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Research Article

Finite-Time Boundedness Analysis for a Class of Switched Linear Systems with Time-Varying Delay

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The problem of finite-time boundedness for a class of switched linear systems with time-varying delay and external disturbance is investigated. First of all, the multiply Lyapunov function of the system is constructed. Then, based on the Jensen inequality approach and the average dwell time method, the sufficient conditions which guarantee the system is finite-time bounded are given. Finally, an example is employed to verify the validity of the proposed method.

1. Introduction

The switched system is a special kind of hybrid dynamic system, composed of a family of subsystems and a switching law specifying the switches between subsystems [1, 2]. The fact that the structure and working mechanism of the switched system are more complex than general systems leads to that the switched system possesses much richer dynamic characteristics. The switched systems are widely applied in engineering practice, such as power system control, robot control, network control, and so forth [3–9].

In practice, switched systems are commonly subjected to time-delay and external disturbance. Due to their significant impact on the performances of switched systems, many scholars have been attracted to investigate the problem. Sun et al. analyzed the asymptotic stability of the switched linear system with time-delay perturbation by using common Lyapunov function and multiple Lyapunov function [3]. Lu and Zhao also investigated the asymptotic stability for switched linear systems with time-delay and proposed an effective method which can direct researchers to choose an appropriate switching law to make sure the system is asymptotic stable [10]. Zhao and Zhang studied the stability of the switched system with time-varying delays based on the average dwell time and time-delay decomposition approaches [11]. For switched systems with time-varying delay, Lian et al. utilized the Lyapunov-Krasovskii function method to design H infinity filter [12]. For switched systems affected by the nonlinear impact and disturbance, Sun used transfer matrix estimation and Gronwall inequality methods to design a feedback law stabilizing system [13]. For the switched system with fixed time-delay and norm bounded disturbance, Lin et al. proposed the finite-time boundedness concept and a method to judge whether the system is finite-time bounded [14].

Up to now, to the best of the authors' knowledge, there are a few papers concerning the finite-time boundedness problem of switched system. For switched systems with time-varying delay and external disturbance, the problem has not yet been discussed by any literature. However, in practical engineering, the time-delays are generally changeable over time, not fixed. In addition, many practical systems are just required that their state trajectories are bounded over a fixed interval. In other words, those systems may be unstable. On the contrary, although some systems are asymptotically stable, they cannot meet the application requirements because of their large transient state amplitudes. Considering the wide application of switched systems with time-varying delay and the requirements for transient behaviors in engineering fields, it is a significant task to investigate finite-time boundedness for switched linear systems with time-varying delay and external disturbance. The main contributions in this paper are listed as follows. (1) For the convenience of processing, a concise definition on the finite-time boundedness is proposed for the switched system. (2) Sufficient conditions of finite-time boundedness for switched linear systems with time-varying delay and external disturbance are given.

2. Preliminaries and Problem Formulation

Consider the following switched linear system with time-varying delay and external disturbance:

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t-h(t)) \\ &\quad + G_{\sigma(t)}w(t), \quad h(t)\geq 0,\ t\geq 0, \\ x(t) &= \varphi(t), \qquad \max|\dot{\varphi}(t)| \leq \rho, \quad \rho\geq 0, \\ &\qquad\qquad t\in[-d,0), \quad d\geq h(0),\end{aligned} \tag{1}$$

where $x(t)$ is state variable and $\sigma(t)$ is the switching law which is a piecewise continuous function with $\sigma(t) \in M = [1,2,\ldots,m]$ which means the switched system is consisted of m subsystems. The ith subsystem is activated when $\sigma(t) = i \cdot A_{\sigma(t)}, B_{\sigma(t)}$, and $G_{\sigma(t)}$ are constant matrices. $h(t)$ represents time-varying delay. $w(t)$ stands for external disturbance. $\varphi(t)$ is the continuous vector-valued initial function on $t \in [-d,0) \cdot \dot{\varphi}(t)$ denotes the derivative of $\varphi(t) \cdot \rho$ is a positive constant.

For the convenience of subsequent processing, assume that the system (1) satisfies the following assumptions.

Assumption 1 (see [14]). The value of external disturbance changes over time, but it satisfies

$$\int_0^{+\infty} w^T(t)\,w(t)\,dt \leq \gamma, \quad \gamma\geq 0,\ \forall t>0. \tag{2}$$

Assumption 2. For the time-varying delay, the following inequalities hold:

$$h(t)\geq 0, \quad \dot{h}(t)\leq k, \quad k<1, \quad h(t)\leq h_{\max}, \tag{3}$$

where k and $h_{\max}$ are positive constants.

Assumption 3. The system state variable does not "jump" at switching instant, that is to say the state trajectory is continuous. In addition, the switching number of $\sigma(t)$ is finite in a limited time interval which implies that the frequency of switching signal is not infinite.

Definition 4 (see [15]). For $T \geq t \geq 0$, let $N_\sigma(t,T)$ denote the switching number of $\sigma(t)$ over $(t,T]$. If

$$N_\sigma(t,T) \leq N_0 + \frac{T-t}{\tau_a} \tag{4}$$

holds for $\tau_a \geq 0$ and an integer $N_0 \geq 0$, then τ_a is called average dwell time.

