoff, Groningen, The Netherlands, 1964.

[17] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1st edition, 1985.

[18] M. G. Krein and M. A. Rutman, "Linear operators leaving invariant a cone in a Banach space," *American Mathematical Society Translation Series 1*, vol. 10, pp. 199–325, 1962.

[19] R. D. Gentry and C. C. Travis, "Comparison of eigenvalues associated with linear differential equations of arbitrary order," *Transactions of the American Mathematical Society*, vol. 223, pp. 167–179, 1976.

[20] K. Schmitt and H. L. Smith, "Positive solutions and conjugate points for systems of differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 2, no. 1, pp. 93–105, 1978.

[21] M. S. Keener and C. C. Travis, "Positive cones and focal points for a class of nth order differential equations," *Transactions of the American Mathematical Society*, vol. 237, p. 331, 1978.

[22] E. C. Tomastik, "Comparison theorems for second order nonselfadjoint differential systems," *SIAM Journal on Mathematical Analysis*, vol. 14, no. 1, pp. 60–65, 1983.

[23] E. Tomastik, "Comparison theorems for conjugate points of nth order nonselfadjoint differential equations," *Proceedings of the American Mathematical Society*, vol. 96, pp. 437–442, 1986.

[24] K. Kreith, "A class of hyperbolic focal point problems," *Hiroshima Mathematical Journal*, vol. 14, no. 1, pp. 203–210, 1984.

[25] D. Hankerson and A. Peterson, "Comparison theorems for eigenvalue problems for nth order differential equations," *Proceedings of the American Mathematical Society*, vol. 104, no. 4, pp. 1204–1211, 1988.

[26] D. Hankerson and J. Henderson, "Positive solutions and extremal points for differential equations," *Applicable Analysis*, vol. 39, no. 2-3, pp. 193–207, 1990.

[27] P. W. Eloe, D. Hankerson, and J. Henderson, "Positive solutions and conjugate points for multipoint boundary value problems," *Journal of Differential Equations*, vol. 95, no. 1, pp. 20–32, 1992.

[28] P. W. Eloe, D. Hankerson, and J. Henderson, "Positive solutions and j-focal points for two point boundary value problems," *Rocky Mountain Journal of Mathematics*, vol. 22, no. 4, pp. 1283–1293, 1992.

[29] P. W. Eloe and J. Henderson, "Focal points and comparison theorems for a class of two point boundary value problems," *Journal of Differential Equations*, vol. 103, no. 2, pp. 375–386, 1993.

[30] P. W. Eloe and J. Henderson, "Focal point characterizations and comparisons for right focal differential operators," *Journal of Mathematical Analysis and Applications*, vol. 181, no. 1, pp. 22–34, 1994.

[31] P. W. Eloe, J. Henderson, and H. B. Thompson, "Extremal points for impulsive Lidstone boundary value problems," *Mathematical and Computer Modelling*, vol. 32, no. 5-6, pp. 687–698, 2000.

[32] P. W. Eloe and B. Ahmad, "Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions," *Applied Mathematics Letters*, vol. 18, no. 5, pp. 521–527, 2005.

[33] G. Diaz, *Applications of cone theory to boundary value problems [Ph.D. dissertation]*, University of Nebraska-Lincoln, 1989.

[34] J. R. Graef and B. Yang, "Positive solutions to a multi-point higher order boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 2, pp. 409–421, 2006.

[35] J. R. Graef, L. Kong, and H. Wang, "Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem," *Journal of Differential Equations*, vol. 245, no. 5, pp. 1185–1197, 2008.

[36] X. Zhang, M. Feng, and W. Ge, "Existence and nonexistence of positive solutions for a class of nth-order three-point boundary value problems in Banach spaces," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 70, no. 2, pp. 584–597, 2009.

[37] P. Zhang, "Iterative solutions of singular boundary value problems of third-order differential equation," *Boundary Value Problems*, vol. 2011, Article ID 483057, 10 pages, 2011.

[38] Y. Sun, Q. Sun, and X. Zhang, "Existence and nonexistence of positive solutions for a higher-order three-point boundary value problem," *Abstract and Applied Analysis*, vol. 2014, Article ID 513051, 7 pages, 2014.

[39] X. Hao, L. Liu, and Y. Wu, "Iterative solution to singular nth-order nonlocal boundary value problems," *Boundary Value Problems*, vol. 2015, article 125, 10 pages, 2015.

[40] B. Lemmens and R. Nussbaum, "Continuity of the cone spectral radius," *Proceedings of the American Mathematical Society*, vol. 141, no. 8, pp. 2741–2754, 2013.

[41] P. Hartman, *Ordinary Differential Equations*, vol. 38 of *SIAM Classics in Applied Mathematics*, Society for Industrial and Applied Mathematics, Philadelphia, Pa, USA, 2nd edition, 2002, previously published as: *Ordinary Differential Equations*, Birkhäuser, Boston, Mass, USA, 2nd edition, 1982.

[42] P. W. Eloe and J. Ridenhour, "Sign properties of Green's functions for a family of two-point boundary value problems," *Proceedings of the American Mathematical Society*, vol. 120, no. 2, pp. 443–452, 1994.

[43] V. Hutson, J. S. Pym, and M. J. Cloud, *Applications of Functional Analysis and Operator Theory*, vol. 200 of *Mathematics in Science and Engineering*, Elsevier, Amsterdam, The Netherlands, 1st edition, 2005.