Research Article

Comparison Theorem for Nonlinear Path-Dependent Partial Differential Equations

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We introduce a type of fully nonlinear path-dependent (parabolic) partial differential equation (PDE) in which the path ω_t on an interval $[0, t]$ becomes the basic variable in the place of classical variables $(t, x) \in [0, T] \times \mathbb{R}^d$. Then we study the comparison theorem of fully nonlinear PPDE and give some of its applications.

1. Introduction

Motivated by uncertainty problems, risk measures, and the superhedging in finance mathematics, Peng [1] systemically established G-expectation theory. In the G-expectation framework, the notion of G-Brownian motion and the corresponding stochastic calculus of Itô's type were established. The key issue is that G-diffusion processes are connected to a large class of fully nonlinear PDEs in the Markov case studied by Peng [2] and Soner et al. [3].

Recently, Dupire [4] and Cont and Fournié [5] introduced a new functional Itô formula which nontrivially generalized the classical one through new notion-path derivatives (see [6, 7] for more general and systematic research). It extends the Itô stochastic calculus to functionals of a given process. It provides an excellent tool for the study of path-dependence. In fact, they showed that a smooth path functional solves a linear path-dependent PDE if its composition with a Brownian motion generates a martingale, which provided a functional extension of the classical Feynman-Kac formula. Moreover, by virtue of a backward stochastic differential equation approach, we can obtain the uniqueness of the smooth solution to the semilinear path-dependent PDE (see also [8–10] and the references therein). However, these methods are mainly based on stochastic calculus.

The objective of this paper is to study fully nonlinear PPDE. We refer to Krylov [11] and Wang [12] for the classical fully nonlinear PDE (see also [13–15]). Peng [16] introduced an approach of frozenness of the main course of the paths where the maximization takes place. This approach is based on techniques of PDE and can be directly applied to treat fully nonlinear path-dependent PDE. The advantage of this PDE approach is that one can treat the solution locally (path by path), whereas stochastic calculus is mainly a global approach. In this paper, we will use this method to obtain a comparison principle of fully nonlinear PPDE. In particular, some properties of the solution to fully nonlinear PPDE are also obtained. We claim that these ideas carry over to much more general frameworks, such as the case of viscosity solution to PPDE. Moreover, this method can have direct applications to stochastic analysis, for example, martingales under a fully nonlinear expectation, stochastic optimal control problems, stochastic games, nonlinear pricing and risk measuring, and backward stochastic differential equations. These more technical details are left to future work and will be presented in forthcoming papers.

The paper is organized as follows. In Section 2, we present some existing results in the theory of Dupire's path derivatives that we will use in this paper. In Section 3, we obtain the comparison theorem of fully nonlinear PPDE and give some of its applications.

2. Preliminaries

In this section, we give an overview of the definitions concerning path derivatives. The following notations are mainly from Dupire [4] and Cont and Fournié [5].

Let $T > 0$ be fixed. For each $t \in [0, T]$, we denote by Λ_t the set of right continuous $\mathbb{R}^d$-valued functions on $[0, t]$. For each $\omega_T \in \Lambda_T$ the value of ω_T at time $s \in [0, T]$ is denoted by $\omega(s)$. Thus $\omega_T = \omega(s)_{0 \le s \le T}$ is a right continuous process on $[0, T]$ and its value at time s is $\omega(s)$. The path of ω_T up to time t is denoted by ω_t; that is, $\omega_t = \omega(s)_{0 \le s \le t} \in \Lambda_t$. Denote $\Lambda = \bigcup_{t \in [0,T]} \Lambda_t$. We sometimes specifically write

$$\omega_t = \omega(s)_{0 \le s < t} = \left(\omega(s)_{0 \le s < t}, \omega(t) \right) \tag{1}$$

to indicate the terminal position $\omega(t)$ of ω_t which often plays a special role in this framework. For each $\omega_t \in \Lambda$ and $x \in \mathbb{R}^d$, denote by $\omega(s)$ the value of ω_t at $s \in [0, t]$ and $\omega_t^x := (\omega(s)_{0 \le s < t}, \omega(t) + x)$ which is also an element of Λ_t.