Definition 5. For a given four positive constants c_1, c_2, T_f, γ, and a switching signal $\sigma(t)$, if

$$\begin{gathered} x^T(\bar{t}_0)\,x(\bar{t}_0) \leq c_1 \Longrightarrow x^T(t)\,x(t) < c_2, \\ c_1 < c_2,\ \forall t\in[0,T_f], \\ \forall w(t): \int_0^{T_f} w^T(s)\,w(s)\,dt \leq \gamma, \end{gathered} \tag{5}$$

then the system (1) is said to be finite-time bounded. Where $x^T(\bar{t}_0)x(\bar{t}_0) = \sup_{-d\leq t\leq 0}\{x^T(t)x(t)\}$, without loss of generality, specify $c_1 = \sup_{-d\leq t\leq 0}\{x^T(t)x(t)\}$.

Remark 6. Definition 5 implies that if the system (1) is finite-time bounded, the state remains within the prescribed bound in the fixed interval. Notice that finite-time boundedness is different from asymptotic stability. The system which is finite-time bounded may not be asymptotically stable while a system is asymptotically stable does not mean it is finite-time bounded either. In a word, there is no necessary relation between them.

Remark 7. The definition of finite-time boundedness in this paper is much more concise than that in [14]. However, they are consistent in essence. By using the definition in this paper, some complex matrix transformations can be avoided in the subsequent mathematical processing.

3. Main Result

Theorem 8. *For system* (1)*, for all* $i \in M$ *and for all* $t \in [0,T_f]$*, assume there exists symmetric positive matrixes* $P_i, R_{i1}, R_{i2}, Q_i, Z_{i1}, Z_{i2}$*, and* H *and positive constants* α*,* $\beta \geq 1$ *such that*

$$\begin{bmatrix} \xi_{11} & P_iB_i + \frac{d}{2}A_i^TZ_iB_i & \frac{2}{d}e^{\alpha d/2}Z_{i,1} & 0 & P_iG_i + \frac{d}{2}A_i^TZ_iG_i \\ * & \frac{d}{2}B_i^TZ_iB_i - (1-k)Q & 0 & 0 & \frac{d}{2}B_i^TZ_iG_i \\ * & * & e^{\alpha d/2}R_i - \frac{2}{d}e^{\alpha d/2}Z_i & \frac{2}{d}e^{\alpha d}Z_{i,2} & 0 \\ * & * & * & -e^{\alpha d}R_{i,2} - \frac{2}{d}e^{\alpha d}Z_{i,2} & 0 \\ * & * & * & * & \frac{d}{2}\left(G_i^TZ_iG_i - H\right) \end{bmatrix} < 0. \tag{6}$$

If the average dwell time satisfies

$$\tau_a < \frac{T_f \ln \beta}{\ln C_1 + \ln \lambda_8 - \ln(\eta_1 C_1 + \eta_2) - \alpha T_f}, \tag{7}$$

then system (1) *is finite-time bounded, where*

$$\begin{aligned}
\xi_{11} &= A_i^T P_i + P_i A_i + R_{i,1} + Q_i \\
&\quad + \frac{d}{2} A_i^T (Z_{i,1} + Z_{i,2}) A_i - \frac{2}{d} e^{\alpha d/2} - \alpha P_i, \\
Z_i &= Z_{i,1} + Z_{i,2}, \quad R_i = R_{i,1} + R_{i,2}, \quad P_i \le \beta P_j, \\
R_{i,1} &\le \beta R_{j,1}, \quad R_{i,2} \le \beta R_{j,2}, \quad Q_i \le \beta Q_j \\
Z_{i,1} &\le \beta Z_{j,1}, \quad Z_{i,2} \le \beta Z_{j,2}, \quad i, j \in [1, 2, \ldots, m], \\
\lambda_1 &= \max_{i \in M} \{\lambda_{\max}(P_i)\}, \qquad \lambda_2 = \max_{i \in M} \{\lambda_{\max}(R_{i,1})\}, \\
\lambda_3 &= \max_{i \in M} \{\lambda_{\max}(R_{i,2})\}, \qquad \lambda_4 = \max_{i \in M} \{\lambda_{\max}(Q_i)\}, \\
\lambda_5 &= \max_{i \in M} \{\lambda_{\max}(Z_{i,1})\}, \qquad \lambda_6 = \max_{i \in M} \{\lambda_{\max}(Z_{i,2})\}, \\
\lambda_7 &= \lambda_{\max}(H), \qquad \lambda_8 = \min_{i \in M} \{\lambda_{\min}(P_i)\}, \\
\eta_1 &= \lambda_1 + \frac{d}{2} e^{\alpha d/2} \lambda_2 + \frac{d}{2} e^{\alpha d} \lambda_3 + h_{\max} e^{\alpha h_{\max}} \lambda_4, \\
\eta_2 &= \frac{d^2}{4} \rho^2 \lambda_5 e^{\alpha d/2} + \frac{d^2}{2} \rho^2 \lambda_6 e^{\alpha d} + \frac{d}{2} \lambda_7 \gamma, \\
C_1 &= \sup_{-d \le t_0 \le 0} \left\{ x^T(\bar{t}_0)\, x(\bar{t}_0) \right\}, \\
C_2 &= \Bigg(\beta^N e^{\alpha T_f} \eta_1 C_1 + \beta^N e^{\alpha T_f} \\
&\quad \times \left(\frac{d^2}{4} \rho^2 \lambda_5 e^{\alpha d/2} + \frac{d^2}{2} \rho^2 \lambda_6 e^{\alpha d} \right. \\
&\quad \left. + \frac{d}{2} e^{\alpha T_f} \lambda_7 \gamma \right) \Bigg) (\lambda_8)^{-1}.
\end{aligned} \tag{8}$$

The left of inequality (6) is a symmetric matrix. Thus, the symmetric terms are denoted by "$*$". $\lambda_{\max}(P_i)$ represents the maximum eigenvalue of P_i.

