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Research Article

Lie Triple Derivations on $\mathcal{J}$-Subspace Lattice Algebras

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We describe the structure of Lie triple derivations on $\mathcal{J}$-subspace lattice algebras. The results can be applied to atomic Boolean subspace lattice algebras and pentagon subspace lattice algebras, respectively.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be an associative algebra, and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. We denote by $Z(\mathcal{M},\mathcal{A})$ the center of $\mathcal{M}$ relative to $\mathcal{A}$; that is, $Z(\mathcal{M},\mathcal{A}) = \{M \in \mathcal{M} : AM = MA \text{ for all } A \in \mathcal{A}\}$. A linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is called a Lie triple derivation if $\delta([[A,B],C]) = [[\delta(A),B],C]+[[A,\delta(B)],C]+[[A,B],\delta(C)]$ for all $A,B,C \in \mathcal{A}$, where $[A,B] = AB - BA$ is the usual Lie product. We say that a Lie triple derivation δ is *standard* if it can be decomposed as a sum of a derivation from $\mathcal{A}$ to $\mathcal{M}$ and a mapping from $\mathcal{A}$ to $Z(\mathcal{M},\mathcal{A})$ vanishing on every double commutator. The standard problem, which has been studied for many years, is to find conditions on $\mathcal{A}$ under which each Lie triple derivation is standard or standard-like. This problem has been investigated for von Neumann algebras in [1], for prime rings in [2], for nest algebras in [3, 4], for TUHF algebras in [5], and for upper triangular algebras in [6]. In this present note, we pursue this line of investigation for $\mathcal{J}$-subspace lattice algebras.

Throughout, all algebras and vector spaces will be over $\mathbb{F}$, where $\mathbb{F}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Given a Banach space X with topological dual X^*, by $B(X)$ we mean the algebra of all bounded linear operators on X. The terms *operator* on X and *subspace* of X will mean "bounded linear map of X into itself" and "norm closed linear manifold of X," respectively. For $A \in B(X)$, denote by A^* the adjoint of A. For any nonempty subset $L \subseteq X$, $L^{\perp}$ denotes its annihilator; that is, $L^{\perp} = \{f \in X^* : f(x) = 0 \text{ for all } x \in L\}$. For $x \in X$ and $f \in X^*$, the rank one operator $x \otimes f$ is defined by $(x \otimes f)z = f(z)x$ for $z \in X$.

A family $\mathcal{L}$ of subspaces of X is called a subspace lattice on X if it contains (0) and X and is complete in the sense that it is closed under the operations $\vee$ (closed linear span) and $\wedge$ (set-theoretic intersection). For a subspace lattice $\mathcal{L}$ on X, the associated subspace lattice algebra $\operatorname{Alg}\mathcal{L}$ is the set of operators on X leaving every subspace in $\mathcal{L}$ invariant; that is

$$\operatorname{Alg}\mathcal{L} = \{A \in B(X) : Ax \in L \text{ for every } x \in L \text{ and for every } L \in \mathcal{L}\}. \tag{1}$$

Obviously, $\operatorname{Alg}\mathcal{L}$ is a unital weakly closed operator algebra. A subalgebra of $\operatorname{Alg}\mathcal{L}$ is called a standard subalgebra if it contains all finite-rank operators in $\operatorname{Alg}\mathcal{L}$.

Subspace lattice algebras are important and mainly consist of nonself-adjoint operator algebras. Completely distributive subspace lattice algebras, commutative subspace lattice algebras, atomic Boolean subspace lattice algebras, pentagon subspace lattice algebras, and so forth have been widely studied. See, for example, [7–11]. Recently, Panaia in [12] introduced a new class of subspace lattices-$\mathcal{J}$-subspace lattice algebras. Several authors have studied $\mathcal{J}$-subspace lattice as well as $\mathcal{J}$-subspace lattice algebras; see, for example, [13–18].