Now consider the function u of path; that is, $u : \Lambda \mapsto \mathbb{R}$. This function $u = u(\omega_t)_{\omega_t \in \Lambda}$ can be also regarded as a family of real valued functions:

$$u\left(\omega_t\right) = u\left(t, \omega_t(s)_{0 \le s \le t}\right) = u\left(t, \omega(s)_{0 \le s < t}, \omega(t)\right),$$
$$\omega_t \in \Lambda_t, \quad t \in [0, T]. \tag{2}$$

Denote $u(\omega_t^x) := u(t, \omega(s)_{0 \le s < t}, \omega(t) + x)$, for $\omega_t \in \Lambda_t$, $x \in \mathbb{R}^d$.

We introduce the distance on Λ. Let $\langle \cdot, \cdot \rangle$ and $| \cdot |$ denote the inner product and norm in $\mathbb{R}^d$. For each $0 \le t \le \bar{t} \le T$ and $\omega_t, \overline{\omega}_{\bar{t}} \in \Lambda$, denote

$$\left\| \omega_t \right\| := \sup_{s \in [0, t]} \left| \omega_t(s) \right|,$$

$$d_\infty \left(\omega_t, \overline{\omega}_{\bar{t}} \right) := \sup_{s \in [0, t \vee \bar{t}]} \left| \omega\left(s \wedge t\right) - \overline{\omega}\left(s \wedge \bar{t}\right) \right| + \left| t - \bar{t} \right|^{1/2}. \tag{3}$$

It is obvious that Λ_t is a Banach space with respect to $\| \cdot \|$. Since Λ is not a linear space, d_∞ is not a norm.

Definition 1 (continuous). A function $u : \Lambda \mapsto \mathbb{R}$ is said to be Λ-continuous at $\omega_t \in \Lambda$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\overline{\omega}_{\bar{t}} \in \Lambda$ with $d_\infty(\omega_t, \overline{\omega}_{\bar{t}}) < \delta$ we have $|u(\omega_t) - u(\overline{\omega}_{\bar{t}})| < \varepsilon$. u is said to be Λ-continuous if it is Λ-continuous at each $\omega_t \in \Lambda$.

Remark 2. In our framework we often regard $u(\omega_t^x)$ as a function of t, ω_t, and x; that is, $u(\omega_t^x) = u(t, \omega(s)_{0 \le s < t}, \omega(t) + x)$. Thus, for a fixed $\omega_t \in \Lambda$, $u(\omega_t^x)$ is regarded as a function of $(t, x) \in [0, T] \times \mathbb{R}^d$.

Definition 3. Given $u : \Lambda \mapsto \mathbb{R}$ and $\omega_t \in \Lambda$, if there exists $p \in \mathbb{R}^d$, such that

$$u\left(\omega_t^x\right) = u\left(\omega_t\right) + \langle p, x \rangle + o\left(|x|\right) \quad \text{as } x \longrightarrow 0, \ x \in \mathbb{R}^d, \tag{4}$$

then we say that u is (vertically) differentiable at ω_t and denote $D_x u(\omega_t) = p$. u is said to be vertically differentiable in Λ if $D_x u(\omega_t)$ exists for each $\omega_t \in \Lambda$. We can similarly define the Hessian $D_{xx} u(\omega_t)$. It is an $\mathbb{S}(d)$-valued function defined on Λ, where $\mathbb{S}(d)$ is the space of all $d \times d$ symmetric matrices.

