Research Article

Traveling Wave Solutions and Infinite-Dimensional Linear Spaces of Multiwave Solutions to Jimbo-Miwa Equation

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The traveling wave solutions and multiwave solutions to $(3 + 1)$-dimensional Jimbo-Miwa equation are investigated in this paper. As a result, besides the exact bounded solitary wave solutions, we obtain the existence of two families of bounded periodic traveling wave solutions and their implicit formulas by analysis of phase portrait of the corresponding traveling wave system. We derive the exact 2-wave solutions and two families of arbitrary finite N-wave solutions by studying the linear space of its Hirota bilinear equation, which confirms that the $(3 + 1)$-dimensional Jimbo-Miwa equation admits multiwave solutions of any order and is completely integrable.

1. Introduction

Various nonlinear partial differential equations (NLPDEs) have been proposed to model different kinds of phenomena in natural and applied sciences such as fluid dynamics, plasma physics, solid-state physics, optical fibers, acoustics, mechanics, biology, and mathematical finance. Obviously, it is of significant importance to study the solutions of such NLPDEs from both theoretical and practical points of view. However, the solution spaces of nonlinear equations are infinite-dimensional and contain diverse solution structures, so it is usually a difficult job to determine the solutions to nonlinear NLPDEs.

A great idea to generate exact solutions of NLPDEs is to reduce the NLPDEs into some algebraic equations by assuming the solutions to have some special forms or satisfy some solvable simpler equations. This can be seen in, for example, the exp-function method [1], the tanh function method [2], the homogeneous balance method [3, 4], the auxiliary function method [5, 6], the sech-function method [7], the sine-cosine method [8, 9], the tanh-coth method [10], the Jacobi elliptic function method [11], the F-expansion method, and the extended F-expansion method [12]. Normally, it is not an easy task to solve these nonlinear algebraic equations reduced from NLPDEs because they involve many parameters. However, the rapid development of symbolic computation makes it relatively easy to solve these algebraic equations [13]. At the same time one should be very careful when applying these methods as different methods might give the same solutions. For example, solutions obtained by using the sech-function method, the tanh-coth method, and the exp-function method are actually the same [14, 15], because $f(\text{sech}^2 x) = f(1 - \tanh^2 x) = f(4/(e^x + e^{-x})^2)$, for any function f.

Recently, the planar dynamical system theorem has been employed to study the traveling wave solutions of NLPDEs [16, 17]. The best advantage of this approach is that the boundedness, periodicity, the shapes of the solutions, and even the singular traveling wave solutions of NLPDEs can be recognized easily from the corresponding orbits of their phase portraits under various different parametric conditions. Also the exact solutions can be derived at the same time. It is worth pointing out that Hirota's bilinear method [18] is also an amazing method to find exact solutions to some NLPDEs. By some independent variable transformation, various nonlinear equations of mathematical physics are transformed into Hirota's bilinear equations [18, 19], which possess some specific properties and thus might be applied

to study the solution sets of nonlinear differential equations. Recently, even some programs have been designed and some algorithms have been proposed on searching for integrable bilinear equations [20–26]. Based on Hirota's bilinear form, soliton solutions were obtained by the Hirota perturbation technique [18], the multiple exp-function algorithm [27], and other methods [28–31]. Even some generalized bilinear form has been proposed recently by Ma [32].

The $(3 + 1)$-dimensional Jimbo-Miwa equation

$$u_{xxxy} + 3(u_x u_y)_x + 2u_{yt} - 3u_{xz} = 0 \qquad (1)$$

was first proposed and studied by Jimbo and Miwa [33]. Recently, this equation has attracted a great deal of attention which mainly focuses on its solutions, integrability properties and symmetries. Ma and Lee [34] proposed a direct approach to solve (1) by using rational function transformations. Li and Dai [35] applied the generalized Riccati equation method to look for its exact solutions. However, in [36], Kudryashov and Sinelshchikov pointed out some mistakes and proved that some solutions in [35] could be written in a uniform form and thus they are not new at all.

The $(3 + 1)$-dimensional Jimbo-Miwa equation can be transformed into the Hirota bilinear equation

$$(D_x^3 D_y + 2D_t D_y - 3D_x D_z) f \cdot f = 0 \qquad (2)$$

through the dependent variable transformation $u = 2(\ln f)_x$.

In this paper, we firstly study the bounded traveling wave solutions of the Jimbo-Miwa equation (1) by investigating the bifurcation and phase portraits of a planar cubic polynomial ordinary differential equation by using the planar dynamical system theory [16, 37]. We then explore the 1-wave and 2-wave solutions and the infinite-dimensional linear spaces of multiwave solutions to Jimbo-Miwa equation by employing Hirota's bilinear method, thus confirming that Jimbo-Miwa equation possesses multiwave solutions of arbitrary order.