Given a subspace lattice $\mathcal{L}$ on Banach space X, put

$$\mathcal{J}(\mathcal{L}) = \{L \in \mathcal{L} : L \neq (0), L_{-} \neq X\}, \tag{2}$$

where $K_- = \vee\{M \in \mathscr{L} : M \not\supseteq K\}$. Call $\mathscr{L}$ a $\mathscr{J}$-subspace lattice on X if

(1) $\vee\{K : K \in \mathscr{J}(\mathscr{L})\} = X$,

(2) $\wedge\{K_- : K \in \mathscr{J}(\mathscr{L})\} = (0)$,

(3) $K \vee K_- = X$, for every $K \in \mathscr{J}(\mathscr{L})$,

(4) $K \wedge K_- = (0)$, for every $K \in \mathscr{J}(\mathscr{L})$.

The simplest example of $\mathscr{J}$-subspace lattice is any pentagon subspace lattice $\mathscr{P} = \{(0), K, L, M, X\}$. Here K, L, and M are subspaces of X satisfying $K \vee L = X$, $K \wedge M = (0)$ and $L \subset M$. In this case, $K_- = M$, $L_- = K$, and $\mathscr{J}(\mathscr{P}) = \{K, L\}$. For further discussion of pentagon subspace lattice see [8, 10]. Another important element of the class of $\mathscr{J}$-subspace lattice is the atomic Boolean subspace lattice. It follows from [15] that every commutative $\mathscr{J}$-subspace lattice on a Hilbert space is an atomic Boolean subspace lattice. However, most $\mathscr{J}$-subspace lattices on Hilbert space are non-commutative.

Therefore, $\mathscr{J}$-subspace lattices as well as $\mathscr{J}$-subspace lattice algebras deserve some attention. In the previous papers [3, 17, 18], we studied algebraic isomorphisms, Jordan isomorphisms, Jordan derivations, and Lie derivations. Here we study Lie triple derivations of $\mathscr{J}$-subspace lattice algebras. Even for pentagon subspace lattice algebras and atomic Boolean subspace lattice algebras, our results are new.

For a subspace lattice $\mathscr{L}$, the relevance of $\mathscr{J}(\mathscr{L})$ is due to the following lemma, which is crucial to what follows.

Lemma 1 (see [11]). *If $\mathscr{L}$ is a subspace lattice on X, then the rank-one operator $x \otimes f$ belongs to Alg $\mathscr{L}$ if and only if there exists a subspace K in $\mathscr{J}(\mathscr{L})$ such that $x \in K$ and $f \in K_-^\perp$, where $K_-^\perp$ means $(K_-)^\perp$.*

From Lemma 1, we can see that if $\mathscr{L}$ is a $\mathscr{J}$-subspace lattice, then Alg $\mathscr{L}$ is rich in rank-one operators. Moreover, finite-rank operators in a $\mathscr{J}$-subspace lattice algebra have nice properties. Given a subspace lattice $\mathscr{L}$, by $\mathscr{F}(\mathscr{L})$ we denote the algebra of all finite-rank operators in Alg $\mathscr{L}$. If $K \in \mathscr{J}(\mathscr{L})$, then we write $\mathscr{F}_1(K) = \{x \otimes f : x \in K, f \in K_-^\perp\}$ and $\mathscr{F}(K) = \langle \mathscr{F}_1(K) \rangle$, the linear manifold spanned by $\mathscr{F}_1(K)$.

Lemma 2 (see [17]). *Let $\mathscr{L}$ be a $\mathscr{J}$-subspace lattice on X. Suppose that A is an operator of rank n in $\mathscr{F}(\mathscr{L})$. Then A can be written as a sum of n rank-one operators in Alg $\mathscr{L}$.*

Recalling that a linear mapping δ of an algebra $\mathscr{A}$ is a local derivation if for every $A \in \mathscr{A}$ there is a derivation d_A, depending on A, such that $\delta(A) = d_A(A)$.