For each $\omega_t \in \Lambda$, $\overline{\omega}_{\bar{t}} \in \Lambda$, with $\bar{t} \ge t$, set

$$\omega_{t,\bar{t}}(r) := \omega(r) \mathbf{1}_{[0,t)}(r) + \omega(t) \mathbf{1}_{[t,\bar{t}]}(r), \quad r \in [0, \bar{t}],$$

$$\omega_t \otimes \overline{\omega}_{\bar{t}}(r) := \omega(r) \mathbf{1}_{[0,t)}(r) + \left(\overline{\omega}(r) - \overline{\omega}(t) + \omega(t)\right) \mathbf{1}_{[t,\bar{t}]}(r),$$
$$r \in [0, \bar{t}]. \tag{5}$$

It is clear that $\omega_{t,\bar{t}} \in \Lambda_{\bar{t}}$ and $\omega_t \otimes \overline{\omega}_{\bar{t}} \in \Lambda_{\bar{t}}$.

Definition 4. For a given $\omega_t \in \Lambda$ if

$$u\left(\omega_{t,s}\right) = u\left(\omega_t\right) + a\left(s - t\right) + o\left(|s - t|\right) \quad \text{as } s \longrightarrow t, \ s \ge t, \tag{6}$$

then we say that $u(\omega_t)$ is (horizontally) differentiable in t at ω_t and denote $D_t u(\omega_t) = a$. u is said to be horizontally differentiable in Λ if $D_t u(\omega_t)$ exists for each $\omega_t \in \Lambda$.

Definition 5. Define $\mathbb{C}^{j,k}(\Lambda)$ as the set of functions u defined on Λ which are j times horizontally and k times vertically differentiable in Λ, such that all these derivatives are Λ-continuous.

Example 6. If $u(\omega_t) = f(t, \omega(t))$ with $f \in C^{1,1}([0, T) \times \mathbb{R})$, then

$$D_t u\left(\omega_t\right) = \partial_t f\left(t, \omega(t)\right), \qquad D_x u\left(\omega_t\right) = \partial_x f\left(t, \omega(t)\right), \tag{7}$$

which is the classic derivative. In general, these derivatives also satisfy the classic properties: linearity, product, and chain rule.

3. Comparison Theorem for Fully Nonlinear PPDE

Now we introduce the following fully nonlinear path-dependent PDE:

$$D_t u\left(\omega_t\right) + G\left(\omega_t, u\left(\omega_t\right), D_x u\left(\omega_t\right), D_{xx}^2 u\left(\omega_t\right)\right) = 0,$$
$$\omega_t \in \Lambda, \quad (8)$$
$$u\left(\omega_T\right) = \Phi\left(\omega_T\right), \quad \omega_T \in \Lambda_T,$$

where $u : \Lambda \mapsto \mathbb{R}$ and $G : \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d) \mapsto \mathbb{R}$ are continuous functions. Moreover, G satisfies the following elliptic conditions: for each $\omega_t \in \Lambda$, $r \in \mathbb{R}$, $p \in \mathbb{R}^d$, $X, Y \in \mathbb{S}(d)$,

$$G\left(\omega_t, r, p, X\right) \ge G\left(\omega_t, r, p, Y\right), \quad \text{whenever } X \ge Y \tag{9}$$

and, for each $r_1, r_2 \in \mathbb{R}$, $p_1, p_2 \in \mathbb{R}^d$, $X_1, X_2 \in \mathbb{S}(d)$, there exists some constant C such that

$$\left| G\left(\omega_t, r_1, p_1, X_1\right) - G\left(\omega_t, r_2, p_2, X_2\right) \right|$$
$$\le C\left(\left| r_1 - r_2 \right| + \left| p_1 - p_2 \right| + \left| X_1 - X_2 \right|\right). \tag{10}$$

Definition 7. A function $u \in \mathbb{C}^{1,2}(\Lambda)$ is called a $\mathbb{C}^{1,2}$-solution of the path-dependent PDE (8) if, for each $\omega_t \in \Lambda$, $t \in [0, T)$, equality (8) is satisfied. u is called a subsolution (resp., supersolution) of (8) if the "=" in (8) is replaced by "≥" (resp., "≤").