2. Traveling Wave Solutions to the $(3 + 1)$-Dimensional Jimbo-Miwa Equation

To investigate the traveling wave solutions to the $(3 + 1)$-dimensional Jimbo-Miwa equation, we make the traveling wave transformation $u(x, y, z, t) = u(\xi)$, $\xi = kx + ly + mz + nt$, under which (1) is reduced to the nonlinear ordinary differential equation

$$k^3 l \frac{d^4 u}{d\xi^4} + 6k^2 l \frac{du}{d\xi} \frac{d^2 u}{d\xi^2} + (2ln - 3km) \frac{d^2 u}{d\xi^2} = 0. \qquad (3)$$

Integrating (1) once with respect to ξ gives

$$k^3 l \frac{d^3 u}{d\xi^3} + 3k^2 l \left(\frac{du}{d\xi}\right)^2 + (2ln - 3km) \frac{du}{d\xi} = g, \qquad (4)$$

where g is an arbitrary constant. Let $v = du/d\xi$; then (1) becomes

$$k^3 l \frac{d^2 v}{d\xi^2} + 3k^2 l v^2 + (2ln - 3km) v = g, \qquad (5)$$

which is a second-order nonlinear ordinary differential equation. We will study the solutions of (5) by planar dynamical system method and thus derive the traveling wave solutions to the $(3 + 1)$-dimensional Jimbo-Miwa equation.

2.1. Bounded Solutions of (5). First, we rewrite (5) in a simpler and more general form, namely,

$$\frac{d^2 v}{d\xi^2} = av^2 + bv + c, \qquad (6)$$

where $a = -3/k$, $b = (3km - 2ln)/(k^3 l)$, and $c = g/(k^3 l)$. We now study the bifurcation and exact solutions of (6). Let $v' = w$; then (6) is equivalent to the dynamical system

$$\begin{aligned} v' &= w, \\ w' &= av^2 + bv + c, \end{aligned} \qquad (7)$$

which has the Hamiltonian

$$H(v, w) = \frac{w^2}{2} - \frac{a}{3}v^3 - \frac{b}{2}v^2 - cv. \qquad (8)$$

Clearly, the phase orbits defined by the vector fields of system (7) determine all solutions of (6). The bounded solutions of (6) correspond to the bounded phase orbits of system (7), which we now investigate. Along the orbit corresponding to $H(v, w) = h$,

$$\left(\frac{dv}{d\xi}\right)^2 = \frac{2a}{3}v^3 + bv^2 + 2cv + 2h. \qquad (9)$$

Consequently, the general formula of the solutions of (6) can be expressed as

$$\int_0^v \frac{dv}{\sqrt{(2av^3/3) + bv^2 + 2cv + 2h}} = \pm \int_{\xi_0}^\xi d\xi. \qquad (10)$$

However, it is not easy to know the properties and the shapes of (10) which actually is determined by the parameters a, b, c, and h. Clearly, the abscissas of equilibrium points of system (7) are the zeros of $av^2 + bv + c = 0$. Obviously, the system has no bounded orbits when $b^2 - 4ac < 0$. We suppose that $b^2 - 4ac > 0$ in order to study the bounded orbits of system (7). Denoting $v_\pm = (-b \pm \sqrt{b^2 - 4ac})/2a$, then $E_+(v_+, 0)$ and $E_-(v_-, 0)$ are two equilibrium points of system (7). By the theory of planar dynamical system, we know that E_+ is a saddle point and E_- is a center. Denote $h_\pm = H(v_\pm, 0)$, and, by careful computation, we get

$$h_\pm = \frac{(b^2 - 4ac)(-b \pm \sqrt{b^2 - 4ac}) + 2abc}{12a^2}. \qquad (11)$$

Obviously, $h_- < h < h_+$. $H(v, w) = h_+$ corresponds to homoclinic orbits, and $H(v, w) = h_-$ corresponds to the center E_- and $H(v, w) = h$, where $h_+ < h < h_-$ corresponds to a family of closed orbits surrounding the center E_-, which are surrounded by a homoclinic orbit. That is to say, (10) defines

bounded solutions if and only if $h_+ \le h < h_-$. To be exact, (10) defines a family of periodic solutions when $h_+ < h < h_-$.