Proof. Construct the multiply Lyapunov function as follows:

$$\begin{aligned}
V(t) &= V_i(t) = V_{i,1}(t) + V_{i,2}(t) + V_{i,3}(t) + V_{i,4}(t), \\
V_{i,1}(t) &= x^T(t) P_i x(t), \\
V_{i,2}(t) &= \int_{t-(d/2)}^{t} x^T(s) e^{-\alpha(s-t)} R_{i,1} x(s)\, ds \\
&\quad + \int_{t-d}^{t-(d/2)} x^T(s) e^{-\alpha(s-t)} R_{i,2} x(s)\, ds, \\
V_{i,3}(t) &= \int_{t-h(t)}^{t} x^T(s) e^{-\alpha(s-t)} Q_i x(s)\, ds, \\
V_{i,4}(t) &= \int_{-d/2}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) e^{-\alpha(s-t)} Z_{i,1} \dot{x}(s)\, ds\, d\theta \\
&\quad + \int_{-d}^{-d/2} \int_{t+\theta}^{t} \dot{x}^T(s) e^{-\alpha(s-t)} Z_{i,2} \dot{x}(s)\, ds\, d\theta.
\end{aligned} \tag{9}$$

Calculate the derivatives of $V_{i,1}(t), V_{i,2}(t), V_{i,3}(t)$, and $V_{i,4}(t)$ as

$$\begin{aligned}
\dot{V}_{i,1}(t) &= x^T(t) \left[A_i^T P_i + P_i A_i \right] x(t) \\
&\quad + x^T(t - h(t)) B_i^T P_i x(t) + w^T(t) G_i^T P_i x(t) \\
&\quad + x^T(t) P_i B_i x(t - h(t)) + x^T(t) P_i G_i w(t).
\end{aligned} \tag{10}$$

Furthermore, it follows that

$$\begin{aligned}
\dot{V}_{i,1}(t) - \alpha V_{i,1} &= x^T(t) \left[A_i^T P_i + P_i A_i \right] x(t) \\
&\quad + x^T(t - h(t)) B_i^T P_i x(t) + w^T(t) G_i^T P_i x(t) \\
&\quad + x^T(t) P_i B_i x(t - h(t)) + x^T(t) P_i G_i w(t) \\
&\quad - \alpha x^T(t) P_i x(t),
\end{aligned} \tag{11}$$

$$\begin{aligned}
\dot{V}_{i,2}(t) &= \alpha V_{i,2}(t) + x^T(t) R_{i,1} x(t) \\
&\quad + x^T\left(t - \frac{d}{2}\right) e^{\alpha d/2} \left[R_{i,2} - R_{i,1} \right] x\left(t - \frac{d}{2}\right) \\
&\quad - x^T(t - d) e^{\alpha d} R_{i,2} x(t - d),
\end{aligned} \tag{12}$$

$$\begin{aligned}
\dot{V}_{i,3}(t) &= \alpha V_{i,3}(t) + x^T(t) Q_i x(t) \\
&\quad - x^T(t - h(t)) \left(1 - \dot{h}(t)\right) \\
&\quad \times e^{\alpha h(t)} Q_i x(t - h(t)) \\
&\le \alpha V_{i,3}(t) + x^T(t) Q_i x(t) \\
&\quad - x^T(t - h(t)) (1 - k) e^{\alpha h(t)} Q_i x(t - h(t)) \\
&\le \alpha V_{i,3}(t) + x^T(t) Q_i x(t) \\
&\quad - x^T(t - h(t)) (1 - k) Q_i x(t - h(t)),
\end{aligned} \tag{13}$$

$$\begin{aligned}
\dot{V}_{i,4}(t) &= \alpha V_{i,4}(t) + \frac{d}{2} \dot{x}^T(t) \left[Z_{i,1} + Z_{i,2} \right] \dot{x}(t) \\
&\quad - \int_{t-(d/2)}^{t} \dot{x}^T(s) e^{-\alpha(s-t)} Z_{i,1} \dot{x}(s)\, ds
\end{aligned}$$

$$
\begin{aligned}
&- \int_{t-d}^{t-(d/2)} x^T(s) e^{-\alpha(s-t)} Z_{i,2} x(s)\, ds \\
&= \alpha V_{i,4}(t) + \frac{d}{2}\left[A_i x(t) + B_i x \right.\\
&\qquad \left. \times (t-h(t)) + G_i w(t)\right]^T \\
&\quad \times \left[Z_{i,1} + Z_{i,2}\right] \\
&\quad \times \left[A_i x(t) + B_i x(t-h(t)) + G_i w(t)\right] \\
&\quad - \int_{t-(d/2)}^{t} \dot{x}^T(s) e^{-\alpha(s-t)} Z_{i,1} \dot{x}(s)\, ds \\
&\quad - \int_{t-d}^{t-(d/2)} x^T(s) e^{-\alpha(s-t)} Z_{i,2} x(s)\, ds.
\end{aligned} \tag{14}
$$