Remark 8. The solution of classical PDE is a special case when $u(\omega_t) = \overline{u}(t, \omega(t))$, $\overline{u} \in C^{1,2}([0, T] \times \mathbb{R})$. Indeed, for each $\omega_t \in \Lambda$ and $t \in [0, T)$,

$$\partial_t \overline{u}(t, \omega(t)) = D_t u(\omega_t),$$

$$\partial_x \overline{u}(t, \omega(t)) = D_x u(\omega_t), \tag{11}$$

$$\partial_{xx}^2 \overline{u}(t, \omega(t)) = D_{xx}^2 u(\omega_t),$$

and thus $\overline{u}(t, x)$ is a classical solution of PDE.

The following result is the so-called comparison principle or comparison theorem of path-dependent PDE.

Theorem 9. *Suppose u_1 is a $\mathbb{C}^{1,2}$-subsolution and u_2 is a $\mathbb{C}^{1,2}$-supersolution. Moreover, u_1 is bounded from above and u_2 is bounded from below. Then the maximum principle holds: if $u_1(\omega_T) \leq u_2(\omega_T)$ for all $\omega_T \in \Lambda_T$, then $u_1(\omega_t) \leq u_2(\omega_t)$ for each $\omega_t \in \Lambda$.*

Remark 10. In the case when $G(\omega_t, r, p, X) = (1/2)\operatorname{tr}[X] + f(\omega_t, r, p)$ for some Lipschitz function f on $\Lambda \times \mathbb{R} \times \mathbb{R}^d$, the above result is the comparison theorem of semilinear path-dependent PDE, which is given by [9].

In order to prove Theorem 9, we will make use of the following definitions.

Definition 11. $u : \Lambda \mapsto R$ is said to be in $\mathrm{USC}_*(\Lambda)$, if it satisfies the following conditions:

(i) for each fixed $\omega_{\widehat{t}} \in \Lambda$, $\overline{u}(t, x) := u((\omega_t^x)_{\widehat{t}, t-\widehat{t}})$ is an upper semicontinuous function of $(t, x) \in [\widehat{t}, T] \times \mathbb{R}^d$,

(ii) for each $\omega_t \in \Lambda$ with $t_i \uparrow t$, $\limsup_{i \to \infty} u(\omega_{t_i}) \leq \sup_x u(\omega_t^x)$.

We also denote $\mathrm{LSC}_*(\Lambda) := \{-u \mid u \in \mathrm{USC}_*(\Lambda)\}$. A function $u \in \mathrm{USC}_*(\Lambda)$ (resp., $u \in \mathrm{LSC}_*(\Lambda)$) is called a $\mathscr{C}$-upper (resp., $\mathscr{C}$-lower) semicontinuous function. $u \in \mathscr{C}(\Lambda) := \mathrm{USC}_*(\Lambda) \cap \mathrm{LSC}_*(\Lambda)$ is called a $\mathscr{C}$-continuous function.

Definition 12. Define $\mathscr{C}^{1,2}(\Lambda)$ as the set of functions $u \in \mathscr{C}(\Lambda) \cup \mathbb{C}(\Lambda)$, such that, for each $\omega_t \in \Lambda$, $D_t u(\omega_t)$, $D_x u(\omega_t)$, $D_{xx} u(\omega_t)$ exist.

Definition 13. A function $u \in \mathscr{C}^{1,2}(\Lambda)$ is called a $\mathscr{C}^{1,2}$-solution of the path-dependent PDE (8) if for each $\omega_t \in \Lambda$, $t \in [0, T)$ equality (8) is satisfied. u is called a $\mathscr{C}^{1,2}$-subsolution (resp., $\mathscr{C}^{1,2}$-supersolution) of (8) if the "=" in (8) is replaced by "≥" (resp., "≤").

Theorem 14. *Suppose u_1 is a $\mathscr{C}^{1,2}$-subsolution of PPDE (8) with G_1 and u_2 is a $\mathscr{C}^{1,2}$-supersolution of PPDE (8) with G_2,*

where $G_1(\omega_t, r, p, X) \leq G_2(\omega_t, r, p, X)$ for each $(\omega_t, r, p, X) \in \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d)$. Moreover, u_1 is bounded from above and u_2 is bounded from below. Then the maximum principle holds: if $u_1(\omega_T) \leq u_2(\omega_T)$ for all $\omega_T \in \Lambda_T$, then $u_1(\omega_t) \leq u_2(\omega_t)$ for each $\omega_t \in \Lambda$.