Lemma 3 (see [3]). *Let $\mathscr{L}$ be a $\mathscr{J}$-subspace lattice on X. Then every local derivation from $\mathscr{F}(\mathscr{L})$ to a standard subalgebra of Alg $\mathscr{L}$ is a derivation.*

Lemma 4 (see [3]). *Let $\mathscr{L}$ be a $\mathscr{J}$-subspace lattice on X. Suppose that $\mathscr{A}$ is in $\mathscr{F}(\mathscr{L})$, S in $\mathscr{A}$, and C in $Z(\mathscr{A})$. If $[A, S] = C$, then $C = 0$.*

2. Lie Triple Derivation on $\mathscr{F}(\mathscr{L})$

The main result in this section reads as follows.

Theorem 5. *Let $\mathscr{L}$ be a $\mathscr{J}$-subspace lattice on X and $\mathscr{A}$ a standard subalgebra of Alg $\mathscr{L}$. Let δ be a Lie triple derivation from $\mathscr{F}(\mathscr{L})$ to $\mathscr{A}$. Then δ is standard.*

For the proof of the theorem, we need some lemmas. In the following, we keep the notation as in the statement of the theorem. Recalling that the statement means that δ is the sum of a derivation from $\mathscr{F}(\mathscr{L})$ to $\mathscr{A}$ and a linear mapping from $\mathscr{F}(\mathscr{L})$ to $Z(\mathscr{A}, \mathscr{F}(\mathscr{L}))$ vanishing on every double commutator. Here $Z(\mathscr{A}, \mathscr{F}(\mathscr{L})) = \{A \in \mathscr{A} : AB = BA$ for all $B \in \mathscr{F}(\mathscr{L})\}$. From [3, Remark 2.5(i)], we know that $Z(\mathscr{A}, \mathscr{F}(\mathscr{L}))$ is equal to $Z(\mathscr{A})$, the center of $\mathscr{A}$.

Lemma 6. *Let $K \in \mathscr{J}(\mathscr{L})$ and P be an idempotent operator in $\mathscr{F}(K)$. Then there are an operator S in $\mathscr{F}(K)$ and a unique operator $\tau(P)$ in $Z(\mathscr{A})$ such that $\delta(P) = [P, S] + \tau(P)$.*

Proof. Set $P_1 = P$ and $P_2 = I - P$. Note that $\mathscr{A}$ does not necessarily contain I; we understand $P_2 B = B - P_1 B$ and $BP_2 = B - BP_1$ for $B \in \mathscr{A}$.

For $A_{11} \in P_1 \mathscr{F}(\mathscr{L}) P_1$, we have that

$$\begin{aligned} 0 &= \delta\left(\left[\left[P_1, A_{11}\right], B\right]\right) \\ &= \left[\left[\delta\left(P_1\right), A_{11}\right], B\right] + \left[\left[P_1, \delta\left(A_{11}\right)\right], B\right]. \end{aligned} \tag{3}$$

Let $C = [\delta(P_1), A_{11}] + [P_1, \delta(A_{11})]$. Since B is arbitrary, we have $C \in Z(\mathscr{A})$. Then

$$\delta\left(P_1\right) A_{11} - A_{11} \delta\left(P_1\right) + P_1 \delta\left(A_{11}\right) - \delta\left(A_{11}\right) P_1 = C. \tag{4}$$

Multiplying this equation by P_1 from both sides, we get

$$P_1 \delta\left(P_1\right) P_1 A_{11} - A_{11} P_1 \delta\left(P_1\right) P_1 = C P_1; \tag{5}$$

that is,

$$\left[P_1 \delta\left(P_1\right) P_1, A_{11}\right] = C P_1. \tag{6}$$

It follows from the Kleinecke-Shirokov theorem (cf. [19, Problem 232]) that CP_1 is quasinilpotent. Let $L \in \mathscr{J}(\mathscr{L})$. Then there exists a scalar $\eta_L \in \mathbb{F}$, such that $Cx = \eta_L x$ for all $x \in L$. Particularly, $CP_1 x = \eta_L P_1 x$; that is, $(\eta_L I - CP_1) P_1 x = 0$. This implies that either $\eta_L = 0$ or $P_1 x = 0$ since CP_1 is quasinilpotent. Consequently, we always have $CP_1 x = 0$ for all $x \in L$. Hence $CP_1 = 0$ since $X = \vee\{L : L \in \mathscr{J}(\mathscr{L})\}$. Then by (5) we have that

$$P_1 \delta\left(P_1\right) P_1 A_{11} = A_{11} P_1 \delta\left(P_1\right) P_1. \tag{7}$$