Due to the Jensen inequality, inequality (15) holds

$$
\begin{aligned}
&\int_{t-(d/2)}^{t} \dot{x}^T(s) e^{-\alpha(s-t)} Z_{i,1} \dot{x}(s)\, ds \\
&\quad \geq \frac{2}{d}\left[x(t) - x\left(t-\frac{d}{2}\right)\right]^T \\
&\quad \times e^{\alpha d/2} Z_{i,1}\left[x(t) - x\left(t-\frac{d}{2}\right)\right], \\
&\int_{t-d}^{t-(d/2)} x^T(s) e^{-\alpha(s-t)} Z_{i,2} x(s)\, ds \\
&\quad \geq \frac{2}{d}\left[x\left(t-\frac{d}{2}\right) - x(t-d)\right]^T \\
&\quad \times e^{\alpha d} Z_{i,2}\left[x\left(t-\frac{d}{2}\right) - x(t-d)\right].
\end{aligned} \tag{15}
$$

By (14) and (15), we obtain

$$
\begin{aligned}
\dot{V}_{i,4}(t) &\leq \alpha V_{i,4}(t) + \frac{d}{2}\left[A_i x(t) + B_i x(t-h(t)) + G_i w(t)\right]^T \\
&\quad \times \left[Z_{i,1} + Z_{i,2}\right]\left[A_i x(t) + B_i x(t-h(t)) + G_i w(t)\right] \\
&\quad - \frac{2}{d}\left[x(t) - x\left(t-\frac{d}{2}\right)\right]^T \\
&\quad \times e^{\alpha d/2} Z_{i,1}\left[x(t) - x\left(t-\frac{d}{2}\right)\right] \\
&\quad - \frac{2}{d}\left[x\left(t-\frac{d}{2}\right) - x(t-d)\right]^T \\
&\quad \times e^{\alpha d} Z_{i,2}\left[x\left(t-\frac{d}{2}\right) - x(t-d)\right].
\end{aligned} \tag{16}
$$

From (11), (12), (13), and (16), it is easy to get

$$
\begin{aligned}
\dot{V}_i(t) - \alpha V_i(t) \leq &\begin{bmatrix} x(t) \\ x(t-h(t)) \\ x\left(t-\frac{d}{2}\right) \\ x(t-d) \\ w(t) \end{bmatrix}^T \\
&\times \begin{bmatrix}
\xi_{11} & P_i B_i + \frac{d}{2} A_i^T Z_i B_i & \frac{2}{d} e^{\alpha d/2} Z_{i,1} & 0 & P_i G_i + \frac{d}{2} A_i^T Z_i G_i \\
* & \frac{d}{2} B_i^T Z_i B_i - (1-k) Q & 0 & 0 & \frac{d}{2} B_i^T Z_i G_i \\
* & * & e^{\alpha d/2} R_i - \frac{2}{d} e^{\alpha d/2} Z_i & \frac{2}{d} e^{\alpha d} Z_{i,2} & 0 \\
* & * & * & -e^{\alpha d} R_{i,2} - \frac{2}{d} e^{\alpha d} Z_{i,2} & 0 \\
* & * & * & * & \frac{d}{2} G_i^T Z_i G_i
\end{bmatrix} \\
&\times \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x\left(t-\frac{d}{2}\right) \\ x(t-d) \\ w(t) \end{bmatrix}.
\end{aligned} \tag{17}
$$

According to the definition of finite-time boundedness, the rest of the proof will be divided into two steps. Under the given conditions, we need to prove that $x^T(t)x(t) < c_2$ and $c_1 < c_2$, respectively.

(i) We will prove that $x^T(t)x(t) < C_2$ holds for all t on $[0, T_f]$.

By (6) and (17), inequality (18) holds

$$\dot{V}_i(t) - \alpha V_i(t) < \frac{d}{2} w^T(t) H w(t). \quad (18)$$

Since $(d/dt)(e^{-\alpha t}V_i(t)) = e^{-\alpha t}[\dot{V}_i(t) - \alpha V_i(t)]$, inequality (18) can be transformed into

$$\frac{d}{dt}\left(e^{-\alpha t} V_i(t)\right) < \frac{d}{2} e^{-\alpha t} w^T(t) H w(t). \quad (19)$$

Let t_k stand for the instant of the Kth switching.

Integrating from t_k to t on both sides of (19), it follows that

$$V_i(t) < e^{\alpha(t-t_k)} V_i(t_k) + \frac{d}{2}\int_{t_k}^{t} w^T(s) e^{\alpha(t-s)} H w(s)\, ds. \quad (20)$$

Notice that $P_i \le \beta P_j$, $R_{i,1} \le \beta R_{j,1}$, $R_{i,2} \le \beta R_{j,2}$, $Q_i \le \beta Q_j$, $Z_{i,1} \le \beta Z_{j,1}$, $Z_{i,2} \le \beta Z_{j,2}$, i, and $j \in [1, 2, \ldots, m]$ and the continuity of $x(t)$, hence (21) holds

$$V_i(t) < \beta e^{\alpha(t-t_k)} V_i(t_{k^-}) + \frac{d}{2}\int_{t_k}^{t} w^T(s) e^{\alpha(t-s)} H w(s)\, ds, \quad (21)$$

where t_{k^-} denotes the instant just before t_k.