It is obvious that $\mathbb{C}^{1,2}(\Lambda) \subset \mathscr{C}^{1,2}(\Lambda)$. Then Theorem 9 is a direct consequence of Theorem 14.

For each $\alpha > 0$ and $\omega_t \in \Lambda$, set

$$w_\alpha(\omega_t) = u_1(\omega_t) - u_2(\omega_t) - \alpha\omega^2(t). \tag{12}$$

In order to prove Theorem 14, we need the following lemma, which is essentially from Peng [16, Lemma 6].

Lemma 15. *If $w_\alpha(\omega_t) > 0$ for some $\omega_t \in \Lambda$, then there exists $\overline{\omega}_{\overline{t}} \in \Lambda$, satisfying $\overline{t} \geq t$, $w_\alpha(\omega_t) \leq w_\alpha(\overline{\omega}_{\overline{t}})$, and $\overline{\omega}_{\overline{t}} = \omega_t \otimes \overline{\omega}_{\overline{t}}$, such that*

$$w_\alpha(\overline{\omega}_{\overline{t}}) = \sup_{\gamma_t \in \Lambda, t \geq \overline{t}} w_\alpha(\overline{\omega}_{\overline{t}} \otimes \gamma_t). \tag{13}$$

Proof. For each $\omega_t \in \Lambda$ and $w_\alpha(\omega_t) > 0$, we can find $\widehat{x} \in \mathbb{R}$ such that

$$w_\alpha(\omega_t^{\widehat{x}}) = \sup_x w_\alpha(\omega_t^x), \tag{14}$$

and for each $\omega_t \in \Lambda$ with $t_i \uparrow t$ we get

$$\limsup_{i \to \infty} w_\alpha(\omega_{t_i}) \leq \sup_x w_\alpha(\omega_t^x). \tag{15}$$

Then the proof is immediate in light of Lemma 6 of Peng [16]. □

Lemma 16. *Let $u \in \mathscr{C}^{1,2}(\Lambda)$ and $\overline{\omega}_{\overline{t}} \in \Lambda$ be given satisfying $u(\overline{\omega}_{\overline{t}}) \geq u(\overline{\omega}_{\overline{t}} \otimes \omega_t)$ for all $\omega_t \in \Lambda$, $t \geq \overline{t}$. Then*

$$D_t u(\overline{\omega}_{\overline{t}}) \leq 0, \qquad D_x u(\overline{\omega}_{\overline{t}}) = 0, \qquad D_{xx}^2 u(\overline{\omega}_{\overline{t}}) \leq 0. \tag{16}$$

Proof. Since $u(\overline{\omega}_{\overline{t}}) \geq u(\overline{\omega}_{\overline{t}, \overline{t}+\delta})$ for each δ, we conclude

$$D_t u(\overline{\omega}_{\overline{t}}) \leq 0. \tag{17}$$

For each $x \in \mathbb{R}^d$,

$$u(\overline{\omega}_{\overline{t}}) = \sup_x u(\overline{\omega}_{\overline{t}}^x), \tag{18}$$

and thus

$$D_x u(\overline{\omega}_{\overline{t}}) = 0, \qquad D_{xx}^2 u(\overline{\omega}_{\overline{t}}) \leq 0, \tag{19}$$

which is the desired result. □

Now we are going to give the proof of Theorem 14.