When $h = h_+$, (10) defines a bounded solution which approaches v_+ as ξ goes to infinity. Actually,

$$\frac{2a}{3}v^3 + bv^2 + 2cv + 2h_+ = \frac{2a}{3}(v - v_+)^2(v - v_0), \quad (12)$$

where $v_0 = -(b + 2\sqrt{b^2 - 4ac})/2a$, so (10) can be reduced to

$$\int_{v_0}^{v} \frac{dv}{(v - v_+)\sqrt{a(v - v_0)}} = \sqrt{\frac{2}{3}}(\xi - \xi_0), \quad (13)$$

from which we can get the exact solution as

$$v = v_+ - (v_+ - v_0)\operatorname{sech}^2\left(\sqrt{\frac{a(v_+ - v_0)}{6}}(\xi - \xi_0)\right). \quad (14)$$

By further simplification, (14) becomes

$$v = v_+ - \frac{3\sqrt{b^2 - 4ac}}{2a}\operatorname{sech}^2\left[\frac{1}{2}(b^2 - 4ac)^{1/4}(\xi - \xi_0)\right], \quad (15)$$

which is an exact bounded solution of (5).

Thus, we have the following lemma.

Lemma 1. *The general second-order ODE (6) has bounded solutions if and only if $b^2 - 4ac > 0$. The bounded solutions can be expressed as (10) in an implicit form. In fact, provided $h_- < h < h_+$, (10) defines a family of bounded periodic solutions and $h = h_+$ defines a bounded solution which approaches v_+ as ξ goes to infinity and can be expressed explicitly as (15), where $v_+ = (-b + \sqrt{b^2 - 4ac})/2a$ and $h_\pm$ is defined by (11).*

2.2. Bounded Traveling Wave Solutions to the (3 + 1)-Dimensional Jimbo-Miwa Equation. According to the analysis and results in Section 2.1, we know that (5) has only two kinds of bounded solutions, among which one is a family of periodic solutions and another is a family of solutions approaching a fixed number as ξ goes to infinity. Note that what we aim to study is the bounded traveling wave solutions to the (3 + 1)-dimensional Jimbo-Miwa equation which are determined by $v = du/d\xi$ and v satisfies (5). So we have to investigate how we can get the bounded solutions to (3) from the bounded solution of (5).

Clearly, $u(\xi) = \int_{\xi_0}^{\xi} v(\xi)d\xi$, and $v(\xi)$ can be expressed implicitly as (10). By the geometry meaning of the integral and the properties of the solutions of (5), we get the traveling wave solutions to the (3+1)-dimensional Jimbo-Miwa equation. To get the bounded solution, we choose the integral constant g to be 0; that is, $c = 0$ in (15), and so

$$u(\xi) = g - \frac{3\sqrt{|b|}}{a}\tanh\left[\frac{\sqrt{|b|}}{2}(\xi - \xi_0)\right]; \quad (16)$$

that is,

$$u(\xi) = g - k\sqrt{\left|\frac{2ln - 3km}{k^3 l}\right|} \\ \times \tanh\left[\frac{1}{2}\sqrt{\left|\frac{2ln - 3km}{k^3 l}\right|}(\xi - \xi_0)\right], \quad (17)$$

which is a family of exact bounded kink traveling wave solutions to the (3 + 1)-dimensional Jimbo-Miwa equation, where g is an arbitrary constant.

However, we may not get bounded solutions from the family of periodic solutions of (5). It is easy to see that if $v(\xi)$ is a periodic solution of (5), then $u(\xi) = \int_{\xi_0}^{\xi} v(\xi)d\xi$ is bounded if and only if $\int_0^T v(\xi)d\xi = 0$, where T is the period of the function $v(\xi)$. Recall that the period of the function $v(\xi)$ which is defined by (10) with $h_- < h < h_+$ continuously depends on the parameters a, b, c, and h. So $\int_0^T v(\xi)d\xi$ continuously depends on the parameters a, b, c, and h too. Let $V(a,b,c,h) = \int_0^T v(\xi)d\xi$; then $V(a,b,c,h)$ is a continuous function of a, b, c, and h. We now prove the existence of the root of $V(a,b,c,h) = 0$ to get the existence of the bounded periodic traveling wave solutions to the (3 + 1)-dimensional Jimbo-Miwa equation.

By the theory of planar dynamical system, when $h_- < h < h_+$,

$$\frac{2a}{3}v^3 + bv^2 + 2cv + 2h_+ = \frac{2a}{3}(v - v_1)(v - v_2)(v - v_3), \quad (18)$$

where $v_1 > v_+ > v_2 > v_- > v_3 > v_0$ in the case of $a > 0$ and $v_0 > v_1 > v_- > v_2 > v_+ > v_3$ in the case of $a < 0$. The periodic solution satisfies $v_+ > v_2 > v(\xi) > v_3 > v_0$ when $a > 0$ and $v_0 > v_1 > v(\xi) > v_2 > v_+$ when $a < 0$. Clearly, $v_0 > 0$ when $a > 0$ and $0 > b > -4\sqrt{ac/3}$. So $v(\xi) > 0$ and thus $V(a,b,c,h) > 0$ when $a > 0$ and $0 > b > -4\sqrt{ac/3}$. However, $v(\xi) < v_+ < 0$ and thus $V(a,b,c,h) < 0$ when $a > 0, c > 0$, and $b > 0$. There must exist at least one zero of $V(a,b,c,h) = 0$ in the region $a > 0$ of the parameter space since $V(a,b,c,h)$ is a continuous function of a, b, c, and h. The same happens in the region $a < 0$ of the parameter space. So, we know that there exist at least two families of bounded periodic traveling wave solutions to the (3 + 1)-dimensional Jimbo-Miwa equation. Thus we have the following theorem.