It is easy to see

$$V_i(t_{k^-}) < e^{\alpha(t_k - t_{k-1})} V_i(t_{k-1}) + \frac{d}{2}\int_{t_{k-1}}^{t_k} w^T(s) e^{\alpha(t-s)} H w(s)\, ds. \quad (22)$$

Then (23) is obtained

$$\begin{aligned} V_i(t) &< \beta^2 e^{\alpha(t-t_{k-1})} V_i\left(t_{(k-1)^-}\right) \\ &\quad + \frac{d}{2}\beta e^{\alpha(t-t_k)} \int_{t_{k-1}}^{t_k} w^T(s) e^{\alpha(t-s)} H w(s)\, ds \\ &\quad + \frac{d}{2}\int_{t_k}^{t} w^T(s) e^{\alpha(t-s)} H w(s)\, ds. \end{aligned} \quad (23)$$

Assume the switching number of $\sigma(t)$ over $[0, T_f]$ is N. (24) is obtained via the iterative calculation

$$\begin{aligned} V_i(t) &< \beta^N e^{\alpha t} V_i(0) + \frac{d}{2}\beta^N e^{\alpha(t-t_1)} \\ &\quad \times \int_{t_0}^{t_1} w^T(s) e^{\alpha(t-s)} H w(s)\, ds + \cdots \\ &\quad + \frac{d}{2}\beta e^{\alpha(t-t_k)} \int_{t_{k-1}}^{t_k} w^T(s) e^{\alpha(t-s)} H w(s)\, ds \\ &\quad + \frac{d}{2}\int_{t_k}^{T_f} w^T(s) e^{\alpha(t-s)} H w(s)\, ds, \end{aligned} \quad (24)$$

$$\begin{gathered} e^{\alpha T_f} \ge e^{\alpha t}, \\ e^{\alpha T_f} > e^{\alpha(T_f - t_1)} > e^{\alpha(T_f - t_2)} > \cdots > e^{\alpha(T_f - t_k)} > 1 \\ \text{for } t \in \left[0, T_f\right], \beta^N \ge \beta^{N-1} \ge \cdots \ge \beta \ge 1. \end{gathered} \quad (25)$$

Thus, it follows that

$$\begin{aligned} V_i(t) &< \beta^N e^{\alpha T_f} V_i(0) + \frac{d}{2}\beta^N e^{\alpha T_f} \\ &\quad \times \int_{t_0}^{t_1} w^T(s) e^{\alpha(t-s)} H w(s)\, ds + \cdots \\ &\quad + \frac{d}{2}\beta^N e^{\alpha T_f} \int_{t_{k-1}}^{t_k} w^T(s) e^{\alpha(t-s)} H w(s)\, ds \\ &\quad + \frac{d}{2} e^{\alpha T_f}\beta^N \int_{t_k}^{t} w^T(s) e^{\alpha(t-s)} H w(s)\, ds, \end{aligned} \quad (26)$$

$$V_i(t) < \beta^N e^{\alpha T_f} V_i(0) + \frac{d}{2} e^{\alpha T_f}\beta^N \int_0^t w^T(s) e^{\alpha(t-s)} H w(s)\, ds. \quad (27)$$

On the other hand, since $1 \le e^{\alpha(t-s)} \le e^{\alpha t} \le e^{\alpha T_f}$ and $H \le \lambda_{\max}(H)$, we have

$$\begin{aligned} &\frac{d}{2} e^{\alpha T_f}\beta^N \int_0^t w^T(s) e^{\alpha(t-s)} H w(s)\, ds \\ &\quad \le \frac{d}{2} e^{2\alpha T_f}\beta^N \lambda_{\max}(H) \int_0^t w^T(s) w(s)\, ds \\ &\quad \le \frac{d}{2} e^{2\alpha t}\beta^N \lambda_{\max}(H)\gamma. \end{aligned} \quad (28)$$

Applying the above inequality to (27), we get

$$V_i(t) < \beta^N e^{\alpha T_f} V_i(0) + \frac{d}{2} e^{2\alpha T_f}\beta^N \lambda_{\max}(H)\gamma. \quad (29)$$

With respect to $V_i(0)$ in (29), it is processed as follows:

$$\begin{aligned}
V_i(0) &= x^T(0)P_ix(0) + \int_{-d/2}^{0} x^T(s)e^{-\alpha s}R_{i,1}x(s)\,ds \\
&\quad + \int_{-d}^{-d/2} x^T(s)e^{-\alpha s}R_{i,2}x(s)\,ds \\
&\quad + \int_{-h(0)}^{0} x^T(s)e^{-\alpha s}Q_ix(s)\,ds \\
&\quad + \int_{-d/2}^{0}\int_{\theta}^{0} \dot{x}^T(s)e^{-\alpha s}Z_{i,1}\dot{x}(s)\,ds\,d\theta \\
&\quad + \int_{-d}^{-d/2}\int_{\theta}^{0} \dot{x}^T(s)e^{-\alpha s}Z_{i,2}\dot{x}(s)\,ds\,d\theta \\
&< \lambda_{\max}(P_i)\sup_{-d\le t\le 0}\{x^T(\bar{t}_0)x(\bar{t}_0)\} \\
&\quad + \frac{d}{2}e^{\alpha d/2}\lambda_{\max}(R_{i,1})\sup_{-d\le t\le 0}\{x^T(\bar{t}_0)x(\bar{t}_0)\} \\
&\quad + \frac{d}{2}e^{\alpha d}\lambda_{\max}(R_{i,2})\sup_{-d\le t\le 0}\{x^T(\bar{t}_0)x(\bar{t}_0)\} \\
&\quad + h_{\max}e^{\alpha h_{\max}}\lambda_{\max}(Q_i)\sup_{-d\le t\le 0}\{x^T(\bar{t}_0)x(\bar{t}_0)\} \\
&\quad + \int_{-d/2}^{0} -\theta\rho^2\lambda_{\max}(Z_{i,1})e^{-\alpha\theta}d\theta \\
&\quad + \int_{-d}^{-d/2} -\theta\rho^2\lambda_{\max}(Z_{i,2})e^{-\alpha\theta}d\theta \\
&< \lambda_{\max}(P_i)\sup_{-d\le t\le 0}\{x^T(\bar{t}_0)x(\bar{t}_0)\} \\
&\quad + \frac{d}{2}e^{\alpha d/2}\lambda_{\max}(R_{i,1})\sup_{-d\le t\le 0}\{x^T(\bar{t}_0)x(\bar{t}_0)\} \\
&\quad + \frac{d}{2}e^{\alpha d}\lambda_{\max}(R_{i,2})\sup_{-d\le t\le 0}\{x^T(\bar{t}_0)x(\bar{t}_0)\} \\
&\quad + h_{\max}e^{\alpha h_{\max}}\lambda_{\max}(Q_i)\sup_{-d\le t\le 0}\{x^T(\bar{t}_0t)x(\bar{t}_0)\} \\
&\quad + \frac{d}{2}\cdot\frac{d}{2}\rho^2\lambda_{\max}(Z_{i,1})e^{\alpha d/2} + \frac{d}{2}\cdot d\rho^2\lambda_{\max}(Z_{i,2})e^{\alpha d}.
\end{aligned} \tag{30}$$

Applying known mathematical relationships to (30), (31) can be obtained as

$$\begin{aligned}
V_i(0) &< \lambda_1C_1 + \frac{d}{2}e^{\alpha d/2}\lambda_2C_1 + \frac{d}{2}e^{\alpha d}\lambda_3C_1 \\
&\quad + h_{\max}e^{\alpha h_{\max}}\lambda_4C_1 + \frac{d^2}{4}\rho^2\lambda_5e^{\alpha d/2} + \frac{d^2}{2}\rho^2\lambda_6e^{\alpha d}.
\end{aligned} \tag{31}$$

Inequality (32) is obtained via (29) and (31) as

$$\begin{aligned}
V_i(t) &< \beta^N e^{\alpha T_f} \\
&\quad \times\left(\lambda_1 + \frac{d}{2}e^{\alpha d/2}\lambda_2 + \frac{d}{2}e^{\alpha d}\lambda_3 + h_{\max}e^{\alpha h_{\max}}\lambda_4\right)C_1 \\
&\quad + \beta^N e^{\alpha T_f}\left(\frac{d^2}{4}\rho^2\lambda_5e^{\alpha d/2} + \frac{d^2}{2}\rho^2\lambda_6e^{\alpha d}\right. \\
&\qquad \left. + \frac{d}{2}e^{\alpha T_f}\lambda_7\gamma\right) \\
&= \beta^N e^{\alpha T_f}\eta_1C_1 + \beta^N e^{\alpha T_f} \\
&\quad \times\left(\frac{d^2}{4}\rho^2\lambda_5e^{\alpha d/2} + \frac{d^2}{2}\rho^2\lambda_6e^{\alpha d} + \frac{d}{2}e^{\alpha T_f}\lambda_7\gamma\right).
\end{aligned} \tag{32}$$

According to the definition of $V_i(t)$, inequality (33) holds

$$\begin{aligned}
V_i(t) &> x^T(t)P_ix(t) \ge \lambda_{\min}(P_i)x^T(t)x(t) \\
&\ge \min_{i\in M}\{\lambda_{\min}(P_i)\}x^T(t)x(t).
\end{aligned} \tag{33}$$

Then the following holds based on (32) and (33):

$$\begin{aligned}
x^T(t)x(t) &< \left(\beta^N e^{\alpha T_f}\eta_1C_1 + \beta^N e^{\alpha T_f}\right. \\
&\quad \times\left(\frac{d^2}{4}\rho^2\lambda_5e^{\alpha d/2} + \frac{d^2}{2}\rho^2\lambda_6e^{\alpha d}\right. \\
&\qquad \left.\left. + \frac{d}{2}e^{\alpha T_f}\lambda_7\gamma\right)\right)(\lambda_8)^{-1} \\
&= C_2.
\end{aligned} \tag{34}$$