Proof of Theorem 14. We first observe that, for $\delta > 0$, the functions defined by $\overline{u}_1 := u_1 - \delta/t$ are a subsolution of

$$D_t \overline{u}_1(\omega_t) + \overline{G}_1(\omega_t, \overline{u}_1(\omega_t), D_x \overline{u}_1(\omega_t),$$

$$D_{xx} \overline{u}_1(\omega_t)) = \frac{\delta}{t^2}, \tag{20}$$

where $\overline{G}_1(\omega_t, r, p, X) := G_1(\omega_t, r + \delta/t, p, X)$. It is easy to check that the function $\overline{G}_1$ satisfies the same conditions as G_1. Since $u_1 \leq u_2$ follows from $\overline{u}_1 \leq u_2$ in the limit $\delta \to 0$, it suffices to prove Theorem 14 under the additional assumption

$$D_t u_1(\omega_t) + G_1(\omega_t, u_1(\omega_t), D_x u_1(\omega_t), D_{xx} u_1(\omega_t)) \geq c,$$

$$c = \frac{\delta}{T^2},$$

$$\lim_{t \to 0} u(\omega_t) = -\infty, \quad \text{uniformly on } [0, T].$$

$$(21)$$

We make the following assumption.

(A) For each $\omega_t \in \Lambda, u, v \in \mathbb{R}, p \in \mathbb{R}^d$, and $Q \in \mathbb{S}(d)$ such that $u \geq v$, we have

$$G(\omega_t, u, p, Q) \leq G(\omega_t, v, p, Q).$$

$$(22)$$

Set $\overline{u}_1(\omega_t) = u_1(\omega_t)e^{-\lambda t}, \overline{u}_2(\omega_t) = u_2(\omega_t)e^{-\lambda t}$, for some $\lambda < -C$. Then it is easy to check that $\overline{u}_i$ $(i = 1, 2)$ is a subsolution or supersolution of

$$D_t \overline{u}_i(\omega_t) + \overline{G}_i(\omega_t, \overline{u}_i(\omega_t), D_x \overline{u}_i(\omega_t), D_{xx} \overline{u}_i(\omega_t)) = 0,$$

$$(23)$$

where, for each $(\omega_t, u, p, Q) \in \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}(d)$, $\overline{G}_i(\omega_t, r, p, Q)$ is given by

$$\overline{G}_i(\omega_t, u, p, Q) = \lambda u + e^{-\lambda t} G_i(\omega_t, e^{\lambda t} u, e^{\lambda t} p, e^{\lambda t} Q), \quad (24)$$

which satisfies the assumption (A). Since $\overline{u}_1 \leq \overline{u}_2$ implies $u_1 \leq u_2$, it suffices to prove Theorem 14 under the additional assumption (A).

Without loss of generality, assume $w_\alpha \leq C^2$. Suppose by the contrary that there exist $\omega_{t_0}^0 \in \Lambda, t_0 < T$, and $\sqrt{\alpha} < c/4(C \vee C^2) \wedge 1$, such that

$$m_0 := w_\alpha(\omega_{t_0}^0) > 0.$$

$$(25)$$

Then, by Lemma 15, there exists $\overline{\omega}_{\overline{t}} \in \Lambda$ such that

$$w_\alpha(\overline{\omega}_{\overline{t}}) = \sup_{\gamma_t \in \Lambda, t \geq \overline{t}} w_\alpha(\overline{\omega}_{\overline{t}} \otimes \gamma_t) \geq m_0.$$

$$(26)$$

Consequently,

$$D_t u_1(\overline{\omega}_{\overline{t}}) - D_t u_2(\overline{\omega}_{\overline{t}}) \leq 0,$$

$$D_x u_1(\overline{\omega}_{\overline{t}}) - D_x u_2(\overline{\omega}_{\overline{t}}) = 2\alpha \overline{\omega}(\overline{t}),$$

$$D_{xx} u_1(\overline{\omega}_{\overline{t}}) - D_{xx} u_2(\overline{\omega}_{\overline{t}}) \leq 2\alpha.$$

$$(27)$$

Since $\alpha \overline{\omega}^2(\overline{t}) \leq C^2$, we have $\alpha \overline{\omega}(\overline{t}) \leq C\sqrt{\alpha}$. Then

$$0 < c$$

$$\leq D_t u_1(\overline{\omega}_{\overline{t}}) + G_1(\overline{\omega}_{\overline{t}}, u_1(\overline{\omega}_{\overline{t}}), D_x u_1(\overline{\omega}_{\overline{t}}), D_{xx} u_1(\overline{\omega}_{\overline{t}}))$$