Theorem 2. *The (3 + 1)-dimensional Jimbo-Miwa equation has two types of bounded traveling wave solutions as given below.*

(1) *The (3 + 1)-dimensional Jimbo-Miwa equation has a family of exact bounded kink traveling wave solutions*

$$u(x, y, z, t) \\ = g - k\sqrt{\left|\frac{2ln - 3km}{k^3 l}\right|} \\ \times \tanh\left[\frac{1}{2}\sqrt{\left|\frac{2ln - 3km}{k^3 l}\right|}(kx + ly + mz + nt - \xi_0)\right], \quad (19)$$

where ξ_0 and g are two arbitrary constants.

(2) *The (3 + 1)-dimensional Jimbo-Miwa equation has at least two families of bounded periodic traveling wave solutions which are determined implicitly by* (10) *and $u(\xi) = \int_{\xi_0}^{\xi} v(\xi) d\xi$, where $\xi = kx + ly + mz + nt$ and ξ_0 is an arbitrary constant.*

3. N-Wave Solutions Linear Subspace of the (3 + 1)-Dimensional Jimbo-Miwa Equation

In this section we study the N-wave solutions to the $(3 + 1)$-dimensional Jimbo-Miwa equation by the linear superposition principle to Hirota bilinear equations which was proposed firstly by Ma and Fan [30].

Let wave variables $\eta_i = \mathbf{k_i} \cdot \mathbf{x}$, $1 \le i \le N$, where $\mathbf{k_i} = \{k_{1,i}, k_{2,i}, k_{3,i}, k_{4,i}\}^T$, $\mathbf{x} = \{x, y, z, t\}^T$. According to the linear superposition principle to Hirota bilinear equations [30, 31], we can get the following sufficient and necessary conditions for span$\{e^{\eta_1}, \ldots, e^{\eta_N}\}$ being a subspace of the Hirota bilinear equation (2).

Theorem 3. *Let $\mathbf{k_i} = \{k_{1,i}, k_{2,i}, k_{3,i}, k_{4,i}\}^T$, $\mathbf{x} = \{x, y, z, t\}^T$, and $f_N = \sum_{i=1}^{N} \epsilon_i e^{\mathbf{k_i} \cdot \mathbf{x}}$. Then, for any constants $\{\epsilon_i, 1 \le i \le N\}$, f_N solves the Hirota bilinear equation* (2) *if and only if*

$$
\begin{aligned}
&\left(k_{1,i} - k_{1,j}\right)^3 \left(k_{2,i} - k_{2,j}\right) + 2\left(k_{4,i} - k_{4,j}\right)\left(k_{2,i} - k_{2,j}\right) \\
&\quad - 3\left(k_{1,i} - k_{1,j}\right)\left(k_{3,i} - k_{3,j}\right) = 0
\end{aligned}
\tag{20}
$$

for any $1 \le j < i \le N$.

It follows from Theorem 3 that f_N is an N-wave solution to the Hirota bilinear equation (2) if (20) holds for any $1 \le j < i \le N$. However, the corresponding $u(x, y, z, t) = 2(f_N)_x / f_N$ is not necessarily an N-wave solution to the $(3 + 1)$-dimensional Jimbo-Miwa equation (1) even if $\{\eta_1, \eta_2, \ldots, \eta_N\}$ is linear independent. In fact,

$$
\begin{aligned}
u\left(x, y, z, t\right) &= 2\frac{(f_N)_x}{f_N} = 2\frac{\sum_{i=1}^{N} \epsilon_i k_{1,i} e^{\mathbf{k_i} \cdot \mathbf{x}}}{\sum_{i=1}^{N} \epsilon_i e^{\mathbf{k_i} \cdot \mathbf{x}}} \\
&= 2\frac{k_{1,1} + \sum_{i=2}^{N} k_{1,i} \epsilon_i' e^{(\mathbf{k_i} - \mathbf{k_1}) \cdot \mathbf{x}}}{1 + \sum_{i=2}^{N} \epsilon_i' e^{(\mathbf{k_i} - \mathbf{k_1}) \cdot \mathbf{x}}} \\
&= 2k_{1,1} + 2\frac{\sum_{i=2}^{N} k_{1,i}' \epsilon_i' e^{\mathbf{k_i'} \cdot \mathbf{x}}}{1 + \sum_{i=2}^{N} \epsilon_i' e^{\mathbf{k_i'} \cdot \mathbf{x}}},
\end{aligned}
\tag{21}
$$