Similarly, for $A_{22} \in P_2 \mathscr{F}(\mathscr{L}) P_2$, by considering $[P_1, A_{22}]$, we have that

$$P_2 \delta\left(P_1\right) P_2 A_{22} = A_{22} P_2 \delta\left(P_1\right) P_2. \tag{8}$$

Now for $A_{12} \in P_1 \mathscr{F}(\mathscr{L}) P_2$, we have that

$$\begin{aligned} \delta\left(A_{12}\right) &= \delta\left(\left[\left[A_{12}, P_1\right], P_1\right]\right) \\ &= \left[\left[\delta\left(A_{12}\right), P_1\right], P_1\right] + \left[\left[A_{12}, \delta\left(P_1\right)\right], P_1\right] \\ &\quad + \left[\left[A_{12}, P_1\right], \delta\left(P_1\right)\right] \end{aligned}$$

$$\begin{aligned}&= P_1\delta(A_{12})P_2 + P_2\delta(A_{12})P_1\\&\quad + A_{12}\delta(P_1)P_1 + P_1\delta(P_1)A_{12}\\&\quad - A_{12}\delta(P_1) + \delta(P_1)A_{12}.\end{aligned} \tag{9}$$

Multiplying this equation by P_1 from the left side and by P_2 from the right side, we get

$$P_1\delta(P_1)P_1A_{12} = A_{12}P_2\delta(P_1)P_2. \tag{10}$$

Similarly, for $A_{21} \in P_2\mathscr{F}(\mathscr{L})P_1$, we have that

$$P_2\delta(P_1)P_2A_{21} = A_{21}P_1\delta(P_1)P_1. \tag{11}$$

Using (7)–(11), it is easy to verify that $\tau(P) = P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 \in Z(\mathscr{A})$. Now let $S = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$. Then $S \in \mathscr{F}(K)$ and $\delta(P_1) = [P_1, S] + \tau(P_1)$. Moreover, by Lemma 4, such $\tau(P_1)$ is unique. □

Lemma 7. *Let $A = x \otimes f$ be in $\mathscr{F}(K)$ with $K \in \mathscr{J}(\mathscr{L})$. Then there are an operator S in $\mathscr{F}(K)$ and a unique operator $\tau(A)$ in $Z(\mathscr{A})$ such that $\delta(A) = [A, S] + \tau(A)$.*

Proof. If $f(x) \neq 0$, then the result follows from the linearity and Lemma 6.

We now suppose that $f(x) = 0$. Take $y \in K$ such that $f(y) = 1$. Obviously, x and y are linearly independent. Set $P = y \otimes f$. Then by Lemma 6, $\delta(P) = [P, S_1] + \tau(P)$, where $S_1 \in \mathscr{F}(K)$ and $\tau(P) \in Z(\mathscr{A})$. We associate a new Lie triple derivation as follows:

$$\Delta(T) = \delta(T) - [T, S_1], \quad T \in \mathscr{F}(\mathscr{L}). \tag{12}$$

Then $\Delta(P) = \tau(P) \in Z(\mathscr{A})$.