$$\leq D_t u_2(\overline{\omega}_{\overline{t}}) + G_2(\overline{\omega}_{\overline{t}}, u_2(\overline{\omega}_{\overline{t}}), D_x u_2(\overline{\omega}_{\overline{t}}) + 2\alpha \overline{\omega}(\overline{t}),$$

$$D_{xx} u_2(\overline{\omega}_{\overline{t}}) + 2\alpha)$$

$$\leq 2C(\alpha + C\sqrt{\alpha}) < c.$$

$$(28)$$

This induces a contradiction and the proof is completed. $\quad\square$

Corollary 17. *The path-dependent PDE (8) has at most one bounded $\mathscr{C}^{1,2}$-solution.*

Lemma 18. *If u is a bounded $\mathscr{C}^{1,2}$-subsolution of PPDE (8) and Φ is a bounded function on Λ_T, then, for each $\omega_t \in \Lambda$,*

$$u(\omega_t) \leq \sup_{\omega_T \in \Lambda_T} \Phi(\omega_T).$$

$$(29)$$

Proof. Since $\sup_{\omega_T \in \Lambda_T} |\Phi(\omega_T)|$ is a $\mathscr{C}^{1,2}$-supersolution of PPDE (8), applying Theorem 14, we have the desired result. $\quad\square$

Remark 19. If $u(\omega_t) = \overline{u}(t, \omega(t))$ is the $\mathbb{C}^{1,2}$-solution of PPDE (8) for some bounded $\overline{u} \in C^{1,2}([0, T) \times \mathbb{R})$, then the above theorem is the classical comparison theorem of PDE of [17].

Example 20. Consider the following PPDE:

$$D_t u(\omega_t) + G(D_{xx}^2 u(\omega_t)) = 0, \quad \omega_t \in \Lambda,$$

$$u(\omega_T) = \Phi(\omega_T), \quad \omega \in \Lambda_T,$$

$$(30)$$

where $G(a) := (1/2)(\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-)$ for some $0 < \underline{\sigma}^2 \leq \overline{\sigma}^2 < \infty$.

If $\Phi(\omega) = \varphi(\omega(t), \omega(T))$ for some bounded Lipschitz function $\varphi \in C(\mathbb{R}^2)$ and $t \in [0, T]$, then we can solve the PPDE (30) by the following method.

First consider the following system of fully nonlinear parabolic partial differential equations, defined on $[0, T] \times \mathbb{R}^2$ and parameterized by $y \in \mathbb{R}$:

$$\partial_s v_1(s, x, y) + G(\partial_{xx}^2 v_1(s, x, y)) = 0,$$

$$v_1(T, x, y) = \varphi(x, y),$$

$$(31)$$

and, then, another one defined on $[0, t] \times \mathbb{R}$:

$$\partial_s v_2(s, x) + G(\partial_{xx}^2 v_2(s, x)) = 0,$$

$$v_2(t, x) = v_1(t, x, x).$$

$$(32)$$

From Krylov [11] and Wang [18], for each $y \in \mathbb{R}, v_1(s, x, y) \in C^{1+\alpha/2, 2+\alpha}([0, T - \kappa] \times \mathbb{R})$, and $v_2(s, x) \in C^{1+\alpha/2, 2+\alpha}([0, t - \kappa] \times \mathbb{R})$, where $\alpha \in (0, 1)$ and $\kappa > 0$.

Denote $u(\omega_s) := v_1(s, \omega(s), \omega(t))1_{t \leq s} + v_2(s, \omega(s))1_{s < t}$; we obtain $u \in \mathscr{C}^{1,2}(\Lambda)$. Applying Theorem 14, u is the unique $\mathscr{C}^{1,2}$-solution of PPDE (30). Indeed, $u(\omega_t)$ is the conditional G-expectation of $\Phi(\omega_T)$ (see [19, 20]).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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