(ii) Next, $C_1 < C_2$ will be demonstrated.
By (7), we have

$$\frac{T_f}{\tau_a} > \frac{\ln C_1 + \ln\lambda_8 - \ln(\eta_1C_1+\eta_2) - \alpha T_f}{\ln\beta}, \tag{35}$$

$$N > \frac{T_f}{\tau_a} > \frac{\ln C_1 + \ln\lambda_8 - \ln(\eta_1C_1+\eta_2) - \alpha T_f}{\ln\beta}, \tag{36}$$

$$\ln(\eta_1C_1+\eta_2) - \ln\lambda_8 > \ln C_1 - N\ln\beta - \alpha T_f, \tag{37}$$

$$\frac{\eta_1C_1+\eta_2}{\lambda_8}e^{\alpha T_f}\beta^N > C_1. \tag{38}$$

On the other hand, due to $e^{\alpha T_f} \geq 1$, there exist the following mathematical relations:

$$\begin{aligned} C_2 &= \left(\beta^N e^{\alpha T_f} \eta_1 C_1 + \beta^N e^{\alpha T_f} \right. \\ &\quad \times \left(\frac{d^2}{4} \rho^2 \lambda_5 e^{\alpha d/2} + \frac{d^2}{2} \rho^2 \lambda_6 e^{\alpha d} \right. \\ &\quad \left. \left. + \frac{d}{2} e^{\alpha T_f} \lambda_7 \gamma \right) \right) (\lambda_8)^{-1} \\ &\geq \left(\beta^N e^{\alpha T_f} \eta_1 C_1 + \beta^N e^{\alpha T_f} \right. \\ &\quad \times \left(\frac{d^2}{4} \rho^2 \lambda_5 e^{\alpha d/2} + \frac{d^2}{2} \rho^2 \lambda_6 e^{\alpha d} \right. \\ &\quad \left. \left. + \frac{d}{2} \lambda_7 \gamma \right) \right) (\lambda_8)^{-1} \\ &= \frac{\eta_1 C_1 + \eta_2}{\lambda_8} e^{\alpha T_f} \beta^N . \end{aligned} \tag{39}$$

Combining (38) and (39), we get $c_2 > c_1$.

By (i) and (ii), the system (1) satisfies the definition of finite-time boundedness under given conditions. This completes the proof of Theorem 8. □

Remark 9. Notice that (6) is not a linear matrix inequality. Thus, it cannot be directly solved via LMI toolbox. Before solving (6), the inequality can be transformed to a linear matrix inequality by specifying the value of α.

4. A Numerical Example

An example is presented to illustrate Theorem 8. Consider

$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} x(t - h(t)) + G_{\sigma(t)} w(t), \quad t \geq 0,$$
$$x(t) = \varphi(t), \quad t \in [-d, 0),$$
$$A_1 = \begin{bmatrix} -1.7 & 1.7 & 0 \\ 1.3 & -1 & 0.7 \\ 0.7 & 1 & -0.6 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0.7 & 0 & -0.6 \\ 1.7 & 0 & -1.7 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1.5 & -1.7 & 0.1 \\ -1.3 & 1 & -0.3 \\ -0.7 & 1 & 0.6 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -1 & -0.3 & 0.1 \\ 1.3 & -0.1 & 0.6 \\ 1.5 & 0.1 & 1.8 \end{bmatrix},$$
$$G_1 = G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$w(t) = \begin{bmatrix} 0.03 \sin(t) \\ 0.02 \cos(2t) \\ 0.015 (\sin(t+1) + \cos(t-2)) \end{bmatrix},$$
$$h(t) = 0.5t, \quad d = 0.2,$$
$$\varphi(t) \equiv \begin{bmatrix} 0.5 & 0.1 & 0 \end{bmatrix}^T, \quad \forall t \in [-0.2, 0],$$
$$\max |\dot{\varphi}(t)| \leq \rho = 0, \quad \dot{h}(t) \leq k = 0.5, \quad C_1 = 0.26. \tag{40}$$

Let $\alpha = 0.02$, $\beta = 1.1$, and $T_f = 10$, then $h(t) \leq h_{\max} = 0.5 * 10 = 5$ and $\int_0^{T_f} w^T(s) w(s) dt \leq \gamma \approx 0.022$.

Solving (6) leads to feasible solutions that

$$P_1 = \begin{bmatrix} 0.8983 & -0.0167 & 0.1555 \\ -0.0167 & 1.0898 & -0.3930 \\ 0.1555 & -0.3930 & 0.9754 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 0.6101 & 0.1828 & -0.1480 \\ 0.1828 & 0.8908 & -0.3026 \\ -0.1480 & -0.3026 & 0.8153 \end{bmatrix},$$
$$R_{1,1} = \begin{bmatrix} 0.7188 & -0.1052 & 0.0283 \\ -0.1052 & 0.7458 & -0.1300 \\ 0.0283 & -0.1300 & 0.7114 \end{bmatrix},$$
$$R_{1,2} = \begin{bmatrix} 1.3854 & -0.1336 & 0.0209 \\ -0.1336 & 1.3613 & -0.1532 \\ 0.0209 & -0.1532 & 1.3368 \end{bmatrix},$$
$$R_{2,1} = \begin{bmatrix} 0.5289 & 0.0089 & -0.0566 \\ 0.0089 & 0.6200 & -0.0083 \\ -0.0566 & -0.0083 & 0.6743 \end{bmatrix},$$
$$R_{2,2} = \begin{bmatrix} 1.1615 & 0.0282 & -0.0465 \\ 0.0282 & 1.2555 & -0.0175 \\ -0.0465 & -0.0175 & 1.3518 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 4.2184 & -0.5908 & -0.1106 \\ -0.5908 & 4.4575 & -0.3066 \\ -0.1106 & -0.3066 & 4.0548 \end{bmatrix},$$
$$Q_2 = \begin{bmatrix} 3.8150 & 0.0518 & 0.0675 \\ 0.0518 & 3.7399 & 0.0356 \\ 0.0675 & 0.0356 & 4.2709 \end{bmatrix},$$
$$Z_{1,1} = \begin{bmatrix} 0.3150 & -0.0492 & -0.0003 \\ -0.0492 & 0.2950 & -0.0639 \\ -0.0003 & -0.0639 & 0.3082 \end{bmatrix},$$
$$Z_{1,2} = \begin{bmatrix} 0.4060 & 0.0002 & 0.0006 \\ 0.0002 & 0.4036 & 0.0013 \\ 0.0006 & 0.0013 & 0.3934 \end{bmatrix},$$
$$Z_{2,1} = \begin{bmatrix} 0.2124 & 0.0176 & -0.0190 \\ 0.0176 & 0.2812 & -0.0034 \\ -0.0190 & -0.0034 & 0.3118 \end{bmatrix},$$
$$Z_{2,2} = \begin{bmatrix} 0.3934 & 0.0070 & 0.0039 \\ 0.0070 & 0.3920 & -0.0131 \\ 0.0039 & -0.0131 & 0.3961 \end{bmatrix},$$