where $\epsilon_i' = \epsilon_i / \epsilon_1$, $k_{1,i}' = k_{1,i} - k_{1,1}$ and $\mathbf{k_i'} = \mathbf{k_i} - \mathbf{k_1}$. Obviously, the function above is an $N - 1$-wave solution to the $(3 + 1)$-dimensional Jimbo-Miwa equation (1) if $\{\eta_i', i = 2, 3, \ldots, N\}$ is independent; that is, $\{\mathbf{k_i'}, i = 2, \ldots, N\}$ is linear independent, solves (20), and satisfies the dispersion relations. Here, $\mathbf{k_i'} = \{k_{1,i}', k_{2,i}', k_{3,i}', k_{4,i}'\}^T$ satisfies the dispersion relations meaning that $k_{1,i}'^3 k_{2,i}' + 2k_{4,i}' k_{2,i}' - 3k_{1,i}' k_{3,i}' = 0$ holds.

Recall that the goal of this paper is to investigate the multiwave solutions to the $(3 + 1)$-dimensional Jimbo-Miwa equation. Combining the conclusion of Theorem 3 with the

analysis above, we can now state the following theorem on the N-wave solution to the $(3 + 1)$-dimensional Jimbo-Miwa equation (1).

Theorem 4. *Let $\mathbf{k_i} = \{k_{1,i}, k_{2,i}, k_{3,i}, k_{4,i}\}^T$, $\mathbf{x} = \{x, y, z, t\}^T$, $1 \le i \le N$. For any constants g and $\{\epsilon_i, 1 \le i \le N\}$, suppose that*

$$
u\left(x, y, z, t\right) = 2\frac{\sum_{i=1}^{N} k_{1,i} \epsilon_i e^{\mathbf{k_i} \cdot \mathbf{x}}}{1 + \sum_{i=1}^{N} \epsilon_i e^{\mathbf{k_i} \cdot \mathbf{x}}} + g.
\tag{22}
$$

Then $u(x, y, z, t)$ is an N-wave solution to the $(3 + 1)$-dimensional Jimbo-Miwa equation (1), if $\{\mathbf{k_1}, \mathbf{k_2}, \ldots, \mathbf{k_N}\}$ is a linear independent solution set of equation (20) satisfying the dispersion relations.

To get other N-wave solutions to the $(3 + 1)$-dimensional Jimbo-Miwa equation, we study the linear independent solution sets to (20) which are required to satisfy the dispersion relations. Actually, (20) is a system possessing $N(N + 1)/2$ coupled equations (plus the dispersion conditions). It is usually not so easy to get the solutions of (20). Fortunately, the number of the equations is 1 when $N = 1$ and it is 3 when $N = 2$, which might make it easy to get the solutions in these two cases. Clearly, 1-wave solution is the traveling wave solution. Let us check what kinds of traveling wave solutions we can obtain by the Hirota bilinear method first.

For the case $N = 1$ in (22), we only need to find the independent solutions $\mathbf{k_1}$ of the dispersion relation

$$
\left(k_{1,1}\right)^3 k_{2,1} + 2k_{4,1} k_{2,1} - 3k_{1,1} k_{3,1} = 0.
\tag{23}
$$

That is to say, we get the traveling wave solutions to the $(3+1)$-dimensional Jimbo-Miwa equation from (20). Suppose that $l_i, i = 1, 2, 3, 4$ are any arbitrary constants, $l_1 \ne 0$ and $l_3 = l_2(l_1^3 + 2l_4)/(3l_1)$; then $\mathbf{k_1} = \{l_1, l_2, l_3, l_4\}^T$ satisfies (23), and thus

$$
u\left(x, y, z, t\right) = 2\frac{l_1 e^{\mathbf{k_i} \cdot \mathbf{x}}}{1 + \epsilon e^{\mathbf{k_i} \cdot \mathbf{x}}} + g,
\tag{24}
$$

where ϵ and g are any arbitrary constants, is a 1-wave solution, that is, traveling wave solution to the $(3+1)$-dimensional Jimbo-Miwa equation.