Let $P_1 = P$ and $P_2 = I - P$. Then

$$\begin{aligned}\Delta(A) &= \Delta([[A, P_1], P_1])\\&= [[\Delta(A), P_1], P_1] = P_1\Delta(A)P_2 + P_2\Delta(A)P_1.\end{aligned} \tag{13}$$

We will show $P_1\Delta(A)P_2 = 0$. For this, we first observe that $P_1\Delta(A)P_2 = y \otimes h$ for some $h \in K_-^{\perp}$. Since

$$\begin{aligned}&[P_1\Delta(A)P_2 + P_2\Delta(A)P_1, A]\\&\quad = [\Delta(A), A] = [\Delta(A), [A, P_1]]\\&\quad = \Delta([A, [A, P_1]]) - [A, [\Delta(A), P_1]]\\&\qquad - [A, [A, \Delta(P_1)]] = -[A, [\Delta(A), P_1]]\\&\quad = -[A, P_2\Delta(A)P_1 - P_1\Delta(A)P_2]\\&\quad = [P_2\Delta(A)P_1 - P_1\Delta(A)P_2, A],\end{aligned} \tag{14}$$

it follows that $[P_1\Delta(A)P_2, A] = 0$; that is, $h(x)y \otimes f - x \otimes h = 0$. Hence $h = 0$ since x and y are linearly independent. So $P_1\Delta(A)P_2 = 0$.

Now by (13), we have $\Delta(A) = P_2\Delta(A)P_1$. Therefore there exists $z \in K$ such that $\Delta(A) = z \otimes f$ and $f(z) = 0$. Choose $g \in K_-^{\perp}$ such that $g(x) = 1$. Then

$$\Delta(A) = z \otimes f = (z \otimes g)(x \otimes f) - (x \otimes f)(z \otimes g). \tag{15}$$

Let $S_2 = z \otimes g$. Then $\Delta(A) = [A, -S_2]$, and so $\delta(A) = [A, S_1 - S_2]$. Thus $S = S_1 - S_2$ and $\tau(A) = 0$ are desired. □

Lemma 8. *Suppose $A = \sum_{k=1}^{n} x_k \otimes f_k \in \mathscr{F}(K)$, with $K \in \mathscr{J}(\mathscr{L})$. Then there are an operator S in $\mathscr{F}(K)$ and a unique operator $\tau(A)$ in $Z(\mathscr{A})$ such that $\delta(A) = [A, S] + \tau(A)$. Moreover, $\tau(A) = \sum_{k=1}^{n} \tau(x_k \otimes f_k)$.*

Proof. By [3, Proposition 2.6], there is a matrix unit $\{y_i \otimes g_j\}_{i,j=1}^{m}$ such that each $x_k \otimes f_k$ belongs to the algebra

$$\mathscr{D} = \left\{C \in \mathscr{F}(K) : C = \sum_{i,j=1}^{m} \lambda_{ij} y_i \otimes g_j, \lambda_{ij} \in \mathbb{F}\right\}. \tag{16}$$

Obviously, $\mathscr{D}$ is a finite-dimensional Banach algebra which is isomorphic $M_m(\mathbb{F})$ via $C \to [\lambda_{ij}]_{m\times m}$. By [1], $\delta|_{\mathscr{D}}$ is standard, that is, $\delta|_{\mathscr{D}} = d + h$, where d is a derivation from $\mathscr{D}$ to $\mathscr{A}$ and h is a linear mapping from $\mathscr{D}$ to $Z(\mathscr{A}, \mathscr{D})$ vanishing on every double commutator. By [7, Lemma 10.7], there is an operator T in $\mathscr{A}$ such that $d(D) = [D, T]$ for all $D \in \mathscr{D}$. Consequently, d is a derivation from $\mathscr{D}$ to $\mathscr{F}(K)$. By [7] again, there is an operator S in $\mathscr{F}(K)$ such that $d(D) = [D, S]$ for all $D \in \mathscr{D}$. Thus for each k, it follows that

$$\delta(x_k \otimes f_k) = [x_k \otimes f_k, S] + h(x_k \otimes f_k). \tag{17}$$

On the other hand, by Lemma 7, for each k, there is an operator S_k in $\mathscr{F}(K)$ such that

$$\delta(x_k \otimes f_k) = [x_k \otimes f_k, S_k] + \tau(x_k \otimes f_k). \tag{18}$$