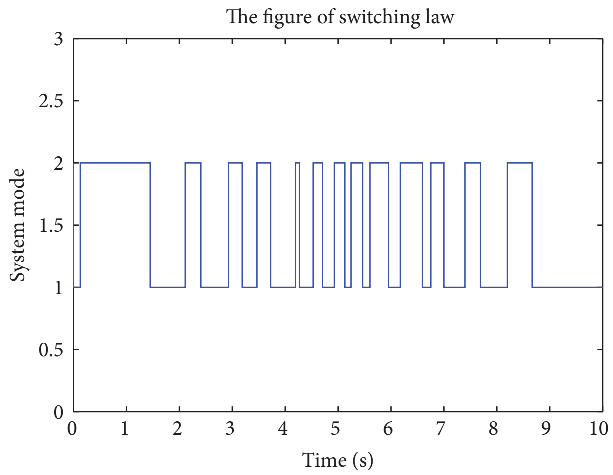

Figure 1: The diagram of switching law.

$$H = \begin{bmatrix} 12.5148 & 0.6115 & 0.1387 \\ 0.6115 & 13.0302 & -0.5248 \\ 0.1387 & -0.5248 & 12.4327 \end{bmatrix},$$
$$\lambda_1 = 1.4539, \quad \lambda_2 = 0.9114, \quad \lambda_3 = 1.5749,$$
$$\lambda_4 = 4.9735, \quad \lambda_5 = 0.2449, \quad \lambda_6 = 0.1192,$$
$$\lambda_7 = 13.5367, \quad \lambda_8 = 0.5200. \tag{41}$$

Further, we get that $C_2 = 2000.6421 > C_1$ and $\tau_a < 1.8263$. The simulation of the numerical example is performed and its results are shown in Figures 1 and 2. From Figure 1, one can get that $\tau_a < 1.8263$ holds. From Figure 2, it is easily found that the value of $x^T(t)x(t)$ remains within C_2 for $t \in [0, T_f]$. So, the system is indeed finite-time bounded over $[0, T_f]$.

5. Conclusion

(1) For the switched linear system, a new definition on finite-time boundedness is proposed which can reduce some complex matrix calculations.

(2) Under given conditions, the sufficient conditions which guarantee the system is finite-time bounded are given for the switched linear system with time-varying delay and external disturbance.

(3) In the future study, a challenging research topic is how to ensure the switched system with time-varying delay remains finite-time bounded for any switching signal.

Conflict of Interests

The authors (Yanke Zhong and Tefang Chen) declare that there is no conflict of interests regarding the publication of this paper.

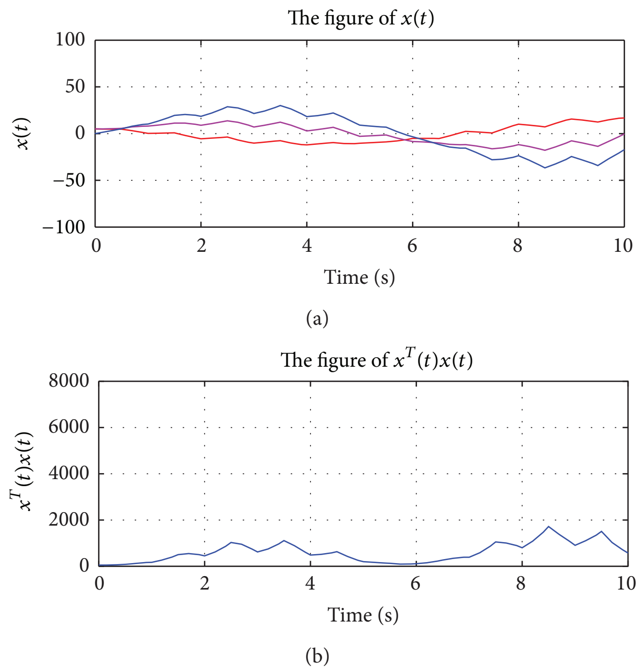

Figure 2: The diagrams of $x(t)$ and $x^T(t)x(t)$.

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