By further computation, (24) could be rewritten as

$$
u\left(x, y, z, t\right) = g - \frac{2l_1}{1 + \epsilon e^{\mathbf{k_i} \cdot \mathbf{x}}}.
\tag{25}
$$

For $\epsilon > 0$,

$$
\frac{2l_1}{1 + \epsilon e^{\mathbf{k_i} \cdot \mathbf{x}}} = l_1 \left[1 - \tanh \frac{1}{2}\left(\mathbf{k_i} \cdot \mathbf{x} + \boldsymbol{\xi_0}\right)\right]
\tag{26}
$$

and, for $\epsilon < 0$,

$$
\frac{2l_1}{1 + \epsilon e^{\mathbf{k_i} \cdot \mathbf{x}}} = l_1 \left[1 - \coth \frac{1}{2}\left(\mathbf{k_i} \cdot \mathbf{x} + \boldsymbol{\xi_0}\right)\right],
\tag{27}
$$

where $\xi_0 = \ln |\epsilon|$ is an arbitrary constant. Consequently, (24) can be rewritten as

$$
u\left(x, y, z, t\right) = g - l_1 \tanh \frac{1}{2}\left(\mathbf{k_i} \cdot \mathbf{x} + \boldsymbol{\xi_0}\right),
\tag{28}
$$

$$
u\left(x, y, z, t\right) = g - l_1 \coth \frac{1}{2}\left(\mathbf{k_i} \cdot \mathbf{x} + \boldsymbol{\xi_0}\right).
\tag{29}
$$

By parametric transformation, we can find that the traveling wave solution (28) is exactly the same as solution (19) in Section 2. However, (29) is a family of unbounded solutions.

3.1. 2-Wave Solutions to the $(3 + 1)$-Dimensional Jimbo-Miwa Equation.

To obtain the 2-wave solutions to the $(3 + 1)$-dimensional Jimbo-Miwa equation, we now study the solutions to the following coupled algebraic equations:

$$k_{1,1}^3 k_{2,1} + 2k_{4,1} k_{2,1} - 3k_{1,1} k_{3,1} = 0,$$

$$k_{1,2}^3 k_{2,2} + 2k_{4,2} k_{2,2} - 3k_{1,2} k_{3,2} = 0,$$

$$(k_{1,2} - k_{1,1})^3 (k_{2,2} - k_{2,1}) + 2 (k_{4,2} - k_{4,1}) (k_{2,2} - k_{2,1})$$

$$-3 (k_{1,2} - k_{1,1}) (k_{3,2} - k_{3,1}) = 0. \tag{30}$$

From (30), we get

$$k_{3,1} = \frac{k_{1,1}^3 + 2k_{4,1}}{3k_{1,1}} k_{2,1}, \tag{31}$$

$$k_{3,2} = \frac{k_{1,2}^3 + 2k_{4,2}}{3k_{1,2}} k_{2,2}, \tag{32}$$

$$k_{3,2} - k_{3,1} = \frac{(k_{1,2} - k_{1,1})^3 + 2 (k_{4,2} - k_{4,1})}{3 (k_{1,2} - k_{1,1})}$$

$$\times (k_{2,2} - k_{2,1}). \tag{33}$$

Clearly, from (31)–(33), we have

$$\frac{(k_{1,2} - k_{1,1})^3 + 2 (k_{4,2} - k_{4,1})}{3 (k_{1,2} - k_{1,1})} (k_{2,2} - k_{2,1})$$

$$= \frac{k_{1,2}^3 + 2k_{4,2}}{3k_{1,2}} k_{2,2} - \frac{k_{1,1}^3 + 2k_{4,1}}{3k_{1,1}} k_{2,1}, \tag{34}$$

from which we get

$$k_{2,2} = \Bigg(\Big(\big(k_{1,1}^3 + 2k_{4,1} \big) (k_{1,2} - k_{1,1})$$

$$- k_{1,1} \left[(k_{1,2} - k_{1,1})^3 + 2 (k_{4,2} - k_{4,1}) \right] \Big)$$

$$\times \Big(\big(k_{1,2}^3 + 2k_{4,2} \big) (k_{1,2} - k_{1,1})$$

$$- k_{1,2} \left[(k_{1,2} - k_{1,1})^3 + 2 (k_{4,2} - k_{4,1}) \right] \Big)^{-1} \Bigg)$$

$$\times \frac{k_{1,1}}{k_{1,2}} k_{2,1}. \tag{35}$$

Now, substituting (35) into (32), we obtain

$$k_{3,2} = \Bigg(\Big(\big(k_{1,1}^3 + 2k_{4,1} \big) (k_{1,2} - k_{1,1})$$

$$- k_{1,1} \left[(k_{1,2} - k_{1,1})^3 + 2 (k_{4,2} - k_{4,1}) \right] \Big)$$

$$\times \Big(\big(k_{1,2}^3 + 2k_{4,2} \big) (k_{1,2} - k_{1,1})$$

$$- k_{1,2} \left[(k_{1,2} - k_{1,1})^3 + 2 (k_{4,2} - k_{4,1}) \right] \Big)^{-1} \Bigg)$$

$$\times \frac{\left(k_{1,2}^3 + 2k_{4,2} \right)}{3k_{1,2}^2} k_{1,1} k_{2,1}. \tag{36}$$

Consequently, we get the 2-wave solutions to the $(3 + 1)$-dimensional Jimbo-Miwa equation and we can now state the following theorem.