By (17) and (18), we get

$$[x_k \otimes f_k, T_k] = C_k, \tag{19}$$

where $T_k = S - S_k$, $C_k = \tau(x_k \otimes f_k) - h(x_k \otimes f_k)$. Since C_k commutes with $x_k \otimes f_k$, it follows from Kleinecke-Shirokov theorem that C_k is quasinilpotent. Moreover, $C_k y_i = \lambda_i y_i$ for some $\lambda_i \in \mathbb{F}$ and $C_k y_i \otimes g_i$ is quasinilpotent since C_k commutes with $y_i \otimes g_i$. This implies that $C_k y_i = 0$, $i = 1, \ldots, m$.

Since $x_k \otimes f_k = (\sum_{i=1}^{m} y_i \otimes g_i)(x_k \otimes f_k)(\sum_{i=1}^{m} y_i \otimes g_i)$, there are i, j such that $f_k(y_i) \neq 0$, $g_j(x_k) \neq 0$. Applying (19) to y_i, we get $f_k(T_k y_i)x_k - f_k(y_i)T_k x_k = C_k y_i = 0$. So $C_k = x_k \otimes g$, where $g = T_k^* f_k - (f_k(T_k y_i)/f_k(y_i))f_k$. Thus $g_j(x_k)y_i \otimes g = (y_i \otimes g_j)(x_k \otimes g) = C_k(y_i \otimes g_j) = 0$. So $g = 0$, and therefore $C_k = 0$ and $[x_k \otimes f_k, T_k] = 0$. Consequently, $[x_k \otimes f_k, S_k] = [x_k \otimes f_k, S]$ and $\tau(x_k \otimes f_k) = h(x_k \otimes f_k)$, $k = 1, \ldots, n$. Thus

$$\begin{aligned}\delta(A) &= \sum_{k=1}^{n} ([x_k \otimes f_k, S] + \tau(x_k \otimes f_k))\\&= [A, S] + \sum_{k=1}^{n} \tau(x_k \otimes f_k).\end{aligned} \tag{20}$$

By Lemma 4, the sum $\sum_{k=1}^{n} \tau(x_k \otimes f_k)$ is independent of the representation of A. Let $\tau(A) = \sum_{k=1}^{n} \tau(x_k \otimes f_k)$. The proof is complete. □

Proof of Theorem 5. Let $A \in \mathscr{F}(\mathscr{L})$. Then there exists a unique finite family of distinct $K_1, K_2, \ldots, K_n$ in $\mathscr{J}(\mathscr{L})$ such that $A = A_1 + A_2 + \cdots + A_n$, $A_i \in \mathscr{F}(K_i)$. By Lemma 8, for each i, there are operators S_i in $\mathcal{F}(K_i)$ and $\tau(A_i)$ in $Z(\mathcal{A})$ such that $\delta(A_i) = [A_i, S_i] + \tau(A_i)$. Let $S = S_1 + S_2 + \cdots + S_n$. Since $S_i A_j = A_j S_i = 0$ if $i \neq j$, it follows that

$$\begin{aligned}\delta(A) &= \sum_{i=1}^{n} \delta(A_i) = \sum_{i=1}^{n} ([A_i, S_i] + \tau(A_i)) \\ &= [A, S] + \sum_{i=1}^{n} \tau(A_i).\end{aligned} \tag{21}$$

Define $\tau(A)$ to be an operator in $Z(\mathcal{A})$ such that $\delta(A) = [A, T] + \tau(A)$ for some T in $\mathcal{F}(\mathcal{L})$. By Lemma 8 and the equation above, we see that $\tau(A)$ is well defined and $\tau(A) = \sum_{i=1}^{n} \tau(A_i)$.