Theorem 5. Let $\mathbf{k_i} = \{k_{1,i}, k_{2,i}, k_{3,i}, k_{4,i}\}^T$, $\mathbf{x} = \{x, y, z, t\}^T$, $i = 1, 2$. For any constants g and $\{\epsilon_i, i = 1, 2\}$,

$$u(x, y, z, t) = 2 \frac{k_{1,1} \epsilon_1 e^{\mathbf{k_1} \cdot \mathbf{x}} + k_{1,2} \epsilon_2 e^{\mathbf{k_2} \cdot \mathbf{x}}}{1 + \epsilon_1 e^{\mathbf{k_1} \cdot \mathbf{x}} + \epsilon_2 e^{\mathbf{k_2} \cdot \mathbf{x}}} + g \tag{37}$$

is a 2-wave solution to the $(3 + 1)$-dimensional Jimbo-Miwa equation (1), where $k_{1,1}$, $k_{1,2}$, $k_{2,1}$, $k_{4,1}$, and $k_{4,2}$ are arbitrary constants and the constants $k_{3,1}$, $k_{2,2}$, and $k_{3,2}$ are determined by (31), (35), and (36), respectively.

3.2. N-Wave Solutions to the $(3 + 1)$-Dimensional Jimbo-Miwa Equation.

To get the N-wave solutions to the $(3 + 1)$-dimensional Jimbo-Miwa equation, we need to investigate the independent solution set $\{\mathbf{k_1}, \mathbf{k_2}, \ldots, \mathbf{k_N}\}$ of (20), which satisfies the dispersion relations. Generally, it is very difficult to get the solution if N is greater than 3 in which case the number of (20) is $N(N + 1)/2$. However, it might be possible to solve these equations by assuming some special relations between these parameters [30, 31].

Following the idea in [30, 31], suppose that $\mathbf{k_i} = \{a_1 k_i^{n_1}, a_2 k_i^{n_2}, a_3 k_i^{n_3}, a_4 k_i^{n_4}\}^T$, $i = 1, 2, \ldots, N$, where $a_j's$ and $n_j's$, $j = 1, 2, 3, 4$, are parameters to be determined later. To get the possible solutions to (20), we choose (n_1, n_2, n_3, n_4) to be $(1, 2, 4, 3)$ or $(1, -1, 1, 3)$.

For the case when $(n_1, n_2, n_3, n_4) = (1, 2, 4, 3)$, substituting $\mathbf{k_i} = \{a_1 k_i^{n_1}, a_2 k_i^{n_2}, a_3 k_i^{n_3}, a_4 k_i^{n_4}\}^T$, $i = 1, 2, \ldots, N$, into (20) gives

$$a_1^3 a_2 (k_i - k_j)^3 (k_i^2 - k_j^2) + 2a_4 a_2 (k_i^3 - k_j^3) (k_i^2 - k_j^2)$$

$$- 3a_1 a_4 (k_i - k_j) (k_i^4 - k_j^4) = 0. \tag{38}$$

Obviously, (38) holds for arbitrary values of k_i and k_j if and only if a_1, a_2, a_3, and a_4 satisfy the two equations

$$2a_2 (a_1^3 + 2a_4) - 3a_1 a_3 = 0,$$

$$a_2 (a_1^3 - 4a_4) = 0. \tag{39}$$

Solving (39), we get $a_4 = (1/4)a_1^3$, $a_3 = a_1^2 a_2$, where a_1 and a_2 are free parameters. It is easy to see that $\mathbf{k_i} = \{a_1 k_i, a_2 k_i^2, a_3 k_i^4, a_4 k_i^3\}^T$, $i = 1, 2, \ldots, N$, satisfy the dispersion condition. Consequently, we know that the $(3 + 1)$-dimensional Jimbo-Miwa equation (1) possesses N-wave solution for any arbitrary positive integer N. Thus, we have the following theorem.