Now define a mapping from $\mathcal{F}(\mathcal{L})$ to $\mathcal{A}$ as $\Delta(A) = \delta(A) - \tau(A)$ for $A \in \mathcal{F}(\mathcal{L})$. Then Δ is linear since τ is linear. Moreover, for each $A \in \mathcal{F}(\mathcal{L})$ there is an operator S such that $\Delta(A) = [A, S]$. Consequently, Δ is a local derivation from $\mathcal{F}(\mathcal{L})$ to $\mathcal{A}$. By [3, Proposition 2.6], Δ is a derivation. Thus δ is standard. This completes the proof. □

Corollary 9. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice and suppose that the dimension of each element in $\mathcal{F}(\mathcal{L})$ is infinite. Let δ be a Lie triple derivation from $\mathcal{F}(\mathcal{L})$ to itself. Then δ is a derivation.*

Proof. By [3, Remark 2.5(iii)], $Z(\mathcal{F}(\mathcal{L})) = 0$ in this case. Hence by Theorem 5, δ is a derivation. □

3. Lie Triple Derivations of $\mathcal{J}$-Subspace Lattice Algebras

In this section, we study Lie triple derivations of whole $\mathcal{J}$-subspace lattice algebras. The principal result describes the structure of those mappings.

Theorem 10. *Let $\mathcal{L}$ be a $\mathcal{J}$-subspace lattice on a Banach space X. Let $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ be a linear mapping. The following is equivalent.*

(i) *δ is a Lie triple derivation.*

(ii) *For each $K \in \mathcal{J}(\mathcal{L})$, there exist an operator T_K in $B(K)$ and a linear functional $\lambda_K : \operatorname{Alg}\mathcal{L} \to \mathbb{F}$ vanishing on every double commutator such that $\delta(A)x = (T_K A - A T_K)x + \lambda_K(A)x$ for all $A \in \operatorname{Alg}\mathcal{L}$ and $x \in K$.*

Proof. (ii) ⇒ (i). This is a straightforward verification.

(i) ⇒ (ii). Obviously, the restriction of δ to $\mathcal{F}(\mathcal{L})$ is a Lie triple derivation. Hence it is standard by Theorem 5. Therefore, there exist a derivation $d : \mathcal{F}(\mathcal{L}) \to \operatorname{Alg}\mathcal{L}$ and a linear mapping $\tau : \mathcal{F}(\mathcal{L}) \to Z(\operatorname{Alg}\mathcal{L})$ vanishing on every double commutator such that $\delta(F) = d(F) + \tau(F)$ for every $F \in \mathcal{F}(\mathcal{L})$.

Fix an element $K \in \mathcal{J}(\mathcal{L})$. Take vectors $x_K \in K$ and $f_K \in K_-^{\perp}$ such that $f_K(x_K) = 1$. Define a linear mapping $T_K : K \to K$ by

$$T_K x = d(x \otimes f_K) x_K \tag{22}$$

for $x \in K$. This is well defined because of $x \otimes f_K \in \mathcal{F}(\mathcal{L})$ for $x \in K$.

For $F \in \mathcal{F}(\mathcal{L})$ and $x \in K$, by (22) we have

$$\begin{aligned}T_K F x &= d(Fx \otimes f_K) x_K \\ &= d(F)(x \otimes f_K) x_K + F d(x \otimes f_K) x_K.\end{aligned} \tag{23}$$

So for all $F \in \mathcal{F}(\mathcal{L})$ and for all $x \in K$,

$$d(F)x = (T_K F - F T_K)x. \tag{24}$$

Let A be in $\operatorname{Alg}\mathcal{L}$. Let x be in K. Then for $F \in \mathcal{F}(\mathcal{L})$ with $F^2 = F$, by (22) and (24), we have

$$\begin{aligned}\delta([[A,F],F])x &= d([[A,F],F])x + \tau([[A,F],F])x \\ &= T_K([[A,F],F])x - ([[A,F],F])T_K x \\ &\quad + \tau([[A,F],F])x \\ &= (T_K AF + T_K FA - 2T_K FAF \\ &\quad - AFT_K - FAT_K + 2FAFT_K)x \\ &\quad + \tau([[A,F],F])x,\end{aligned} \tag{25}$$

$$\begin{aligned}\delta([[A,F],F])x &= [[\delta(A),F],F]x + [[A,\delta(F)],F]x \\ &\quad + [[A,F],\delta(F)]x \\ &= [[\delta(A),F],F]x + [[A,d(F)],F]x \\ &\quad + [[A,F],d(F)]x \\ &= (\delta(A)F - 2F\delta(A)F + F\delta(A) \\ &\quad + Ad(F)F - d(F)AF - FAd(F) \\ &\quad + Fd(F)A + AFd(F) - FAd(F) \\ &\quad - d(F)AF + d(F)FA)x \\ &= (\delta(A)F - 2F\delta(A)F + F\delta(A) \\ &\quad + A(T_K F - FT_K)F - (T_K F - FT_K)AF \\ &\quad - FA(T_K F - FT_K) + F(T_K F - FT_K)A \\ &\quad + AF(T_K F - FT_K) - FA(T_K F - FT_K) \\ &\quad - (T_K F - FT_K)AF \\ &\quad + (T_K F - FT_K)FA)x \\ &= (\delta(A)F - 2F\delta(A)F + F\delta(A) + AT_K F \\ &\quad - 2T_K FAF + 2FT_K AF \\ &\quad - 2FAT_K F + 2FAFT_K - FT_K A \\ &\quad - AFT_K + T_K FA)x.\end{aligned} \tag{26}$$

Comparing these two equations, we get

$$\begin{aligned}(\delta(A)-T_KA+AT_K)Fx &= F(2\delta(A)F-\delta(A)-2T_KAF\\ &\quad +2AT_KF-T_KFA-AT_K)x\\ &\quad +\tau([[A,F],F])x.\end{aligned} \tag{27}$$

Taking $f \in K_-^{\perp}$ with $f(x)=1$ and then putting $F = x\otimes f$ in the last equation, we get

$$\begin{aligned}(\delta(A)-(T_KA-AT_K))x &= \lambda_1(A,x,f)x+\tau([[A,F],F])x,\end{aligned} \tag{28}$$

where $\lambda_1(A,x,f) = f((2\delta(A)F-\delta(A)-2T_KAF+2AT_KF-T_KFA-AT_K)x)$. Since $\tau([[A,F],F])$ is in the center of $Z(\mathrm{Alg}\,\mathscr{L})$, it follows from [3, Remark 2.5] that $\tau([[A,F],F])x = \lambda_2(A,x,f)x$ for some $\lambda_2(A,x,f) \in \mathbb{F}$. Consequently, $(\delta(A)-(T_KA-AT_K))x$ is a scalar multiple of x. Since this holds for each $x \in K$, it follows easily that $\delta(A)-(T_KA-AT_K)$ viewed as a linear mapping from K to K is a scalar multiple of the identify on K. Namely, there exists a scalar $\lambda_K(A)$ such that

$$(\delta(A)-(T_KA-AT_K))x=\lambda_K(A)x \tag{29}$$

for each $x \in K$. One can easily see that $\lambda_K : \mathrm{Alg}\,\mathscr{L} \to \mathbb{F}$ is linear and vanishes on every double commutator.

It remains to verify the boundedness of T_K. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence of vectors in K, $\lim_{n\to\infty}x_n = 0$, and $\lim_{n\to\infty}T_Kx_n = x_0$. For any $f \in K_-^{\perp}$,

$$\begin{aligned}f(x_n)d(x_K\otimes f_K)x_K &= d((x_K\otimes f)(x_n\otimes f_K))x_K\\ &= d(x_K\otimes f)x_n\\ &\quad +(x_K\otimes f)d(x_n\otimes f_K)x_K\\ &= d(x_K\otimes f)x_n+f(T_Kx_n)x_K.\end{aligned} \tag{30}$$

Taking the limit, we get that $f(x_0)=0$. Since x_0 is in K and $f \in K_-^{\perp}$ is arbitrary, $x_0 = 0$. So T_K is closed. The Closed Graph theorem gives the boundedness of T_K, completing the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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