Theorem 6. *For any arbitrary positive integer N and for any different k_i, $i = 1, 2, \ldots, N$,*

$$
\begin{aligned}
&u\left(x, y, z, t\right) \\
&= 2a_1 \frac{\sum_{i=1}^{N} k_i \epsilon_i e^{a_1 k_i x + a_2 k_i^2 y + a_1^2 a_2 k_i^4 z + (1/4) a_1^3 k_i^3 t}}{1 + \sum_{i=1}^{N} \epsilon_i e^{a_1 k_i x + a_2 k_i^2 y + a_1^2 a_2 k_i^4 z + (1/4) a_1^3 k_i^3 t}} + g
\end{aligned}
\tag{40}
$$

is a family of N-wave solutions to the $(3 + 1)$-dimensional Jimbo-Miwa equation (1), *where ϵ_i, $i = 1, 2, \ldots, N$, and a_1 and a_2 are arbitrary constants.*

For the case when $(n_1, n_2, n_3, n_4) = (1, -1, 1, 3)$, substituting $\mathbf{k_i} = \{a_1 k_i, a_2 k_i^{-1}, a_3 k_i, a_4 k_i^3\}^T$, $i = 1, 2, \ldots, N$, into (20) gives

$$
\begin{aligned}
&a_1^3 a_2 \left(k_i - k_j\right)^3 \left(k_i^{-1} - k_j^{-1}\right) + 2a_4 a_2 \left(k_i^3 - k_j^3\right) \left(k_i^{-1} - k_j^{-1}\right) \\
&\quad - 3a_1 a_4 \left(k_i - k_j\right) \left(k_i^3 - k_j^3\right) = 0.
\end{aligned}
\tag{41}
$$

Clearly, (41) holds for arbitrary values of k_i and k_j if and only if a_1, a_2, a_3, and a_4 satisfy

$$
\begin{aligned}
2a_2 \left(a_1^3 - a_4\right) - 3a_1 a_3 &= 0, \\
a_2 \left(a_1^3 + 2a_4\right) &= 0.
\end{aligned}
\tag{42}
$$

Solving (42), we obtain $a_4 = -a_1^3/2$, $a_3 = a_1^2 a_2$, where a_1 and a_2 are free parameters. Unfortunately, unlike the case when $(n_1, n_2, n_3, n_4) = (1, 2, 4, 3)$ above, it is easy to see that $\mathbf{k_i} = \{a_1 k_i, a_2 k_i^{-1}, a_3 k_i, a_4 k_i^3\}^T$, $i = 1, 2, \ldots, N$, do not satisfy the dispersion condition any more. However, if we let $\mathbf{k_0} = \{a_1, a_2, a_3, a_4\}^T$, then, for any $i = 1, 2, \ldots, N$,

$$
\begin{aligned}
\mathbf{k_i'} = \mathbf{k_i} - \mathbf{k_0} = \Big\{ &a_1 \left(k_i - 1\right), a_2 \left(k_i^{-1} - 1\right), \\
&a_3 (k_i - 1), a_4 (k_i^3 - 1) \Big\}^T
\end{aligned}
\tag{43}
$$

satisfies the dispersion relations because $\{\mathbf{k_0}, \mathbf{k_1}, \mathbf{k_2}, \ldots, \mathbf{k_N}\}$ is a solution set of (20).

Consequently, besides the family of N-wave solution (40), the $(3+1)$-dimensional Jimbo-Miwa equation (1) has another group of N-wave solution for any arbitrary positive integer N and so we have the following theorem.

Theorem 7. *For any arbitrary positive integer N and for any different k_i, $i = 1, 2, \ldots, N$,*

$$
\begin{aligned}
&u\left(x, y, z, t\right) \\
&= 2a_1 \frac{\sum_{i=1}^{N} \left(k_i - 1\right) \epsilon_i e^{a_1(k_i-1)x + a_2(k_i^{-1}-1)y + a_1^2 a_2(k_i-1)z - (1/2)a_1^3(k_i^3-1)t}}{1 + \sum_{i=1}^{N} \epsilon_i e^{a_1(k_i-1)x + a_2(k_i^{-1}-1)y + a_1^2 a_2(k_i-1)z - (1/2)a_1^3(k_i^3-1)t}} \\
&\quad + g
\end{aligned}
\tag{44}
$$

is a family of N-wave solutions to the $(3 + 1)$-dimensional Jimbo-Miwa equation (1), *where ϵ_i, $i = 1, 2, \ldots, N$, and a_1 and a_2 are arbitrary constants.*

4. Concluding Remarks

The dynamical system theory was employed to study the traveling wave solutions of the $(3 + 1)$-dimensional Jimbo-Miwa equation (1). The multiwave solutions were investigated by studying the linear space of the corresponding Hirota bilinear equation. The exact formulas of two families of multiwave solutions of any order were obtained as well. Wazwaz [38] employed Hirota's bilinear method and derived 1-wave and 2-wave solutions to this equation and stated that it is completely integrable and it admits multiple-soliton solutions of any order. In this paper we explicitly obtained two different families of N-wave solutions to the $(3 + 1)$-dimensional Jimbo-Miwa equation (1) given by (40) and (44) and this confirmed the statement given in [38] that the Jimbo-Miwa equation (1) admits multiple-wave solutions of any order N.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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